

# Chapter 3

## Free Modal Algebras Revisited: The Step-by-Step Method

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*In memory of Leo Esakia and Dito Pataraidze*

**Abstract** We review the step-by-step method of constructing finitely generated free modal algebras. First we discuss the global step-by-step method, which works well for rank one modal logics. Next we refine the global step-by-step method to obtain the local step-by-step method, which is applicable beyond rank one modal logics. In particular, we show that it works well for constructing the finitely generated free algebras for such well-known modal systems as **T**, **K4** and **S4**. This yields the notions of one-step algebras and of one-step frames, as well as of universal one-step extensions of one-step algebras and of one-step frames. We show that finitely generated free algebras for **T**, **K4** and **S4** and their dual spaces can be obtained by iterating the universal one-step extensions of one-step algebras and of one-step frames. In the final part of the chapter we compare our construction with recent literature, especially with [11] which undertakes a very similar approach.

**Keywords** Free modal algebras · One-step algebras · One-step frames · Duality · Step-by-step method

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### 3.1 Introduction

Having a good description of finitely generated free algebras is an important tool for investigating propositional logics: free algebras give an insight on the shape of formulae and on the deduction mechanism that is independent of the particular syntactic methodology used for introducing a logical calculus. With a clear combinatorial and conceptual description of free algebras in mind, one can investigate better metatheoretical properties like admissibility of inference rules, solvability of equations, definability and interpretability matters, etc.

For modal logics (or some other non-classical logics such as intuitionistic logic), one way to characterize finitely generated free algebras is to use the relevant properties of their dual spaces: many of these algebras are atomic [3, 4, 9], thus restricting dual spaces to atoms still gives the possibility of having a representation theorem. The spaces of the atoms become the so-called ‘universal models’ and finitely generated free algebras can be described as the algebras of definable subsets of these models. Atoms, in turn, generate ‘irreducible’ or ‘definable’ finite models (along the suggestions of [21, 27]): such models can be described inductively, using for instance the height of the model. This line of research has been largely explored in a long series of papers in the 1970s and 1980s by the Georgian school (see, e.g., [15, 16, 24]), the Russian school (see, e.g., [30, 31]), and the Italian logician F. Bellissima (see, e.g., [3–5]). We refer to [6, 10] for an overview.

Still, duality can be used in another way to get descriptions of finitely generated free algebras: formulae naturally come equipped with a complexity measure, the measure counting maximum nested ‘intensional’ operators occurring in them. By ‘intensional’ operators we mean modal operators (or implication in the context of intuitionistic or relevance logic); non-intensional operators are the Boolean connectives (or a subset of them) and it is well known that only a finite number of combinations of such operators can be applied (up to logical equivalence) to a finite set of given formulae. Thus, finitely generated free algebras have a ‘dual profinite’ description as chain colimits of finite algebras defined by imposing complexity bounds. By finite duality, these finite algebras admit a description as finite discrete spaces and the intensional operators (which are only partially defined on them) induce a kind of combinatorial structure. The investigation of such combinatorial structure paves the way to a new, different description of finitely generated free algebras, a description that we call a *step-by-step* description: its essence is in fact the dual explanation of what it means to enrich a given set of formulae by one application of intensional operators followed by the finite closure with respect to the non-intensional operators. Of course, the whole construction should not destroy previously introduced intensional operators, that is why it applies to a ‘one-step algebra’ and results into an updated ‘one-step algebra’.

The origin of these step-by-step constructions is two-fold: from the logical point of view they describe normal forms (in the sense of K. Fine [17]), and from the coalgebraic point of view they correspond to free coalgebra constructions ([1, 2, 8, 11, 20]). However, coalgebraic constructions work well for rank one logics (e.g.,

$\mathbf{K}$  and  $\mathbf{D}$  in modal logic), but become unclear or quite involved when arbitrary subvarieties/logics are involved. Our plan is to exploit discrete dualities and the step-by-step combinatorics to illustrate potential applications of the method outside the rank one case.

We start the chapter by reviewing the general idea of the step-by-step method which, via duality, results in the dual description of free  $\mathbf{K}$ -algebras and free algebras for rank one modal logics. We then extend this method to logics of higher rank using the new notion of a one-step algebra and its dual notion of a one-step frame. We adjust these ideas to particular non-rank one logics such as  $\mathbf{T}$ ,  $\mathbf{K4}$  and  $\mathbf{S4}$  obtaining (duals of) finitely generated free algebras of these logics in a transparent and modular way. Our construction follows closely the method developed in [11] for various non-rank one modal logics. We discuss the similarities and differences of our construction with that of [11], as well as with those of [20, 22]. We also list a number of (challenging) open problems.

In the chapter we discuss finitely generated free Heyting algebras only briefly, and refer the interested reader to [7, 19] for details. However, we believe that one-step algebras and one-step frames have a potential to play an important role in the theory of free algebras in various varieties of Heyting algebras.

We conclude our introduction by pointing out that another co-author of this paper was going to be Dito Pataraiia. He developed interest towards the step-by-step method after Leo Esakia drew his attention to [19] where free Heyting algebras were described via this method. Both Dito and Leo were very interested in this method. Dito's interest towards this construction was mostly determined by its use in proving that every Heyting algebra can be realized as the subobject classifier of an elementary topos. Dito gave a few talks about this important theorem and his colleagues are now trying to reconstruct his very involved and original proof, a part of which essentially uses the step-by-step method. Dito had a number of deep observations on the step-by-step construction for free Heyting and modal algebras, and many of them were supposed to form part of this chapter. Leo was interested in this method as it gives an alternative and useful perspective on Esakia spaces of free Heyting algebras (see [18] for more details on this). In fact, Esakia duality for Heyting algebras plays a prominent role in this and nearly all other approaches that apply the ideas of duality to various constructions of Heyting algebras. With great sadness for their loss, but with a lot of admiration for their outstanding scientific achievements, their unique character and personality, we would like to dedicate this chapter to the memory of Leo Esakia and Dito Pataraiia.

## 3.2 The Global Step-by-Step Method

In this section we recall the global step-by-step method and construct free  $\mathbf{K}$ -algebras. As we will see, this method works nicely for rank one modal logics, but its extension to arbitrary modal logics, although possible, is quite laborious and involved [20]. We recall that a modal formula is of *rank one* if each occurrence of atomic formulas

(i.e., propositional variables or constants) is under the scope of exactly one modal operator; moreover, the constants  $\perp$ ,  $\top$ , may appear in rank one formulae without being under the scope of any modal operator. A modal logic is of rank one if it can be axiomatized by rank one modal formulae. In Sect. 3.3 we will refine the global step-by-step method to the local step-by-step method that works for modal logics of higher rank. This will result in a neat description of finitely generated free algebras for the well-known modal systems **T**, **K4** and **S4**.

Recall that a *modal algebra* is a pair  $(B, \diamond)$ , where  $B$  is a Boolean algebra and  $\diamond : B \rightarrow B$  is a unary operation satisfying  $\diamond 0 = 0$  and  $\diamond(a \vee b) = \diamond a \vee \diamond b$  for each  $a, b \in B$ . A modal algebra  $(B, \diamond)$  is called a **T**-algebra if  $a \leq \diamond a$  and a **K4**-algebra if  $\diamond \diamond a \leq \diamond a$  for each  $a \in B$ . Finally,  $(B, \diamond)$  is an **S4**-algebra if it is both a **T**-algebra and a **K4**-algebra.

### 3.2.1 Algebraic View

The method we discuss in this section is taken essentially from [1, 20] (see also [8]). Thus, we only sketch the construction and refer the interested reader to any of [1, 8, 20] for details.

Given a Boolean algebra  $B$ , we let  $V(B)$  denote the free Boolean algebra generated by the set  $\{\diamond a : a \in B\}$  and quotiented by the two axioms defining modal algebras. Alternatively  $V(B)$  is the free Boolean algebra over the join-semilattice  $\vee$ , 0-reduct (equivalently  $\wedge$ , 1-reduct) of  $B$ . That is, the map  $i_{\diamond}^B : B \rightarrow V(B)$  (mapping  $a \in B$  to  $\diamond a \in V(B)$ ) is such that it is a join-semilattice morphism (preserves  $\vee$  and 0), and for any Boolean algebra  $A$ , any join-semilattice morphism  $h : B \rightarrow A$  can be extended uniquely to a Boolean homomorphism  $h^T : V(B) \rightarrow A$  so that  $h^T \circ i_{\diamond}^B = h$ . Actually,  $V$  can be turned into an endofunctor on the category of Boolean algebras in a standard way: by letting  $V(f)$  (for  $f : B \rightarrow C$ ) be  $(i_{\diamond}^C \circ f)^T$ . Note that the correspondence  $h \mapsto h^T$  is bijective and, as a consequence, modal algebras can be equivalently defined as Boolean algebras  $B$  equipped with semilattice morphisms  $\diamond^T : V(B) \rightarrow B$  (we shall exploit this fact below).

Let  $B_0$  be the free Boolean algebra on  $n$ -generators. For each  $k \geq 0$  we let

$$B_{k+1} = B_0 + V(B_k),$$

where  $+$  means the coproduct in the category of Boolean algebras. As Boolean algebras are locally finite, the coproduct of two finite Boolean algebras is again finite.

We define the maps  $i_k : B_k \rightarrow B_{k+1}$  and  $\diamond_k^T : V(B_k) \rightarrow B_{k+1}$  as follows. Let  $\diamond_k^T$  be the second injection into the coproduct, and let  $i_k$  be defined recursively as follows:  $i_0$  is the first coproduct injection and  $i_{k+1}$  is  $id + V(i_k)$ . Let  $B_{\infty}$  be the colimit of the following diagram in the category of Boolean algebras and Boolean homomorphisms

$$B_0 \xrightarrow{i_0} B_1 \rightarrow \cdots \rightarrow B_k \xrightarrow{i_k} B_{k+1} \rightarrow \cdots \quad (3.1)$$

**Proposition 1** For each  $k \geq 0$  we have  $\diamond_{k+1}^T \circ V(i_k) = i_{k+1} \circ \diamond_k^T$ . Therefore,  $\{\diamond_k^T : k \geq 0\}$  can be extended to a map  $\diamond_\infty^T : V(B_\infty) \rightarrow B_\infty$ .

*Proof (Sketch)* That  $\diamond_{k+1}^T \circ V(i_k) = i_{k+1} \circ \diamond_k^T$  holds can be easily seen by a direct computation (recall that the two diamonds are just the coproduct injections). To define  $\diamond_\infty^T$  one can then use the fact that  $V$  commutes with chain (more generally with filtered) colimits: thus, we can assume that the domain of  $\diamond_\infty^T$  is the colimit of

$$V(B_0) \xrightarrow{V(i_0)} V(B_1) \rightarrow \cdots \rightarrow V(B_k) \xrightarrow{V(i_k)} V(B_{k+1}) \rightarrow \cdots \quad (3.2)$$

in the category of Boolean algebras and Boolean homomorphisms. Now the maps  $\diamond_k^T : V(B_k) \rightarrow B_{k+1}$  form vertical maps from the chain (3.2) to the chain (3.1) commuting the related squares, hence it induces a colimit map which is our  $\diamond_\infty^T$ .  $\square$

Let  $\diamond_k : B_k \rightarrow B_{k+1}$  be the map that corresponds to  $\diamond_k^T : V(B_k) \rightarrow B_{k+1}$  and let  $\diamond_\infty : B_\infty \rightarrow B_\infty$  be the map that corresponds to  $\diamond_\infty^T : V(B_\infty) \rightarrow B_\infty$ . Then we have the following characterization of finitely generated free modal algebras.

**Proposition 2** The algebra  $\langle B_\infty, \diamond_\infty \rangle$  is the free modal algebra on  $n$  generators.

*Proof* See [1, 8, 20].  $\square$

Let  $L$  be a normal modal logic and  $\mathbf{V}_L$  the corresponding variety of modal algebras. We also let  $\text{Ax}(L)$  be a (finite or infinite) equational axiomatization of  $\mathbf{V}_L$ . We will now briefly sketch how to extend the above method to obtain finitely generated free  $\mathbf{V}_L$ -algebras.

If we try to quotient (3.1) by the axioms of  $\text{Ax}(L)$ , we need to interpret modal formulae into the steps of a chain colimit algebra and then take a quotient of the algebras in the chain. The definition of such interpretation must take into account the fact that the axioms have arbitrary modal rank, hence the interpretation involves many algebras at a time. If the axioms have modal rank one, the situation simplifies because we can modify uniformly the whole construction, by taking instead of  $V$  a suitable quotient of it [8, 20]. Examples of logics of rank one include  $\mathbf{D} = \mathbf{K} + \diamond\top = \mathbf{K} + \Box p \rightarrow \diamond p$ ,  $\mathbf{K} + \diamond p \rightarrow \Box p$  and  $\mathbf{K} + \Box p \leftrightarrow \diamond p$ .

### 3.2.2 Dual View

For the purposes of our paper it suffices to restrict ourselves to the discrete duality between finite modal algebras and finite relational structures. Almost all the results can be generalized to the infinite case by defining an appropriate Stone topology on relational structures (see, e.g., [18]). We chose to stick to the finite duality to keep the arguments simple.

We recall that there is a one-to-one correspondence between join-preserving maps between finite Boolean algebras and relations on their dual finite sets. In fact, the category of finite Boolean algebras and  $\vee$ , 0-preserving maps is dually equivalent to the category of finite sets and binary relations (see, e.g., [25, 28]).

Let  $X$  be a set and  $\wp(X)$  its powerset. Then it is easy to see that for each  $Y$ , any relation  $R \subseteq Y \times X$  uniquely corresponds to a map  $f : Y \rightarrow \wp(X)$  defined by  $f(y) = R(y) = \{x \in X : yRx\}$ . Throughout this paper we will use twofold notations for binary relations interchangeably: as subsets of a cartesian product or as maps into the powerset.

Given a finite set  $X$ , let  $\ni$  be the relation on  $\wp(X) \times X$  defined by  $U \ni x$  iff  $x \in U$  for each  $U \in \wp(X)$  and  $x \in X$ . Then  $X$ ,  $\wp(X)$  and  $\ni$  have the following universal property: for each finite  $Y$  and  $R \subseteq Y \times X$ , there exists a unique map  $f : Y \rightarrow \wp(X)$  (defined by  $f(y) = R(y)$ ) such that we have  $R = \ni \circ f$ . The last equation refers to relational composition, i.e. it means that for each  $x \in X$ ,  $y \in Y$  we have  $yRx$  iff (there is  $S \in \wp(X)$  such that  $x \in S$  and  $S = f(y)$ ) iff  $x \in f(y)$ .

*Remark 1* In the general case we need to consider Stone spaces and continuous relations and maps. But the same correspondence holds in this case as well. That is, if  $Y$  is a Stone space (i.e., the dual of a Boolean algebra) and  $R$  is a continuous relation (i.e., the dual of a join-preserving map), then  $f$  is also a continuous map (i.e., the dual of a Boolean algebra homomorphism).

Translating this into algebraic terms gives us that the join-semilattice morphism dual to  $\ni$  and the Boolean algebra dual to  $\wp(X)$  satisfy the universal property of  $V(B)$  and  $i_{\diamond}^B$  discussed in the previous section. Thus, as the universal property defines an object uniquely up to an isomorphism, we obtain the following theorem.

**Proposition 3** ([20, Prop. 2.1], [32]) *Let  $B$  be a finite Boolean algebra and  $X$  its dual finite set. Then the algebra  $V(B)$  is dual to  $\wp(X)$ . Moreover, the map  $i_{\diamond}^B : B \rightarrow V(B)$  is dual to the relation  $\ni \subseteq \wp(X) \times X$ .*

*Remark 2* We note that this result can be generalized to the infinite case by considering Stone spaces and continuous maps and relations, and by taking the Vietoris space instead of the finite powerset. We also refer to [26, Chap. III.4] for generalizations of this result to compact regular frames.

Now we are ready to construct the duals of free modal algebras. Let  $X_0$  be a  $2^n$ -element set (the dual of  $B_0$ ) and (because of the duality of  $\wp$  and  $V$  (Proposition 3) and of  $\times$  and  $+$ ) let

$$X_{k+1} = X_0 \times \wp(X_k).$$

**Proposition 4** *The sequence  $(X_k)_{k < \omega}$  with the maps  $\pi_k : X_0 \times \wp(X_k) \rightarrow X_k$  defined by*

$$\pi_0(x, U) = x, \quad \pi_k(x, U) = (x, \pi_{k-1}[U])$$

*is dual to the sequence  $(B_k)_{k < \omega}$  with the maps  $i_k : B_k \rightarrow B_{k+1}$ . In particular, the  $\pi_k$  are surjective. Moreover, the relation  $R_k \subseteq (X_0 \times \wp(X_k)) \times X_k$  defined by*

$$(x, U)R_k y \text{ iff } y \in U$$

is dual to  $\diamond_k : B_k \rightarrow B_{k+1}$ .

**Theorem 1** *Let  $X_\omega$  be the limit in the category of Stone spaces and continuous maps of the diagram  $(X_k)_{k \in \omega}$  with the maps  $\pi_{k+1} : X_{k+1} \rightarrow X_k$ . Let also  $R_\omega$  be the limit of  $(R_k)_{k \in \omega}$  in the category of Stone spaces and continuous relations defined by  $(x_i)_{i \in \omega} R_\omega (y_i)_{i \in \omega}$  iff  $x_{k+1} R_k y_k$  for each  $k \in \omega$ . Then  $(X_\omega, R_\omega)$  is (isomorphic in a suitable category to) the dual of the free modal algebra  $(B_\omega, \diamond_\omega)$ .*

Thus, via the global step-by-step method we described finitely generated free modal algebras and their dual spaces.

### 3.3 The Local Step-by-Step Method

The construction presented in the previous section is very useful for logics axiomatized by rank one equations. It, however, also has some drawbacks. For example, there is no manageable way to apply it to the well-known extensions of  $\mathbf{K}$  such as  $\mathbf{K4}$  and  $\mathbf{S4}$  (it works for  $\mathbf{T}$ , but for the other systems the adaptation is involved, see [20]). The point is that the definition of step  $k + 1$  mentions not only step  $k$  but also step 0, which is rather unnatural. In this section we introduce a refinement of the construction. From an algebraic point of view, the new construction may be considered as just a trivial variant of the former one. Nevertheless it induces better constructions at the dual level. Its distinguishing feature is that the construction is local in that it relies on the universal property of the one-step construction (see [11, 19, 22] for similar ideas).

#### 3.3.1 Algebraic View

As we pointed out in the introduction, the essence of the method we propose is to build free algebras in steps; a single step (taken *independently* on the whole chain of steps needed to build the free algebra as a colimit) applies the modal operators to the existing propositions and embeds the actual propositions into the new ones. This leads naturally to a *two-sorted* viewpoint: we have one algebra for the actual propositions and another one collecting actual and new propositions; moreover the two algebras are connected by an embedding and a diamond. All this is formally captured by the following definition.

##### Definition 1

1. A *one-step modal algebra* is a quadruple  $(A_0, A_1, i_0, \diamond_0)$ , where  $A_0, A_1$  are Boolean algebras,  $i_0 : A_0 \rightarrow A_1$  is a Boolean morphism, and  $\diamond_0 : A_0 \rightarrow A_1$  is

a semilattice morphism. The algebras  $A_0, A_1$  are called the *source* and the *target* Boolean algebras of the one-step modal algebra  $(A_0, A_1, i_0, \diamond_0)$ .

2. A *one-step extension* of the one-step modal algebra  $(A_0, A_1, i_0, \diamond_0)$  is a one-step modal algebra  $(A_1, A_2, i_1, \diamond_1)$  (i.e., it is a one-step modal algebra whose source is the same as the target of  $(A_0, A_1, i_0, \diamond_0)$ ) satisfying  $i_1 \diamond_0 = \diamond_1 i_0$ .
3. The *universal one step-extension* of  $(A_0, A_1, i_0, \diamond_0)$  is a one-step extension  $(A_1, A_2, i_1, \diamond_1)$  such that for every other one-step extension  $(A_1, A'_2, i'_1, \diamond'_1)$ , there is a unique Boolean morphism  $\mu : A_2 \rightarrow A'_2$  such that  $\mu \circ i_1 = i'_1$  and  $\mu \circ \diamond_1 = \diamond'_1$ . The meaning of the universal one-step extension is that it represents the general solution to the problem of adding  $\diamond_1 a_1$  for all  $a_1 \in A_1$  while keeping (through  $i_1$ ) the  $\diamond_0 a_0$  for  $a \in A_0$ .

Universal one-step extensions exist and are easily built through pushouts:

**Proposition 5** *The universal one-step extension of  $(A_0, A_1, i_0, \diamond_0)$  is given by the following pushout taken in the category of Boolean algebras and Boolean homomorphisms.*

$$\begin{array}{ccc} V(A_0) & \xrightarrow{V(i_0)} & V(A_1) \\ \diamond_0^T \downarrow & & \downarrow \diamond_1^T \\ A_1 & \xrightarrow{i_1} & A_2 \end{array}$$

where  $A_2 = A_1 +_{V(A_0)} V(A_1)$ .

*Proof* Immediate by the universal property of pushouts.  $\square$

Let  $B_0$  be the free Boolean algebra on  $n$ -generators. We define a new sequence  $B_k$  by using pushouts. In the new sequence, the algebras  $B'_0, B'_1$  and the morphisms  $i'_0, \diamond_0^T$  are as before; for  $k \geq 1$ , we have instead

$$B'_{k+1} := B'_k +_{V(B'_{k-1})} V(B'_k)$$

where  $i'_k, \diamond_k^T$  are the canonical maps into the pushout

$$\begin{array}{ccc} V(B'_{k-1}) & \xrightarrow{\diamond_{k-1}^T} & B'_k \\ V(i'_{k-1}) \downarrow & & \downarrow i'_k \\ V(B'_k) & \xrightarrow{\diamond_k^T} & B'_{k+1} \end{array}$$

Let  $B'_\infty$  be the colimit of the diagram



$$B_0 \xrightarrow{i'_0} B'_1 \rightarrow \dots \rightarrow B'_k \xrightarrow{i'_k} B'_{k+1} \rightarrow \dots \quad (3.3)$$

Then we have

**Proposition 6**  $\{\diamond_k^T : k \geq 0\}$  can be extended to a map  $\diamond_\infty^T : V(B'_\infty) \rightarrow B'_\infty$  such that  $(B'_\infty, \diamond_\infty)$  is the  $n$ -generated free modal algebra.

*Proof (Sketch)* That  $\diamond_{k+1}^T \circ V(i'_k) = i'_{k+1} \circ \diamond_k^T$  holds now comes directly from the commutativity of the above pushout square. Thus,  $\diamond_\infty^T$  can be defined in the colimit like in the proof of Proposition 1. To show that the construction gives finitely generated free modal algebras, one can use Proposition 5, along the lines of e.g. [22] (alternatively, it is possible to show inductively that the construction is isomorphic to the global step-by-step construction of Proposition 2).  $\square$

### 3.3.2 Dual View

The dual construction is described through the notion of a one-step frame.

**Definition 2** A *one-step frame* is a quadruple  $(X, Y, f, R)$ , where  $X, Y$  are sets,  $f : X \rightarrow Y$  is a map and  $R \subseteq X \times \wp(Y)$  is a relation between  $X$  and  $Y$ .

The dual of a finite one-step frame  $(X, Y, f, R)$  is the one-step modal algebra  $(\wp(Y), \wp(X), f^*, \diamond_R)$ , where  $f^*$  is the inverse image operation and  $\diamond_R$  is the semi-lattice morphism associated with  $R$  (for  $A \subseteq Y$ , we have  $\diamond_R(A) = \{x \in X \mid R(x) \cap A \neq \emptyset\}$ ). Every finite one-step modal algebra is the dual of a finite one-step frame (again, to extend this duality beyond the finite case, Stone spaces, continuous maps, continuous relations and Vietoris spaces are needed).

We can now dualize the local construction of Proposition 6 as follows: the dual of  $i'_k$  is given by a map  $f_k : X_{k+1} \rightarrow X_k$  between finite sets; the dual of  $\diamond_k$  is a map  $R_k : X_{k+1} \rightarrow \wp(X_k)$  (alternatively a relation  $R_k \subseteq X_{k+1} \times X_k$ ). Then the dual of  $(B'_k, B'_{k+1}, i'_k, \diamond_k)$  is determined by the following diagram:

$$\begin{array}{ccc}
 X_{k+1} & \xrightarrow{R_k} & \wp(X_k) \\
 \downarrow f_k & & \downarrow \wp(f_{k-1}) \\
 X_k & \xrightarrow{R_{k-1}} & \wp(X_{k-1}) \\
 \downarrow f_{k-1} & & \\
 X_{k-1} & & 
 \end{array}$$

where the square is a pullback, and for each map  $f : X \rightarrow Y$ , we assume that  $\wp(f) = f[\cdot] : \wp(X) \rightarrow \wp(Y)$  is the direct image of  $f$ . Thus, we have

$$X_{k+1} = \{(x, S) \mid x \in X_k, S \in \wp(X_k), R_{k-1}(x) = f_{k-1}(S)\},$$

with

$$f_k(x, S) = x$$

and

$$R_k(x, S) = S.$$

We must also consider the dual of the first step of the chain leading to free modal algebras. This is more simple, being just a coproduct not a pushout. In short, if  $X_0$  is the finite set dual to the Boolean algebra  $B_0$ , the dual of the one-step modal algebra  $(B_0, B_0 + V(B_0), i_0, \diamond_0)$  (where recall that  $i_0, \diamond_0$  are the two coproduct injections) is the one-step frame

$$(X_0 \times \wp(X_0), X_0, f_0, R_0) \quad (3.4)$$

where  $f_0$  is the first projection and we have  $R_0(x, S) = S$  for all  $x \in X_0, S \subseteq X_0$ .

### 3.3.3 Adding Equations

The main advantage of the second method is that, when building  $B'_{k+1}$ , it refers only to  $B'_k$  (and not also to  $B'_0$ ): this makes descriptions of quotients modulo further equations easier. Suppose in fact that we are given some axioms  $\text{Ax}(L)$  for a logic  $L$ . Following a suggestion by Coumans and van Gool [11], we can rewrite an axiom in the form of a quasi-identity

$$t = 1 \rightarrow v = 1 \quad (3.5)$$

where the terms/formulae  $t, v$  have modal complexity less or equal to one (i.e., nested modal operators do not occur). To achieve this, one can repeat the following ‘flattening’ of quasi-equations  $E \rightarrow v = 1$  sufficiently many times (we start with  $E = \emptyset$ ): take a subterm  $\diamond v'$  of  $v$ , pick a fresh variable  $x$  and replace  $E \rightarrow v = 1$  with  $E \cup \{x = \diamond v'\} \rightarrow v(x/\diamond v') = 1$ . Finally, quasi-identities having many premises can be turned into single-premise quasi-identities by taking conjunctions.

The quasi-equations of this kind can be interpreted in a one-step modal algebra  $(A_0, A_1, i_0, \diamond_0)$ : once an assignment  $\mathfrak{a}$  of variables to members of  $A_0$  is given, we can recursively define the element  $t^{\mathfrak{a}} \in A_1$  for every term  $t$  having modal complexity 0 or 1. An equation  $t = 1$  of modal complexity at most 1 is valid in  $(A_0, A_1, i_0, \diamond_0)$  iff  $t^{\mathfrak{a}} = 1$  holds in  $A_1$  for every  $\mathfrak{a}$ ; similarly one can define validity of quasi-equations. Thus, an  $\text{Ax}(L)$ -one-step modal algebra is a one-step modal algebra where all quasi-equations belonging to  $\text{Ax}(L)$  are valid; notice that this notion is relative not just to

a logic  $L$ , but to a set  $\text{Ax}(L)$  of quasi-equations (i.e. of inference rules) chosen in order to axiomatize  $L$ .

Validity of conditions such as (3.5) can be forced by taking a quotient; since (3.5) is a quasi-equation (and not just an equation), we need a quotient which is iterated: one just quotients  $A_1$  by the filter generated by all  $v^a$  such that  $t^a = 1$  holds in  $A_1$  (varying  $a$ ) and then repeats this procedure  $\omega$ -times (or just sufficiently many finite times if  $A_1$  is finite). In the end, one gets a one-step modal algebra  $(A_0, A_1/F, q \circ i_0, q \circ \diamond_0)$  (where  $F$  is the filter obtained in the end of the iteration and  $q : A_1 \rightarrow A_1/F$  is the canonical map onto the quotient) that satisfies (3.5) and is universal with this property.

We define a chain

$$B_0 \xrightarrow{i''_0} B''_1 \rightarrow \dots \rightarrow B''_k \xrightarrow{i''_k} B''_{k+1} \rightarrow \dots \quad (3.6)$$

of Boolean algebras equipped with semilattice morphisms

$$B_0 \xrightarrow{\diamond_0} B''_1 \rightarrow \dots \rightarrow B''_k \xrightarrow{\diamond_k} B''_{k+1} \rightarrow \dots \quad (3.7)$$

satisfying the conditions  $\diamond_{k+1} \circ i''_k = i''_{k+1} \circ \diamond_k$  (equivalently,  $\diamond_{k+1}^T \circ V(i''_k) = i''_{k+1} \circ \diamond_k^T$ ) and such that for every  $k \geq 0$ , the one-step modal algebra  $(B''_k, B''_{k+1}, i''_k, \diamond_k)$  satisfies  $\text{Ax}(L)$ . This is done by the same construction as in Proposition 6, with the only difference that we also apply the aforementioned quotient by  $\text{Ax}(L)$ ; that is, we define  $B''_{k+1}$  by taking a pushout

$$B''_k +_{V(B''_{k-1})} V(B''_k)$$

followed by a quotient by  $\text{Ax}(L)$ . Let  $B''_\infty$  be the colimit of this diagram. Then:

**Proposition 7** *For each  $k \geq 0$ , we have that  $\diamond_{k+1}^T \circ i''_k = i''_{k+1} \circ \diamond_k^T$ . Therefore,  $\{\diamond_k^T : k \geq 0\}$  can be extended to a map  $\diamond_\infty^T : V(B''_\infty) \rightarrow B''_\infty$ , so that the algebra  $(B''_\infty, \diamond_\infty)$  is the  $n$ -generated free  $\mathbf{V}_L$ -algebra.*

*Proof* That  $(B''_\infty, \diamond_\infty)$  is free is proved in the same way as in Proposition 6. That we quotient each approximant by  $\text{Ax}(L)$  guarantees that  $(B''_\infty, \diamond_\infty)$  satisfies  $\text{Ax}(L)$ , and hence is a  $\mathbf{V}_L$ -algebra.  $\square$

What is not guaranteed in general here is that the maps  $i''_k$  are injective; this is unavoidable, giving the fact that there are undecidable logics:

**Proposition 8** *If for each  $k \geq 0$  the maps  $i''_k : B''_k \rightarrow B''_{k+1}$  are injective, then  $B''_k$  is isomorphic to a Boolean subalgebra of all terms of complexity  $k$  of the free  $\mathbf{V}_L$ -algebra, and moreover the logic  $L$  is decidable.*

*Proof* Given terms  $t, u$  whose complexity is less than, say  $k$ , we can define their canonical realizations  $[t], [u] \in B''_\infty$  and  $[t]_k, [u]_k \in B''_k$  (this is quite straightforward

and intuitive, see, e.g., [20]); notice also that we have  $\iota_k([t]_k) = [t]$  and  $\iota_k([u]_k) = [u]$ . Under the obvious identification of terms and propositional formulae, it is evident that (since  $B_\infty$  is the free  $V_L$ -algebra)  $[t] = [u]$  holds iff  $t \leftrightarrow u$  is provable in the logic  $L$ . Since  $\iota_k$  is a function, we have that  $[t]_k = [u]_k$  implies  $[t] = [u]$ ; according to the standard algebraic colimit construction, the converse is true in case the maps  $i''_k$  for  $k \leq \tilde{k}$  are all injective, whence the claim of the proposition. The statement about decidability is clear: to check whether an identity  $t = 1$  holds, it is sufficient to inspect whether  $[t]_k = 1$  holds, where  $k$  is the complexity of  $t$ .  $\square$

The above relation  $[t]_k = [u]_k$  is quite interesting from the proof-theoretic point of view: it means that it is possible to establish  $t \leftrightarrow u$  via a proof involving formulae whose complexity does not exceed  $k$ . The existence of a proof whose complexity is bounded by the size of the formula to be proved is an evidence for a nice proof-theoretic behavior of the given axiomatization for a logic. It is also quite a desirable property sufficient to entail decidability. Thus the above one-step algebraic approach provides an interesting tool also from a purely proof-theoretic perspective.

From the dual point of view, one should try to understand in terms of dual one-step frames what it means for a one-step algebra  $(A_0, A_1, i_0, \diamond_0)$  to satisfy a set of quasi-equations (3.5). For this, one needs to develop the *one-step correspondence theory*. The goal of this one-step correspondence theory is to characterize in the two-sorted predicate language for one-step frames what it means for a one-step frame that the dual one-step modal algebra satisfies  $\text{Ax}(L)$  (ideally, the characterization should be manageable and possibly first-order).

Once this is understood, one has to understand further what it means from the dual point of view to build a quotient making a one-step algebra  $(A_0, A_1, i_0, \diamond_0)$  an algebra satisfying the quasi-equations (3.5) occurring in  $\text{Ax}(L)$ . In view of the applications, it is sufficient to characterize the case in which  $A_0, A_1$  are both finite. Armed by this characterization, if one is able to prove that the duals of the  $i''_k$  are surjective, one can conclude that the logic is decidable. If the duals of the  $i''_k$  are not surjective, one can try with a different axiomatization of the logic  $L$ . In conclusion, the duality task is threefold:

- (dt1) develop one-step correspondence theory;
- (dt2) have a nice characterization of the dual of the following operation: take a finite one-step algebra  $(A_0, A_1, i_0, \diamond_0)$  satisfying  $\text{Ax}(L)$ , build the universal one-step extension of it and make it a one-step algebra satisfying  $\text{Ax}(L)$  again (we call this the *one step-Ax(L)-extension* of  $(A_0, A_1, i_0, \diamond_0)$ );
- (dt3) have a nice characterization of the dual of the following operation: take a finite set  $X_0$ , build the one-step modal algebra dual to the one-step frame (3.4) and make it a one-step algebra satisfying  $\text{Ax}(L)$ .

Usually (dt3) is quite easy, while (dt1)-(dt2) are different for each particular logic. We will discuss some cases in detail below.

### 3.4 Free $\mathbf{K4}$ -Algebras

We start by considering task (dt1) for  $\mathbf{K4}$ . As we will see, it is accomplished by Proposition 9 below. We say that a one step-frame *validates* a quasi-equation if the corresponding one-step modal algebra  $(\wp(Y), \wp(X), f^*, \diamond_R)$  validates this quasi equation.

**Definition 3** A one-step frame  $(X, Y, f, R)$  is *transitive* if it validates the  $\mathbf{K4}$ -quasi-equation

$$a \leq \diamond b \Rightarrow \diamond a \leq \diamond b \quad (3.8)$$

i.e., if

$$f^*(A) \subseteq \diamond_R(B) \Rightarrow \diamond_R(A) \subseteq \diamond_R(B) \quad (3.9)$$

holds for all  $A, B \subseteq Y$ .

For  $S \subseteq X$  and  $x \in X$ , define  $S_x := \{\tilde{x} \in S \mid R(\tilde{x}) \subseteq R(x)\}$ .

**Proposition 9** A one-step frame  $(X, Y, f, R)$  is transitive iff

$$y \in R(x) \Rightarrow f^*(y) \cap X_x \neq \emptyset \quad (3.10)$$

holds for all  $x \in X, y \in Y$ .

*Proof* Assume (3.10) and pick,  $A, B, x$  such that  $f^*(A) \subseteq \diamond_R(B)$  and  $x \in \diamond_R(A)$ . The goal is to show that  $x \in \diamond_R(B)$ . From  $x \in \diamond_R(A)$ , we obtain a  $y \in R(x) \cap A$ . Pick  $z \in f^*(y) \cap X_x$ . Since  $f(z) = y \in A$ , we have  $z \in f^*(A) \subseteq \diamond_R(B)$ , hence  $R(z) \cap B \neq \emptyset$ , giving also  $R(x) \cap B \neq \emptyset$  (because  $z \in X_x$ ). Thus,  $x \in \diamond_R(B)$ , as desired. Conversely, assume (3.9) and pick  $y \in R(x)$ . If  $f^*(y) \cap X_x$  is empty, then for every  $z \in f^*(y)$ , there is  $w \in R(z)$  such that  $w \notin R(x)$ . Let  $A := \{y\}$  and let  $B$  be the complement of  $R(x)$ . Then  $f^*(A) \subseteq \diamond_R(B)$ , hence  $\diamond_R(A) \subseteq \diamond_R(B)$ . Since  $y \in R(x)$ , it follows that  $x \in \diamond_R(B)$ , i.e. that  $R(x) \cap B$  is not empty, a contradiction because  $B$  is the complement of  $R(x)$ .  $\square$

If  $(X, Y, f, R)$  is a one-step frame, there is the largest  $X^\sharp \subseteq X$  such that  $(X^\sharp, Y, f, R)$  is transitive: in fact, the set of all  $\tilde{X} \subseteq X$  such that for all  $x \in \tilde{X}$

$$\forall y \in Y (y \in R(x) \Rightarrow f^*(y) \cap \tilde{X}_x \neq \emptyset)$$

(with  $f, R$  restricted to  $\tilde{X}$  in the domain) is closed under unions and hence has the largest element  $X^\sharp$ . A *subframe* of a one-step frame is obtained by restricting functions and relations to some subset of a given frame. The largest subset  $X^\sharp$  gives rise to the one-step subframe  $(X^\sharp, Y, f|_{X^\sharp}, R|_{X^\sharp})$  (obtained by restricting  $f, R$  to  $X^\sharp$  in the domain) that corresponds to the quotient modulo the quasi-equation (3.8). In general, one cannot say more than that: we just need to characterize the one-step subframe arising in tasks (dt2)–(dt3).

For (dt3) the situation is trivial: given any finite set  $X_0$ , the one-step frame (3.4) is already transitive, so the one-step transitive subframe we are looking for is the whole one-step frame in this case.

Task (dt2) requires to characterize the universal one-step **K4**-extension of a finite transitive one-step frame (the universal one-step extension of a finite transitive one-step frame is obviously defined to be the dual of the universal one-step **K4**-extension of the corresponding dual finite one-step **K4**-algebra).

In short, we get the following notion. Given a transitive finite one-step frame  $(X, Y, f, R)$ , the *universal one-step **K4**-extension* of it is the largest transitive one-step frame  $(X^\sharp, X, f', R') \subseteq (X', X, f', R')$ , where  $X', f', R'$  are defined as follows

- $X' = \{(x, S) \in X \times \wp(X) \mid R(x) = f(S)\}$ ;
- $f'(x, S) = x$ ;
- $R'(x, S) = S$ .

According to the above definitions,  $X^\sharp$  is the largest  $\widehat{X} \subseteq X'$  such that

$$(x, S) \in \widehat{X} \Rightarrow (\forall y \in S \exists S' \subseteq S (y, S') \in \widehat{X}). \quad (3.11)$$

To fully accomplish task (dt2), we need here a better explicit characterization of  $X^\sharp$ .

A subset  $S \subseteq X$  of a one-step frame  $(X, Y, f, R)$  is said to be *transitive* (abbreviated  $Tr(S)$ ) if  $(S, Y, f|_S, R|_S)$  is a transitive one-step frame (by  $(-)|_S$  we denote the restriction of a relation or of a function to a subset  $S$  of its domain).

**Proposition 10**  $X^\sharp = \{(x, S) \mid R(x) = f(S) \ \& \ R(S) \subseteq R(x) \ \& \ Tr(S)\}$ .

*Proof* We must show that (1)  $X^\sharp$  satisfies condition (3.11); (2) if  $\widehat{X}$  satisfies condition (3.11), then  $\widehat{X} \subseteq X^\sharp$ .

Ad (1): Take  $(x, S) \in X^\sharp$  and  $y \in S$ . Define

$$S' = \{\widehat{y} \in S \mid f(\widehat{y}) \in R(y) \ \& \ R(\widehat{y}) \subseteq R(y)\}. \quad (3.12)$$

We show that  $(y, S') \in X^\sharp$ . First,  $R(S') \subseteq R(y)$  and  $f(S') \subseteq R(y)$  are immediate from the definition of  $S'$ . To show that  $R(y) \subseteq f(S')$ , notice that since  $S$  is transitive, for all  $z \in R(y)$  there is  $y_z \in S$  such that  $f(y_z) = z$  and  $R(y_z) \subseteq R(y)$ . This shows that  $z \in f(S')$ . It remains to show that  $S'$  is transitive. Let  $\widehat{y} \in S'$  and  $w \in R(\widehat{y})$ . Since  $S' \subseteq S$  and  $S$  is transitive, there is  $s \in S$  such that  $f(s) = w$  and  $R(s) \subseteq R(\widehat{y})$ . We only need to prove that  $s \in S'$ , i.e. that (a)  $f(s) \in R(y)$  and (b)  $R(s) \subseteq R(y)$ . Since  $\widehat{y} \in S'$ , we have  $R(\widehat{y}) \subseteq R(y)$  and this implies  $f(s) \in R(y)$  (because  $f(s) = w \in R(\widehat{y})$ ). Thus, (a) holds. For (b), observe that  $\widehat{y} \in S'$  implies  $R(\widehat{y}) \subseteq R(y)$ . Since we also have  $R(s) \subseteq R(\widehat{y})$ , we obtain  $R(s) \subseteq R(y)$ , i.e. (b) holds.

Ad (2): Let  $\widehat{X}$  satisfy (3.11) and let  $(x, S) \in \widehat{X}$ . We show that  $(x, S) \in X^\sharp$ . To show that  $R(S) \subseteq R(x)$ , take  $y \in S$ . Then, according to (3.11), there is  $S' \subseteq S$  such that  $(y, S') \in \widehat{X}$ . Thus,  $R(y) = f(S') \subseteq f(S) = R(x)$ , and consequently,  $R(y) \subseteq R(x)$ . So  $R(S) \subseteq R(x)$  ( $y \in S$  is arbitrary), as required. It remains to verify that  $S$  is transitive. Consider  $y \in S$  and  $z \in R(y)$ . Then there is  $S'$  such that

$(y, S') \in \widehat{X}$ , which implies  $R(y) = f(S')$ . Therefore, there is  $s \in S'$  such that  $f(s) = z$ . However,  $R(S') \subseteq R(y)$  follows from  $(y, S') \in \widehat{X}$  (we just proved that this applies to all members of  $\widehat{X}$ ), hence  $R(s) \subseteq R(y)$ .  $\square$

The following proposition says that we can also apply Proposition 8 in this case.

**Proposition 11** *If  $(X, Y, f, R)$  is a transitive finite one-step frame, then  $f'$  restricted to  $X^\sharp$  is surjective.*

*Proof* It is sufficient to show that for every  $x \in X$ , we have that the pair  $(x, S)$  belongs to  $X^\sharp$ , where  $S$  is given by

$$S = \{\widehat{y} \in X \mid f(\widehat{y}) \in R(x) \ \& \ R(\widehat{y}) \subseteq R(x)\}.$$

This follows from the fact that  $X$  is transitive (in the same way as case (i) above).  $\square$

Let  $X_0$  be a  $2^n$ -element set and let

$$X_1 = X_0 \times \wp(X_0), \quad X_{k+1} = X_k^\sharp \quad (k \geq 1).$$

We also let  $f_k : X_{k+1} \rightarrow X_k$  and  $R_k : X_{k+1} \rightarrow \wp(X_k)$  be defined by  $f_k(x, S) = x$  and  $R_k(x, S) = S$ . Then using duality and Propositions 7, 8, 10, and 11 we arrive at the following result (we refer to the statement of Theorem 1 for the indication of the appropriate categories where limits below are taken in):

**Theorem 2** *The limit  $(X_\omega, R_\omega)$  of the sequence  $\{(X_k, X_{k+1}, f_k, R_k) : k < \omega\}$  is (isomorphic to) the dual of the free  $n$ -generated **K4**-algebra. Moreover, each  $X_k$  is dual to the algebra of all **K4**-equivalent terms of complexity  $k$ .*

### 3.5 Free S4-Algebras

We first deal with the **T**-case.

**Definition 4** A one-step frame  $(X, Y, f, R)$  is *reflexive* if it validates the **T**-equation

$$a \leq \diamond a$$

i.e., if

$$f^*(A) \subseteq \diamond_R(A) \tag{3.13}$$

holds for all  $A \subseteq Y$ .

Task (dt1) is accomplished by the following easy proposition:

**Proposition 12** *A one-step frame  $(X, Y, f, R)$  is reflexive iff*

$$f(x) \in R(x) \tag{3.14}$$

*holds for all  $x \in X$ .*

*Proof* Suppose  $x \in f^*(A)$  for some  $A \subseteq Y$ . By (3.14),  $f(x) \in R(x)$  and so  $R(x) \cap A \neq \emptyset$ . Thus,  $x \in \diamond_R(A)$ , satisfying (3.13). Conversely, suppose  $x \in X$  is such that  $f(x) \notin R(x)$ . Let  $A = X \setminus R(x)$ . Then  $f(x) \in A$  and  $R(x) \cap A = \emptyset$ . So  $x \in f^*(A)$  and  $x \notin \diamond_R(A)$ . Therefore,  $f^*(A) \not\subseteq \diamond_R(A)$ , refuting (3.13).  $\square$

Clearly the largest reflexive one-step subframe of a one-step frame  $(X, Y, f, R)$  is obtained by taking the subset of  $X$  formed by those  $x$  such that  $f(x) \in R(x)$  and by restricting  $f$  and  $R$  to it. This observation accomplishes also task (dt3). For (dt2), we have an obvious notion of the *universal one-step  $\mathbf{T}$ -extension* of a reflexive one-step frame  $(X, Y, f, R)$ . This is the largest reflexive one-step frame  $(X^\sharp, X, f', R') \subseteq (X', X, f', R')$ , where  $X', f', R'$  are defined as follows

- $X' = \{(x, S) \in X \times \wp(X) \mid R(x) = f(S)\}$ ;
- $f'(x, S) = x$ ;
- $R'(x, S) = S$ .

We immediately have that

**Proposition 13**  $X^\sharp = \{(x, S) \mid R(x) = f(S) \ \& \ x \in S\}$ .

*Proof* The proof is similar to the proof of Proposition 10. We must show that  $X^\sharp$  is the largest subset  $\widehat{X} \subseteq X'$  satisfying the condition

$$(x, S) \in \widehat{X} \Rightarrow x \in S. \tag{3.15}$$

But this immediately follows from the definition of  $X^\sharp$ .  $\square$

As a consequence, we obtain:

**Proposition 14** *If  $(X, Y, f, R)$  is a reflexive finite one-step frame such that  $f$  is surjective, then  $f'$  restricted to  $X^\sharp$  is surjective.*

*Proof* Let  $x \in X$ . We need to find  $S \subseteq X$  such that  $R(x) = f(S)$  and  $x \in S$ . Let  $S = f^*(R(x))$ . Then, as  $f$  is surjective, we have  $f(S) = f(f^*(R(x))) = R(x)$ . As  $X$  is reflexive, by Proposition 12,  $f(x) \in R(x)$ . So  $x \in f^*(R(x)) = S$ . Therefore,  $(x, S) \in X^\sharp$  and  $f'(x, S) = x$ . Thus,  $f'$  is surjective.  $\square$

Then if  $X_k$ 's,  $f_k$ 's and  $R_k$ 's are as in the previous section (of course  $X^\sharp$  is as in Proposition 13), we have the following:

**Theorem 3** *The limit  $(X_\omega, R_\omega)$  of the sequence  $\{(X_k, X_{k+1}, f_k, R_k) : k < \omega\}$  is (isomorphic to) the dual of the free  $n$ -generated  $\mathbf{T}$ -algebra. Moreover, each  $X_k$  is dual to the algebra of all  $\mathbf{T}$ -equivalent terms of complexity  $k$ .*



Finally, the relevant tasks for **S4** are obtained by combining the above results for **T** and **K4**. The combination is not entirely straightforward because we need to carefully revisit all proofs. More precisely, the *universal one-step S4-extension* of a reflexive and transitive one-step frame  $(X, Y, f, R)$  is the largest reflexive and transitive one-step frame  $(X^\sharp, X, f', R') \subseteq (X', X, f', R')$ , where  $X', f', R'$  are defined as follows

- $X' = \{(x, S) \in X \times \wp(X) \mid R(x) = f(S)\}$ ;
- $f'(x, S) = x$ ;
- $R'(x, S) = S$ .

We have:

**Proposition 15**  $X^\sharp = \{(x, S) \mid R(x) = f(S) \ \& \ R(S) \subseteq R(x) \ \& \ Tr(S) \ \& \ x \in S\}$ .

*Proof* We can repeat word by word the proof of Proposition 10 (we only need to observe that if  $(X, Y, f, R)$  is reflexive, then  $S'$  as defined in (3.12) is such that  $y \in S'$ ).  $\square$

Next we let the  $X_k$ 's,  $f_k$ 's and  $R_k$ 's be the same as in the previous section with  $X^\sharp$  as in Proposition 15. Then, since we can prove surjectivity by the same argument used in Proposition 11, we obtain the following:

**Theorem 4** *The limit  $(X_\omega, R_\omega)$  of the sequence  $\{(X_k, X_{k+1}, f_k, R_k) : k < \omega\}$  is (isomorphic to) the dual of the free  $n$ -generated **S4**-algebra. Moreover, each  $X_k$  is dual to the algebra of all **S4**-equivalent terms of complexity  $k$ .*

### 3.6 Comparison to Other Approaches

In the recent literature, various approaches have been proposed in order to build free modal algebras as chain colimits iterating some one-step construction. In the end, all constructions must be isomorphic, however the differences they induce in the dual combinatorics of finite frames might be significant, especially from the point of view of concrete manageability and ease of manipulation. This is why it is worth trying to make a closer comparison. The origins of the global step-by-step method for the basic modal logic **K** go back to Abramsky's 1988 British Logic Colloquium talk (the paper [1] based on this talk appeared a decade later). In [20] a detailed account of the global step-by-step method for **K** is given and, moreover, it is extended to other modal logics. In [8] the same construction is put in the context of coalgebra, underlying direct applications to rank one modal logics and is generalized to the basic positive modal logic.

In [22] the step-by-step approach to the construction of free algebras is applied to the **S4** case. The solution adopted there is to build a chain of finite **S4**-algebras

$$(B_0, \diamond_0) \xrightarrow{i_0} (B_1, \diamond_1) \rightarrow \cdots \rightarrow (B_k, \diamond_k) \xrightarrow{i_k} (B_{k+1}, \diamond_{k+1}) \rightarrow \cdots \quad (3.16)$$

where the morphisms  $i_k$  are *continuous* and *relatively open*, which means that they fully preserve ‘old’ diamonds (i.e. diamonds of the kind  $\diamond_k i_{k-1}(a)$ ) and just semi-preserve ‘last’ (i.e. the other) diamonds. In essence, this means that in each algebra there are ‘core’ diamonds that are defined once and for all and some other diamonds that are defined ‘by completion’ in a temporary fashion. The merit of this construction is that the duals of the  $(B_k, \diamond_k)$  are relatively nicely defined preordered sets. However, it is not clear how far this technique (implicitly relying on the existence of a kind of ‘S4-completion by continuity’) can be pushed because its adaptation to general logics looks unclear.

To overcome this potential problem, in [11] free algebras are built up from a chain such as (3.16), where the algebras  $(B_k, \diamond_k)$  are *partial*, i.e. diamonds are only partially defined there. In addition, it is required that the image of  $i_k$  is in the domain of the next diamond  $\diamond_{k+1}$ . If we view a partial algebra  $(B, \diamond)$  as a pair

$$B_0 \subseteq B, \quad B_0 \xrightarrow{\diamond} B$$

(where  $B_0$  is the Boolean subalgebra of  $B$  which is the domain of  $\diamond$ ), then it is clear that

$$(B_0, B, i, \diamond)$$

is a one-step modal algebra in our sense (here  $i$  is the inclusion morphism). This is essentially the alternative description of partial algebras as described in [11, Remark 2.2]. The only difference is that our definition of a one-step modal algebra *does not require injectivity of  $i$* . This difference might be considered quite immaterial – and in fact it is whenever one is able to prove that in the end the duals of the  $i_k$  are surjective maps (see Propositions 11,14). However, taking the injectivity requirement out is essential for *our recipe based on the tasks* (dt1)-(dt2)-(dt3) to be formulated and applied. Thus, this approach supplies an algebraic tool to investigate proof-theoretic aspects, as explained in Proposition 8.

The step-by-step method for free Heyting algebras was first introduced in [19]. As an application of this method, it was shown that free finitely generated Heyting algebras are in fact bi-Heyting algebras. In [7] a modular approach to this construction was developed. It is similar to the local step-by-step approach of [11] and of the current paper. It is our understanding that the technique of [7] can be rephrased in terms of one-step Heyting algebras and the corresponding one-step frames.

### 3.7 Open Problems

We conclude by listing some open problems. The basic question is whether the local step-by-step method applies (or can be adjusted) to other important modal logics. As possible candidates we suggest the logic  $\mathbf{wK4} = \mathbf{K} + (\diamond\diamond p \rightarrow p \vee \diamond p)$  of weakly transitive frames [13] or, more generally, the logics of  $n$ -transitive frames [29].

Another challenging test case is the provability logic **GL**. It is known that **GL** is axiomatized over **K4** by a one-step rule

$$(\Box p \leq p) \rightarrow (p = 1),$$

see [12, 23]. Thus, the technique developed in Sect. 3.3 can be applied to **GL**. Also [12, 14] give a similar axiomatization of the logic **Grz** over **K4** by the rule

$$(\Box(p \rightarrow \Box p) \leq p) \rightarrow (p = 1).$$

Strictly speaking this is not a one-step rule, but we can up to equivalence flatten the rule to

$$(q = p \rightarrow \Box p) \& (\Box q \leq p) \rightarrow (p = 1)$$

so that it fits our purposes. We leave it as an open problem whether free **GL** and **Grz**-algebras could be described using the technique developed in this paper.

Finally, our results (Proposition 9 and 12) suggest that there might exist some Sahlqvist-like correspondence for one-step frames. An investigation of this correspondence is an interesting topic on its own, but it will also undoubtedly shed a new light on the problem of obtaining free algebras for modal logics via the step-by-step method.

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