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Morava K-theory rings for the groups $G_{38},\cdots,G_{41}$ of order 32

by

Malkhaz Bakuradze and Mamuka Jibladze*

Abstract

B. Schuster [19] proved that the *mod* 2 Morava K-theory $K(s)^*(BG)$ is evenly generated for all groups G of order 32. For the four groups G of order 32 with the numbers 38, 39, 40 and 41 in the Hall-Senior list [11], the ring $K(2)^*(BG)$ has been shown to be generated as a $K(2)^*$-module by transferred Euler classes. In this paper, we show this for arbitrary s and compute the ring structure of $K(s)^*(BG)$. Namely, we show that $K(s)^*(BG)$ is the quotient of a polynomial ring in 6 variables over $K(s)^*(pt)$ by an ideal for which we list explicit generators.

Key Words: Transfer, Morava K-theory.

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1. Introduction and Statements

Let $K(s)^*$, $s > 1$, be the s-th Morava K-theory at 2. In this paper we compute the ring structure of $K(s)^*(BG)$ for the four groups $G = G_{38},...,G_{41}$ from the Hall-Senior list [11], by showing that $K(s)^*(BG)$ is the quotient of a polynomial ring $K(s)^*(pt)[a,b,c,x_2,y_2,T]$ by a certain ideal R for which we give explicit generators.

A finite group G is said to be good [12] if $K(s)^*(BG)$ is generated as a $K(s)^*$-module by transfers of Euler classes of complex representations. Special effort was needed to find an example of a group not good in this sense [17]. For the additive structure, the principal calculational tool is the Atiyah-Hirzebruch spectral sequence [2, 3] and the Serre SS [17]. Even if the additive structure is calculated, the multiplicative structure is still a delicate task. It is not always determined by representation theory, i.e., G does not have exact Chern approximation in the terminology of Strickland [29]. Also the presentation of $K(s)^*(BG)$ in terms of the formal group law and splitting principle [16] is not always convenient. This clearly indicates that the part of the relations which can be derived from the properties of the transfer should play decisive role in determining the whole ring structure.

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In the current paper we will consider four groups $G = G_{38},\cdots,G_{41}$ of order 32 from the Hall-Senior list [11]. It is proved in [19] that $K(s)^*(BG)$ is evenly generated and for $s=2$ is generated by Euler classes and transferred Euler classes. One consequence of our main theorem below is that this is true for any s. We obtain generators for the ideal R above by using the formula for transferred Euler class from [7] and follow a certain plan, which proved to be sufficient to handle the 2-groups D, SD, QD, Q [9], [6] and modular p-groups [4]. For a discussion of the ring structure of all other groups of order 32 see [19], [20].

Let G be one of the groups

$$\begin{aligned}
G_{38} &= \langle \mathbf{a},\mathbf{b},\mathbf{c} \mid \mathbf{a}^4 = \mathbf{b}^2 = \mathbf{c}^4 = [\mathbf{a},\mathbf{b}] = 1, \mathbf{cac}^{-1} = \mathbf{ac}^2, \mathbf{cbc}^{-1} = \mathbf{a}^2\mathbf{b}\rangle,\\
G_{39} &= \langle \mathbf{a},\mathbf{b},\mathbf{c} \mid \mathbf{a}^4 = \mathbf{b}^4 = \mathbf{c}^2 = [\mathbf{a},\mathbf{b}] = 1, \mathbf{cac} = \mathbf{a}^3, \mathbf{cbc} = \mathbf{a}^2\mathbf{b}^3\rangle,\\
G_{40} &= \langle \mathbf{a},\mathbf{b},\mathbf{c} \mid \mathbf{a}^4 = \mathbf{b}^4 = 1, \mathbf{c}^2 = \mathbf{b}^2, [\mathbf{a},\mathbf{b}] = 1, \mathbf{c}^{-1}\mathbf{ac} = \mathbf{a}^3, \mathbf{c}^{-1}\mathbf{bc} = \mathbf{a}^2\mathbf{b}^3\rangle,\\
G_{41} &= \langle \mathbf{a},\mathbf{b},\mathbf{c} \mid \mathbf{a}^4 = \mathbf{b}^4 = \mathbf{c}^2 = [\mathbf{a},\mathbf{b}] = 1, \mathbf{cac} = \mathbf{a}^3\mathbf{b}^2, \mathbf{cbc} = \mathbf{a}^2\mathbf{b}\rangle.
\end{aligned}$$

Let H be the maximal abelian subgroup of index two $\langle \mathbf{a},\mathbf{b},\mathbf{c}^2\rangle \cong C_4\times C_2\times C_2$ for $G = G_{38}$ and $\langle \mathbf{a},\mathbf{b}\rangle \cong C_4\times C_4$ for all other cases. Let λ, μ and ν denote complex line bundles over BH. For $H \lhd G_{38}$, let

$$\lambda(\mathbf{a}) = i, \mu(\mathbf{b}) = \nu(\mathbf{c}^2) = -1, \lambda(\mathbf{b}) = \lambda(\mathbf{c}^2) = \mu(\mathbf{a}) = \mu(\mathbf{c}^2) = \nu(\mathbf{a}) = \nu(\mathbf{b}) = 1,$$

be the pullbacks of the canonical complex line bundles along the projections onto the first, second and third factor of H respectively. For all other cases, let

$$\lambda(\mathbf{a}) = \nu(\mathbf{b}) = i, \lambda(\mathbf{b}) = \nu(\mathbf{a}) = 1,$$

be the pullbacks of the canonical complex line bundles along the projections onto the first and second factor of H respectively.

The quotient of G by the center is isomorphic to $C_2\times C_2\times C_2$. The projections on the three factors induce three line bundles α, β and γ respectively.

Let us denote Chern classes by

$$x_i = \begin{cases} c_i(Ind_H^G(\lambda)) & \text{for } G = G_{39}, G_{40}\\ c_i(Ind_H^G(\nu)) & \text{for } G = G_{38}, G_{41};\end{cases}$$

$$y_i = \begin{cases} c_i(Ind_H^G(\nu)) & \text{for } G = G_{39}, G_{40}\\ c_i(Ind_H^G)(\lambda) & \text{for } G = G_{38}, G_{41};\end{cases}$$

$$a = \begin{cases} c_1(\alpha) & \text{for } G = G_{38}, G_{41}\\ c_1(\alpha\beta) & \text{for } G = G_{39}, G_{40};\end{cases}$$

$$b = \begin{cases} c_1(\beta) & \text{for } G = G_{38}, G_{39}, G_{40} \\ c_1(\alpha\beta) & \text{for } G = G_{41}; \end{cases}$$

$$c = c_1(\gamma), \text{for all cases.}$$

Let $Tr^* : K(s)^*(BH) \to K(s)^*(BG)$ be the transfer homomorphism [1] associated to the double covering $\rho : BH \to BG$ and let

$$T = Tr^*(c_1(\lambda)c_1(\nu)).$$

Note that by [14], $K(s)^*(pt)$ is the Laurent polynomial ring in one variable, which is usually denoted in our situation by $\mathbb{F}_2[v_s, v_s^{-1}]$, where $\mathbb{F}_2$ is the field of 2 elements.

Our main result is the following

Theorem 1.1 *Let G be one of the groups $G_{38},...,G_{41}$. Then*

i) $K(s)^*(BG) \cong K(s)^*[a,b,c,x_2,y_2,T]/R$, where the ideal R is generated by a^{2^s}, b^{2^s}, c^{2^s},

$$c(c + x_1 + v_s \sum_{i=1}^{s-1} c^{2^s-2^i} x_2^{2^{i-1}}),\ \ c(c + y_1 + v_s \sum_{i=1}^{s-1} c^{2^s-2^i} y_2^{2^{i-1}}),$$

$$a(a + x_1 + v_s \sum_{i=1}^{s-1} a^{2^s-2^i} x_2^{2^{i-1}}),\ \ b(b + y_1 + v_s \sum_{i=1}^{s-1} b^{2^s-2^i} y_2^{2^{i-1}}),$$

$$(c + x_1 + v_s \sum_{i=1}^{s-1} c^{2^s-2^i} x_2^{2^{i-1}})(b + y_1 + v_s \sum_{i=1}^{s-1} b^{2^s-2^i} y_2^{2^{i-1}}) + v_s b^{2^s-1} T,$$

$$(c + y_1 + v_s \sum_{i=1}^{s-1} c^{2^s-2^i} y_2^{2^{i-1}})(a + x_1 + v_s \sum_{i=1}^{s-1} a^{2^s-2^i} x_2^{2^{i-1}}) + v_s a^{2^s-1} T,$$

$$T^2 + T x_1 y_1 + x_2 y_1 (c + y_1 + v_s \sum_{i=1}^{s-1} c^{2^s-2^i} y_2^{2^{i-1}}) + x_1 y_2 (c + x_1 + v_s \sum_{i=1}^{s-1} c^{2^s-2^i} x_2^{2^{i-1}}),$$

$$T(a + x_1 + v_s \sum_{i=1}^{s-1} a^{2^s-2^i} x_2^{2^{i-1}}) + v_s a^{2^s-1} x_2 (c + y_1),$$

$$T(b + y_1 + v_s \sum_{i=1}^{s-1} b^{2^s-2^i} y_2^{2^{i-1}}) + v_s b^{2^s-1} y_2 (c + x_1),\ \ cT, \text{ and}$$

$$v_s^2 x_2^{2^s} + \begin{cases} a^2 + b^2 + ac + v_s abc^{2^s-1} & \text{for } G = G_{39}, G_{40}, G_{41} \\ c^2 + ac & G = G_{38} \end{cases}$$

$$v_s^2 y_2^{2^s} + \begin{cases} a^2 + bc + v_s abc^{2^s-1} & \text{for } G = G_{38}, G_{41} \\ b^2 + bc & G = G_{39} \\ b^2 + c^2 + bc & G = G_{40}, \end{cases}$$

where

$$x_1 = v_s(x_2 + v_s x_1 x_2^{2^{s-1}})^{2^{s-1}} + \begin{cases} a & \text{for } G = G_{38} \\ b + c + v_s(bc)^{2^{s-1}} & G = G_{39}, G_{40}, G_{41}; \end{cases}$$

$$y_1 = v_s(y_2 + v_s y_1 y_2^{2^{s-1}})^{2^{s-1}} + \begin{cases} c & \text{for } G = G_{39} \\ 0 & G = G_{40} \\ a + b + c + & \\ v_s(ab + bc + ac)^{2^{s-1}} & G = G_{38}, G_{41}. \end{cases}$$

ii) Some other relations are
$a^2c = ac^2,\ b^2c = bc^2,\ x_1^{2^s} = a^{2^{s-1}}c^{2^{s-1}},\ y_1^{2^s} = b^{2^{s-1}}c^{2^{s-1}}$.

The rest of the paper is organized as follows. Section 2 presents some preliminaries. In Section 3 we treat the representation theory of the groups under consideration. In Section 4 we derive the relations of Theorem 1.1. Section 5 is devoted to the most difficult part of the proof of Theorem 1.1. Namely for each of our groups (see Lemma 5.2, Lemma 5.6 for G_{39}, G_{40} and Lemma 5.4, Lemma 5.7 for G_{38}, G_{41}) we prove that certain monomials in $a, b, c, x_1, x_2, y_1, y_2, T$ form a basis of $K(s)^*(BG)$ as a free $K(s)^*$-module. It follows c, a, b, x_2, y_2, T are $K(s)^*$-algebra generators as x_1 and y_1 are decomposable in these elements. Finally we prove that the relations in Section 4 provide a complete set of defining relations. For the reader's convenience Section 6 discusses some papers on the subject.

2. Preliminaries

Let $H \lhd G$ be of index 2. Consider the double covering $\pi : BH \to BG$. Let

$$Tr_\pi^* = Tr^*(H, G) : K(s)^*(BH) \to K(s)^*(BG)$$

be the associated transfer homomorphism induced by the stable transfer map [1], [15], [10]. We will need the following transfer formula from [7].

Let $\xi \to BH$ be a complex line bundle and $\xi_\pi = Ind_H^G(\xi)$ be its Atiyah transfer. Then

$$c_1(\xi_\pi) = c_1(\psi) + v_s \sum_{i=1}^{s-1} c_1(\psi)^{2^s - 2^i} c_2(\xi_\pi)^{2^{i-1}} + Tr_\pi^*(c_1(\xi)), \qquad (1)$$

where $\psi \to BG$ is the pullback of the canonical line bundle over $B\mathbb{Z}/2$ along the map $BG \to B\mathbb{Z}/2$ classifying π.

Let

$$u = \begin{cases} c_1(\lambda) & \text{for } G = G_{39}, G_{40} \\ c_1(\nu) & \text{for } G = G_{38}, G_{41} \end{cases}$$

and

$$v = \begin{cases} c_1(\lambda) & \text{for } G = G_{38}, G_{41} \\ c_1(\nu) & \text{for } G = G_{39}, G_{40}. \end{cases}$$

For the Chern classes u, v and $Tr^* = Tr^*(H,G)$ the formula (1) implies

$$Tr^*(u) = c + x_1 + v_s \sum_{i=1}^{s-1} c^{2^s - 2^i} x_2^{2^{i-1}} \tag{2}$$

and

$$Tr^*(v) = c + y_1 + v_s \sum_{i=1}^{s-1} c^{2^s - 2^i} y_2^{2^{i-1}}. \tag{3}$$

One has the following approximation formula for the formal group law in Morava K-theory ([9], Lemma 2.2 ii)).

$$F(x,y) = x + y + v_s \Phi(v_s, x, y)^{2^{s-1}}, \tag{4}$$

where $\Phi(v_s, x, y) = xy + v_s (xy)^{2^{s-1}} (x+y)\, modulo\, (xy)^{2^{s-1}} (x+y)^{2^{s-1}}$.

We also will need the following

Lemma 2.1 *The tensor square of a complex plane vector bundle ζ has the following total Chern class $C(\zeta^{\otimes 2}) = (1 + c_1^2(det \zeta))(1 + v_s c_1^{2^s}(\zeta) + v_s^2 c_2^{2^s}(\zeta))$.*

Proof: Use the splitting principle and write formally

$$\zeta = \xi_1 + \xi_2$$

and

$$c_1(\zeta) = t_1 + t_2;\, c_2(\zeta) = t_1 t_2.$$

We have that the i-th Chern class ($i = 1,2,3,4$) on the right hand side of the bundle relation

$$(\xi_1 + \xi_2) \otimes (\xi_1 + \xi_2) = \xi_1^2 + \xi_2^2 + 2\xi_1 \otimes \xi_2 \tag{5}$$

is the i-th elementary symmetric function in $F(t_1,t_1), F(t_2,t_2), F(t_1,t_2), F(t_1,t_2)$. That is

$$c_i(\zeta^2) = \sigma_i(F(t_1,t_1), F(t_2,t_2), F(t_1,t_2), F(t_1,t_2)).$$

Hence we have for the first Chern class

$$c_1(\zeta^2) = v_s t_1^{2^s} + v_s t_2^{2^s} = v_s c_1^{2^s}(\zeta).$$

For the second Chern class we have

$$c_2(\zeta^2) = v_s t_1^{2^s} v_s t_2^{2^s} + F^2(t_1,t_2) = v_s^2 c_2^{2^s}(\zeta) + c_1^2(det\zeta).$$

Similarly, the third and fourth Chern classes are

$$v_s(t_1^{2^s} + t_2^{2^s})F^2(t_1,t_2) = v_s c_1^{2^s}(\zeta) c_1^2(det\zeta)$$

and

$$v_s t_1^{2^s} v_s t_2^{2^s} F^2(t_1,t_2) = v_s^2 c_2^{2^s}(\zeta) c_1^2(det\zeta)$$

respectively. □

3. Bundle relations

Let us give some relations of bundles over BG we will need. We omit the proofs since they are completely standard and easily follow from the definitions and from Frobenius reciprocity of the transfer in complex K-theory.

Let $\rho : BH \to BG$ be the double covering $\rho = \rho(H,G)$ and let $\lambda_! = Ind_H^G(\lambda)$ and $\nu_! = Ind_H^G(\nu)$ in each of the four cases.

$\mathbf{G = G_{38}}$; $H = \langle \mathbf{a}, \mathbf{b}, \mathbf{c}^2 \rangle$.

Determinants.
$det(\lambda_!) = \alpha\beta\gamma, det(\nu_!) = \alpha$ and

Restrictions.
i) $\rho^*\alpha = \lambda^2$, $\rho^*\beta = \mu$, $\rho^*\gamma = 1$;

ii) $\rho^*\lambda_! = \lambda + \lambda\mu$, $\rho^*\nu_! = \nu + \lambda^2\nu$;

Product relations.
iii) $\beta\lambda_! = \lambda_!$, $\gamma\lambda_! = \lambda_!$;

iv) $\alpha\nu_! = \nu_!$, $\gamma\nu_! = \nu_!$;

v) $(\nu_!)^2 = 1 + \alpha + \gamma + \alpha\gamma$;

vi) $(\lambda_!)^2 = \alpha + \alpha\gamma + \alpha\beta + \alpha\beta\gamma$.

The first relation of iv) suggests that $\nu_!$ should be also transferred from some line bundle for the 2-covering corresponding to α. Namely we have

Lemma 3.1 *Let* $A = \langle \mathbf{a}^2, \mathbf{b}, \mathbf{c} \rangle$ *and let* ν' *be represented by* $\nu'(\mathbf{a}^2) = 1$, $\nu'(\mathbf{b}) = 1$, $\nu'(\mathbf{c}) = i$. *Then* $Ind_H^G(\nu) = Ind_A^G(\nu')$.

Similarly the second relation of iii) suggests

Lemma 3.2 *Let* $B = \langle \mathbf{a}, \mathbf{c} \rangle \cong \langle \mathbf{c} \rangle \rtimes \langle \mathbf{a} \rangle$ *and let* λ' *be represented by* $\lambda'(\mathbf{a}) = i$, $\lambda'(\mathbf{c}) = 1$. *Then* $Ind_H^G(\lambda) = Ind_B^G(\lambda')$.

$\mathbf{G = G_{39}}$, $H = \langle \mathbf{a}, \mathbf{b} \rangle \cong C_4 \times C_4$.

Determinants.
$det(\lambda_!) = \beta\gamma, det(\nu_!) = \gamma$.

Restrictions.
i) $\rho^*\alpha = \lambda^2$, $\rho^*\beta = \nu^2$, $\rho^*\gamma = 1$;

ii) $\rho^*\lambda_! = \lambda + \lambda^3\nu^2$, $\rho^*\nu_! = \nu + \nu^3$.

Product relations.
iii) $\alpha\beta\lambda_! = \lambda_!$, $\gamma\lambda_! = \lambda_!$;

iv) $\beta\nu_! = \nu_!$, $\gamma\nu_! = \nu_!$;

v) $(\lambda_!)^2 = \alpha + \beta + \alpha\gamma + \beta\gamma$;

vi) $(\nu_!)^2 = 1 + \beta + \gamma + \beta\gamma$.

The first relation of iv) suggests that $\nu_!$ should be also transferred from some line bundle for the 2-covering corresponding β. Namely we have

Lemma 3.3 *Let* $B = \langle \mathbf{a}, \mathbf{b}^2, \mathbf{c} \rangle$ *and let* ν' *be represented by* $\nu'(\mathbf{a}) = 1$, $\nu'(\mathbf{b}^2) = -1$, $\nu'(\mathbf{c}) = 1$, *Then* $Ind_H^G(\nu) = Ind_B^G(\nu')$.

The second relation of iii) suggests

Lemma 3.4 *Let* $AB = \langle \mathbf{a}^2, \mathbf{ab}, \mathbf{c} \rangle$ *and let* λ' *be represented by* $\lambda'(\mathbf{a}^2) = -1$, $\lambda'(\mathbf{ab}) = i$, $\lambda'(\mathbf{c}) = 1$. *Then* $Ind_H^G(\lambda) = Ind_{AB}^G(\lambda')$.

$\mathbf{G = G_{40}}$, $H = \langle \mathbf{a}, \mathbf{b} \rangle \cong C_4 \times C_4$.

G_{40} has the same character table as G_{39}. The only difference is in the determinants of $\lambda_!$ and $\nu_!$, while restrictions, products, and the two lemmas above are the same.

$$det(\lambda_!) = \beta\gamma, det(\nu_!) = 1.$$

$\mathbf{G = G_{41}}$, $H = \langle \mathbf{a}, \mathbf{b} \rangle \cong C_4 \times C_4$.

Determinants.
$det(\lambda_!) = \beta\gamma, det(\nu_!) = \alpha\beta\gamma.$

Restrictions.
i) $\rho^*\alpha = \lambda^2$, $\rho^*\beta = \nu^2$, $\rho^*\gamma = 1$;

ii) $\rho^*\lambda_! = \lambda + \lambda^3\nu^2$, $\rho^*\nu_! = \nu + \lambda^2\nu$.

Product relations.
iii) $\gamma\lambda_! = \alpha\beta\lambda_! = \lambda_!$;

iv) $\alpha\nu_! = \gamma\nu_! = \nu_!$.

v) $(\lambda_!)^2 = \alpha + \beta + \alpha\gamma + \beta\gamma$;

vi) $(\nu_!)^2 = \beta + \alpha\beta + \beta\gamma + \alpha\beta\gamma$.

Lemma 3.5 *We can replace the group G_{40} by G_{41} in Lemma 3.4.*

Also for $\nu_!$ one has

Lemma 3.6 $\nu_! = Ind_A^G(\nu')$, *where* $A = \langle \mathbf{a}^2, \mathbf{b}, \mathbf{c} \rangle$, *and* $\nu'(\mathbf{a}^2) = 1$, $\nu'(\mathbf{b}) = i$, $\nu'(\mathbf{c}) = 1$.

4. Relations of Theorem 1.1

Clearly the relations

$$a^{2^s} = b^{2^s} = c^{2^s} = 0.$$

are immediate consequences of the bundle relations $\alpha^2 = \beta^2 = \gamma^2 = 1$ for all cases.

The 4th and 5th relations follow from (2) and (3) respectively.

For the 6th relation

$$a(a + y_1 + v_s \sum_{i=1}^{s-1} a^{2^s-2^i} y_2^{2^{i-1}}) = 0$$

consider the double covering $\rho : BH \to BG$ in each of the four cases and apply formula (1) and Lemma 3.1, 3.3, 3.3 or 3.5 for G_{38}, G_{39}, G_{40}, or G_{41} respectively. For example if $G = G_{38}$ formula (1) and Lemma 3.1 imply that the second factor of the relation is the transfer of $c_1(\nu')$

$$Tr^*(c_1(\nu')) = (a + y_1 + v_s \sum_{i=1}^{s-1} a^{2^s-2^i} y_2^{2^{i-1}}).$$

Then $aTr^*(c_1(\nu')) = Tr^*(\rho^*(a)c_1(\nu')) = Tr^*(0 \cdot c_1(\nu')) = 0.$ □

Similarly for the 7th relation

$$b(b + x_1 + v_s \sum_{i=1}^{s-1} b^{2^s-2^i} x_2^{2^{i-1}}) = 0$$

apply formula (1) and Lemma 3.2, 3.4, 3.4 or 3.6 for G_{38}, G_{39}, G_{40}, or G_{41} respectively. □

Now note that the 4th and 6th relations imply $a^2c = ac^2$, the first relations of Theorem 1.1 ii). For this multiply the 4th relation by a and the 6th relation by c. The sum of these terms equals $a^2c + ac^2$ up to an invertible factor. Similarly the 5th and 7th relation imply $b^2c = bc^2$, the second relation of Theorem 1.1 ii).

For the decompositions of $v_s^2 x_2^{2^s}$, $v_s^2 y_2^{2^s}$, (also for the formulas for $x_1^{2^s}$ and $y_1^{2^s}$ of Theorem 1.1 ii) we need the material of Section 3. Namely we have to apply Lemma 2.1 to all induced representations given in Section 3 and take into account that their determinants can written in terms of the bundles α,β,γ. For example for $G = G_{38}$

$$v_s^2 x_2^{2^s} = c^2 + ac, \; x_1^{2^s} = a^{2^{s-1}} c^{2^{s-1}}$$

and

$$v_s^2 y_2^{2^s} = a^2 + bc + v_s abc^{2^s-1}, \; y_1^{2^s} = b^{2^{s-1}} c^{2^{s-1}}$$

are the consequences of the product relations v) and vi). Let us prove the first two relations. Equate Chern classes in the bundle relation of v). Then for the first Chern classes we get

$$v_s x_1^{2^s} = a + c + a + c + v_s a^{2^{s-1}} c^{2^{s-1}} = v_s a^{2^{s-1}} c^{2^{s-1}}.$$

For the decomposition of $v_s^2 x_2^{2^s}$ apply the equation for the second Chern classes:

$$\begin{aligned} v_s^2 x_2^{2^s} &= c_2(\nu_!^2) + c_1(det\, \nu_!)^2 \\ &= c_2(1 + \alpha + \gamma + \alpha\gamma) + c_1(\alpha)^2 = ac + (a + c)F(a,c) + a^2 \\ &= c^2 + ac + v_s(a + c)(ac)^{2^{s-1}} \\ &= c^2 + ac \end{aligned}$$

since $a^2c = ac^2$.

Note also

$$a^k c^i = 0, b^k c^i = 0, k + i > 2^s. \tag{6}$$

The 8th and 9th relations:

$$(c + x_1 + v_s \sum_{i=1}^{s-1} c^{2^s-2^i} x_2^{2^{i-1}})(b + y_1 + v_s \sum_{i=1}^{s-1} b^{2^s-2^i} y_2^{2^{i-1}}) = v_s T b^{2^s-1}.$$

Proof: Let $G = G_{38}$. Consider the diagram

$$\begin{array}{ccc} B\langle \mathbf{a}, \mathbf{c}^2 \rangle & \longrightarrow & B\langle \mathbf{a}, \mathbf{c} \rangle \\ \downarrow{\scriptstyle \rho_\mu} & & \downarrow{\scriptstyle \rho_\beta} \\ B\langle \mathbf{a}, \mathbf{b}, \mathbf{c}^2 \rangle & \xrightarrow[\rho_\gamma]{} & BG. \end{array} \tag{7}$$

Then the left hand side of our relation is equal to

$$\begin{array}{ll} Tr_\gamma^*(u)(b + y_1 + v_s \sum_{i=1}^{s-1} b^{2^s - 2^i} y_2^{2^{i-1}}) & \text{by transfer formula (1)} \\ = Tr_\gamma^*(u) Tr_\beta^*(c_1(\lambda')) & \text{by Lemma 3.2} \\ = Tr_\gamma^*(u \cdot \rho_\gamma^* Tr_\beta^*(c_1(\lambda'))) & \text{by Frobenius reciprocity of the transfer} \\ = Tr_\gamma^*(u \cdot Tr_\mu^*(\rho_\mu^*(v))) & \text{by the double coset formula and Lemma 3.2} \\ = Tr_\gamma^*(Tr_\mu^*(\rho_\mu^*(uv))) & \text{by Frobenius reciprocity} \\ = Tr_\gamma^*(uv \cdot Tr_\mu^*(1)) & \\ = Tr_\gamma^*(uv \cdot v_s c_1^{2^s - 1}(\mu)) & \text{by the formula for } Tr^*(1) \\ = v_s T b^{2^s - 1} & \text{by the definitions of } \beta, \mu, \text{ and } T. \quad \Box \end{array}$$

Similarly

$$(c + y_1 + v_s \sum_{i=1}^{s-1} c^{2^s - 2^i} y_2^{2^{i-1}})(a + x_1 + v_s \sum_{i=1}^{s-1} a^{2^s - 2^i} x_2^{2^{i-1}}) + v_s T a^{2^s - 1}.$$

Proof: Consider the diagram

$$\begin{array}{ccc} B\langle \mathbf{a}^2, \mathbf{b}, \mathbf{c}^2 \rangle & \longrightarrow & B\langle \mathbf{a}^2, \mathbf{b}, \mathbf{c} \rangle \\ \downarrow{\scriptstyle \rho_{\lambda 2}} & & \downarrow{\scriptstyle \rho_\alpha} \\ B\langle \mathbf{a}, \mathbf{b}, \mathbf{c}^2 \rangle & \xrightarrow[\rho_\gamma]{} & BG. \end{array} \tag{8}$$

With this notation the left hand side of the above relation is equal to

$$\begin{array}{ll} Tr_\gamma^*(v) Tr_\alpha^*(c_1(v')) & \text{by Lemma 3.1 and formula (1)} \\ = Tr_\gamma^*(v \cdot \rho_\gamma^* Tr_\alpha^*(c_1(v'))) & \text{by Frobenius reciprocity of the transfer} \\ = Tr_\gamma^*(v \cdot Tr_{\lambda^2}^*(\rho_{\lambda^2}^*(u))) & \text{by the double coset formula and Lemma 3.1} \\ = Tr_\gamma^*(Tr_{\lambda^2}^*(\rho_{\lambda^2}^*(uv))) & \text{by Frobenius reciprocity} \\ = Tr_\gamma^*(uv \cdot Tr_{\lambda^2}^*(1)) & \\ = Tr_\gamma^*(uv \cdot v_s c_1^{2^s - 1}(\lambda^2)) & \text{by the formula for } Tr^*(1) \\ = v_s T a^{2^s - 1} & \text{by the definitions of } \alpha, \lambda, \text{ and } T. \quad \Box \end{array}$$

For the other cases of G the proof is analogous. We just have to apply Lemma 3.3 and Lemma 3.4 for $G = G_{39}$ or G_{40}, and Lemma 3.5 and Lemma 3.6 for G_{41}.

The same applies to 11th and 12th relations and may be proved for all four groups simultaneously. In each case we will arrive at

$$T(a + x_1 + v_s \sum_{i=1}^{s-1} a^{2^s-2^i} x_2^{2^{i-1}}) = v_s a^{2^s-1} Tr^*(u^2 v)$$

or

$$T(b + y_1 + v_s \sum_{i=1}^{s-1} b^{2^s-2^i} y_2^{2^{i-1}}) = v_s b^{2^s-1} Tr^*(uv^2).$$

Therefore we will need that for the involution $t \in C_2 = G/H$ one has by Frobenius reciprocity

i) $Tr^*(u^2 v) = Tr^*(uv(u + tu) - vutu)) = Tr^*(uv)x_1 - Tr^*(v)x_2$,
ii) $Tr^*(uv^2) = Tr^*(uv(v + tv) - uvtv)) = Tr^*(uv)y_1 - Tr^*(u)y_2$.

Here are the details for $G = G_{38}$. Apply again the diagram (8).

$T(a + x_1 + v_s \sum_{i=1}^{s-1} a^{2^s-2^i} x_2^{2^{i-1}})$	
$= Tr_\gamma^*(uv)Tr_\alpha^*(c_1(\nu'))$	by Lemma 3.1 and formula (1)
$= Tr_\gamma^*(uv \cdot \rho_\gamma^* Tr_\alpha^*(c_1(\nu')))$	by Frobenius reciprocity of the transfer
$= Tr_\gamma^*(uv \cdot Tr_{\lambda^2}^*(\rho_{\lambda^2}^*(u)))$	by the double coset formula and Lemma 3.1
$= Tr_\gamma^*(Tr_{\lambda^2}^*(\rho_{\lambda^2}^*(u^2 v)))$	by Frobenius reciprocity
$= Tr_\gamma^*(u^2 v \cdot Tr_{\lambda^2}^*(1))$ $= Tr_\gamma^*(u^2 v \cdot v_s c_1^{2^s-1}(\lambda^2))$	by the formula for $Tr^*(1)$
$= v_s T(u^2 v) a^{2^s-1}$	by the definitions of α and λ

and the above equality i) gives

$$T(a + x_1 + v_s \sum_{i=1}^{s-1} a^{2^s-2^i} x_2^{2^{i-1}}) + v_s a^{2^s-1} T x_1$$
$$+ v_s a^{2^s-1} x_2 (c + y_1 + v_s \sum_{i=1}^{s-1} c^{2^s-2^i} y_2^{2^{i-1}}) = 0.$$

Then the second summand is zero by the 6th relation. The third summand is equal to $v_s a^{2^s-1} x_2 (c + y_1)$ by (6). This gives the 11th relation.

Similarly, applying the diagram (7) and the above equality ii) we have

$$T(b+y_1+v_s\sum_{i=1}^{s-1}b^{2^s-2^i}x_2^{2^{i-1}})+b^{2^s-1}Ty_1+b^{2^s-1}y_2(c+x_1+v_s\sum_{i=1}^{s-1}c^{2^s-2^i}x_2^{2^{i-1}})=0.$$

Again the second summand is zero by the 7th relation and the third summand is equal to $b^{2^s-1}y_2(c+x_1)$ and we get the 12th relation. □

The relation $cT=0$ is easy.

Proof: $cT\equiv cTr_\gamma^*(uv)=Tr_\gamma^*(uv\gamma^*(c))=Tr_\gamma^*(uv\cdot 0)=0.$ □

Now let us prove the 10th relation.

Proof: Let $u'=tu$ and $v'=tv$ be as above and $Tr^*=Tr_\gamma^*$. Then

$$\begin{aligned}Tr^*(uv)+Tr^*(uv')&=Tr^*(u(v+v'))=Tr^*(u)Tr^*(v),\\ Tr^*(uv)Tr^*(uv')&=Tr^*(uv(uv'+u'v))\\ &=Tr^*(u^2vv')+Tr^*(v^2uu')=Tr^*(u^2)y_2+Tr^*(v^2)x_2.\end{aligned}$$

Also

$$\begin{aligned}Tr^*(u^2)&=Tr^*(u(u+u')-uu')=Tr^*(u)x_1-Tr^*(1)x_2,\\ Tr^*(v^2)&=Tr^*(v(v+v')-vv')=Tr^*(v)y_1-Tr^*(1)y_2.\end{aligned}$$

Now we apply these formulas and take into account $Tr^*(1)x_1=Tr^*(1)y_1=0$. This gives the quadratic equation in $T=Tr_\gamma^*(uv)$

$$\begin{aligned}T^2=T(c+x_1+v_s\sum_{i=1}^{s-1}c^{2^s-2^i}x_2^{2^{i-1}})(c+y_1+v_s\sum_{i=1}^{s-1}c^{2^s-2^i}y_2^{2^{i-1}})\\ +x_2y_1(c+y_1+v_s\sum_{i=1}^{s-1}c^{2^s-2^i}y_2^{2^{i-1}})+x_1y_2(c+x_1+v_s\sum_{i=1}^{s-1}c^{2^s-2^i}x_2^{2^{i-1}}).\end{aligned}$$

Now to get the 10th relation apply $cT=0$. □

The decompositions for x_1 and y_1 are the consequences of the formula (4) applied to the determinants of $\nu_!$ and $\lambda_!$.

We need $(x_1x_2)^{2^{2s-2}}=0$ and $(y_1y_2)^{2^{2s-2}}=0$. It follows from the relations ii) of Theorem 1.1 that moreover we have $(x_1x_2)^{2^s}=(y_1y_2)^{2^s}=0$: decompositions $x_1^{2^s}=(ac)^{2^{s-1}}$ and $y_1^{2^s}=(bc)^{2^{s-1}}$ imply

$$x_1^{2^s}a=x_1^{2^s}c=x_1^{2^s}b^2=y_1^{2^s}b=y_1^{2^s}c=y_1^{2^s}a^2=0$$

since $a^2c=ac^2$, $b^2c=bc^2$ and $a^{2^s}=b^{2^s}=c^{2^s}=0$.

That is, all the terms of the above decomposition of $x_2^{2^s}$ annihilate $x_1^{2^s}$. Similarly for y_1 and y_2. It is also clear that for computing the Euler classes of the determinants (see Section 3) in each case we need only the initial fragment of the formal group law $F(x,y) = x + y + v_s(xy)^{2^{s-1}}$.

For instance consider G_{38}: $det(\lambda_!) = \alpha\beta\gamma$, $C(\lambda_!) = 1 + y_1 + y_2$, $a = e(\alpha)$, $b = e(\beta)$, $c = e(\gamma)$, hence $e(det\lambda_!) = F(a, F(b,c)) = a + b + c + v_s(ab + ac + bc)^{2^{s-1}}$.

For G_{41} we have different a and b, but $det(\lambda_!) = \beta\gamma = \alpha(\alpha\beta)\gamma$, hence $e(det(\lambda_!)) = F(a, F(b,c))$ as for G_{38}. □

5. Invariants

Let α, β, λ, ν, $\lambda_!$, $\nu_!$ be as above. We need the action of the involution $t \in G/H = C_2$ on $K(s)^*(BH)$. For simplicity we will ignore the powers of v_s. Also we will denote the restrictions of the generators of Theorem 1.1 to $K(s)^*(BH)$ with the same symbols but with bars.

Lemma 5.1 *Let $G = G_{39}, G_{40}$. Then*
$t(u) = u + u^{2^s} + v^{2^s} + (uv)^{2^{s-1}} + u^{2^{s-1}(1+2^s)} + u^{2^{s-1}}v^{2^{2s-1}}$;
$t(v) = v + v^{2^s} + v^{2^{s-1}(1+2^s)}$.

Proof: We need the action of the involution on bundles in Section 3.

$$t(\lambda) = \lambda^3\nu^2 = \lambda(\lambda\nu)^2, t(\nu) = \nu^3 = \nu(\nu)^2.$$

Note that the initial segment of the formal group law suffices. Namely as $\lambda^4 = \nu^4 = 1$ we can apply the formula $F(y,z) = y + z + (yz)^{2^{s-1}}$ modulo $z^{2^{2(s-1)}}$ (see [8], Lemma 5.3)

$$\begin{aligned} t(u) &= F(u, F(u^{2^s}, v^{2^s})) \\ &= u + (u^{2^s} + v^{2^s} + (uv)^{2^{2s-1}}) + u^{2^{s-1}}(u^{2^s} + v^{2^s})^{2^{s-1}} \\ &= u + u^{2^s} + v^{2^s} + (uv)^{2^{2s-1}} + u^{2^{s-1}(1+2^s)} + u^{2^{s-1}}v^{2^{2s-1}}. \end{aligned}$$

Similarly for v

$$t(v) = F(v, v^{2^s}) = v + v^{2^s} + v^{2^{s-1}(1+2^s)}.$$

□

We recall that $K(s)^*(BH) = \mathbb{F}_p[v_s, v_s^{-1}][u,v]/(u^{4^s}, v^{4^s})$. This is because of the Künneth isomorphisms, $K(s)^*(BC_4 \times BC_4) = K(s)^*(BC_4) \otimes K(s)^*(BC_4)$, whereas Morava K-theories for cyclic groups are the truncated polynomials [18]. So in particular $K(s)^*(BC_4) = K(s)^*[u]/u^{4^s}$.

Then we have by definition, where bar is defined as in the first paragraph of this section, that

$$\begin{aligned}
&\bar{a} = F(u^{2^s}, v^{2^s}) = u^{2^s} + v^{2^s} + (uv)^{2^{2s-1}};\\
&\bar{b} = v^{2^s};\\
&\bar{x_1} = u + t(u) = u^{2^s} + v^{2^s} + (uv)^{2^{2s-1}} + u^{2^{s-1}(1+2^s)} + u^{2^{s-1}} v^{2^{2s-1}};\\
&\bar{x_2} = ut(u) = u(u + u^{2^s} + v^{2^s} + (uv)^{2^{2s-1}} + u^{2^{s-1}(1+2^s)} + u^{2^{s-1}} v^{2^{2s-1}});\\
&\bar{y_1} = v + t(v) = v^{2^s} + v^{2^{s-1}(1+2^s)};\\
&\bar{y_2} = vt(v) = v(v + v^{2^s} + v^{2^{s-1}(1+2^s)});\\
&\bar{T} = uv + t(uv) = uv +\\
&\quad (u + u^{2^s} + v^{2^s} + (uv)^{2^{2s-1}} + u^{2^{s-1}(1+2^s)} + u^{2^{s-1}} v^{2^{2s-1}})(v + v^{2^s} + v^{2^{s-1}(1+2^s)}).
\end{aligned}$$

Note that as $u^{2^{2s}} = v^{2^{2s}} = 0$ one has

$$\bar{x}_1^{2^s} = \bar{y}_1^{2^s} = \bar{x}_2^{2^{2s-1}} = \bar{y}_2^{2^{2s-1}} = 0.$$

To describe all invariants we need the following

Lemma 5.2 *Let $G = G_{39}, G_{40}$. Then*

i) $K(s)^*(BH)$ is a free $K(s)^*(\bar{x_2}, \bar{y_2})/(\bar{x_2}^{2^{2s-1}}, \bar{y_2}^{2^{2s-1}})$ module generated by $1, u, v, uv$;

ii) $K(s)^*$-rank of $K(s)^*(BH)^{C_2}$ is $16^s/2 + 4^s/2$ and a basis is

1) $\bar{x_2}^i \bar{y_2}^j uv$,

2) $\bar{x_2}^i \bar{y_2}^j u$, $i, j \geq 2^{2s-1} - 2^{s-1}$ *in 1), 2);*

3) $\bar{x_2}^i \bar{y_2}^j v$, $j \geq 2^{2s-1} - 2^{s-1}$;

4) $\bar{x_2}^i \bar{y_2}^j u + \bar{x_2}^i \bar{y_2}^{j-2^{s-1}} (\bar{x_2}^{2^{s-1}} + \bar{y_2}^{2^{s-1}} \sum_{k=1}^{s} \bar{y_2}^{2^{2s-1}-2^{2s-k}+2^{s-k}-2^{s-1}}$
$+ \bar{x_2}^{2^{2s-2}} \bar{y_2}^{2^{2s-2}}) v$, $i < 2^{2s-1} - 2^{s-1}$, $j \geq 2^{s-1}$;

5) $\bar{x_2}^i \bar{y_2}^j u + \bar{x_2}^i \bar{y_2}^j (\sum_{k=1}^{s} \bar{y_2}^{2^{2s-1}-2^{2s-k}+2^{s-k}-2^{s-1}}) v + \bar{x_2}^{i-2^{s-1}} \bar{y_2}^{j+2^{2s-1}-2^{s-1}} uv$,
$i \geq 2^{2s-1} - 2^{s-1}$, $j < 2^{s-1}$;

6) $\bar{x_2}^i \bar{y_2}^j u + \bar{x_2}^i \bar{y_2}^j (\sum_{k=1}^{s} \bar{y_2}^{2^{2s-1}-2^{2s-k}+2^{s-k}-2^{s-1}}) v$,
$i \geq 2^{2s-1} - 2^{s-1}$, $2^{s-1} \leq j < 2^{2s-1} - 2^{s-1}$;

7) $\bar{x_2}^i \bar{y_2}^j$;

$i, j < 2^{2s-1}$ *in 1)-7).*

iii) There is a decomposition of $K(s)^*(BH)$ into free and trivial C_2 modules, such that a basis for the trivial module is $\bar{x_2}^i\bar{y_2}^j$, $\bar{a}\bar{x_2}^i\bar{y_2}^j$ $i < 2^s, j < 2^{s-1}$.

iv) $K(s)^*(BG)$ is generated by c,a,b,x_2,y_2,T as a $K(s)^*$-algebra.

Proof: i) It suffices to prove that $\{\bar{x_2}^i\bar{y_2}^i\}\times\{1,u,v,uv\}$, $i,j < 2^{2s-1}$ is a $K(s)^*$ generating set. Clearly it will be a $K(s)^*$ basis by its number of elements. Any polynomial in u,v can be uniquely written as $f_0+f_1u+f_2v+f_3uv$, $f_i = f_i(\bar{x_2},\bar{y_2})$ as follows. Because $u^2 = u\bar{x_1}-\bar{x_2}$ and $v^2 = v\bar{y_1}-\bar{y_2}$ any polynomial in u,v can be uniquely written as $g_0+g_1u+g_2v+g_3uv$ where $g_i = g_i(\bar{x_1},\bar{y_1},\bar{x_2},\bar{y_2})$. But

$$\bar{y_1} = \bar{y_2}^{2^{s-1}}. \tag{9}$$

This follows from the decomposition of $\bar{y_1}$ of Theorem 1.1 as $\bar{y_2}^{2^{2s-1}} = 0:$

$$\bar{y_1} = \bar{y_2}^{2^{s-1}} + \bar{y_1}^{2^{s-1}}\bar{y_2}^{2^{2s-2}} = \bar{y_2}^{2^{s-1}}\, modulo\, \bar{y_2}^{2^{2s-2}}\bar{y_2}^{2^{2s-2}} = \bar{y_2}^{2^{2s-1}}.$$

Similarly the decomposition of $\bar{x}_1$ of Theorem 1.1 implies

$$\begin{aligned}\bar{x_1} &= \bar{b} + \bar{x_2}^{2^{s-1}} + \bar{x_1}^{2^{s-1}}\bar{x_2}^{2^{2s-2}} \\ &= \bar{b} + \bar{x_2}^{2^{s-1}} + \bar{b}^{2^{s-1}}\bar{x_2}^{2^{2s-2}}\, modulo\, \bar{x_2}^{2^{2s-2}}\bar{x_2}^{2^{2s-2}} \\ &= \bar{b} + \bar{x_2}^{2^{s-1}} + \bar{b}^{2^{s-1}}\bar{x_2}^{2^{2s-2}} \qquad\qquad (\text{as } \bar{x_2}^{2^{2s-1}} = 0).\end{aligned}$$

Then

$$u^{2^s} = u\bar{x_1}^{2^s-1} + \sum_{i=1}^{s}\bar{x_1}^{2^s-2^i}\bar{x_2}^{2^{i-1}} \tag{10}$$

follows inductively from $u^2 = u\bar{x}_1+\bar{x}_2$. We can replace $u,\bar{x}_i$ by $v,\bar{y}_i$ in (10) and get (by inductive argument again) from $v^2 = v\bar{y}_1+\bar{y}_2$ and (9)

$$\bar{b} = v^{2^s} = \sum_{i=1}^{s}\bar{y_2}^{2^{2s-1}+2^{s-i}-2^{2s-i}} + v\bar{y_2}^{2^{2s-1}-2^{s-1}}. \tag{11}$$

Then by Theorem 1.1 $\bar{b}^2 = \bar{y_2}^{2^s}$ and we get for $\bar{x_1}$

$$\bar{x_1} = \bar{x_2}^{2^{s-1}} + \bar{y_2}^{2^{s-1}}\sum_{i=1}^{s}\bar{y_2}^{2^{2s-1}-2^{2s-i}+2^{s-i}-2^{s-1}} + \bar{x_2}^{2^{2s-2}}\bar{y_2}^{2^{2s-2}} + v\bar{y_2}^{2^{2s-1}-2^{s-1}}. \tag{12}$$

ii) First let us write $\bar{T}$ in our basis. Note $t(u)t(v) = (\bar{x_1}-u)(\bar{y_1}-v) = \bar{x_1}\bar{y_1} - \bar{x_1}v - \bar{y_1}u + uv$ and we have $\bar{T} = uv + t(uv) = \bar{x_1}\bar{y_1} - \bar{x_1}v - \bar{y_1}u$. But

$$\bar{x_1}v = (\bar{x_2}^{2^{s-1}} + \bar{y_2}^{2^{s-1}}\sum_{i=1}^{s}\bar{y_2}^{2^{2s-1}-2^{2s-i}+2^{s-i}-2^{s-1}} + \bar{x_2}^{2^{2s-2}}\bar{y_2}^{2^{2s-2}})v + \bar{y_2}^{2^{2s-1}-2^{s-1}+1}$$

as $v^2 = v\bar{y_2}^{2^{s-1}} + \bar{y_2}$ and $\bar{y_2}^{2^{2s-1}} = 0$. Then (12) and (9) imply

$$\begin{aligned}\bar{T} &= (\bar{x_2}^{2^{s-1}} + \bar{y_2}^{2^{s-1}}\sum_{i=1}^{s}\bar{y_2}^{2^{2s-1}-2^{2s-i}+2^{s-i}-2^{s-1}} + \bar{x_2}^{2^{2s-2}}\bar{y_2}^{2^{2s-2}})\bar{y_2}^{2^{s-1}}\\ &+ \bar{y_2}^{2^{2s-1}-2^{s-1}+1}\\ &+ (\bar{x_2}^{2^{s-1}} + \bar{y_2}^{2^{s-1}}\sum_{i=1}^{s}\bar{y_2}^{2^{2s-1}-2^{2s-i}+2^{s-i}-2^{s-1}} + \bar{x_2}^{2^{2s-2}}\bar{y_2}^{2^{2s-2}})v\\ &+ \bar{y_2}^{2^{s-1}}u.\end{aligned} \tag{13}$$

Now let $g = f_0 + f_1u + f_2v + f_3uv$, $f_i = f_i(\bar{x_2}, \bar{y_2})$ be an invariant, that is, $g \in Ker(1+t)$. Then

$$f_1\bar{x_1} + f_2\bar{y_1} + f_3\bar{T} = 0.$$

Taking into account the decompositions (12), (9) and (13) we have

$$\begin{aligned} f_3\bar{y_2}^{2^{s-1}} &= 0, \text{therefore}\\ f_3\bar{x_2}^{2^{s-1}} &= f_1\bar{y_2}^{2^{2s-1}-2^{s-1}},\\ f_2\bar{y_2}^{2^{s-1}} &= f_1(\bar{x_2}^{2^{s-1}} + \bar{y_2}^{2^{s-1}}\sum_{i=1}^{s}\bar{y_2}^{2^{2s-1}-2^{2s-i}+2^{s-i}-2^{s-1}} + \bar{x_2}^{2^{2s-2}}\bar{y_2}^{2^{2s-2}}).\end{aligned} \tag{14}$$

Now by the third equation of (14) we have two possible cases:
a) f_1 has a factor $\bar{y_2}^{2^{s-1}}$ and we can restore f_2 modulo summands of type 3). Also $f_3 = 0$ modulo summands of type 1). Therefore g is decomposable into sum of elements of types 1), 3), 4), 7);
b) f_1 annihilates $\bar{x_2}^{2^{s-1}}$, that is f_1 has a factor $\bar{x_2}^{2^{2s-1}-2^{s-1}}$. Hence right hand side of the third equation of (14) has a factor $\bar{y_2}^{2^{s-1}}$ and we can restore f_2 modulo summands of type 3). Then by the second equation of (14) we can restore f_3 modulo summands of type 1). Hence g is decomposable into elements of types 1), 3), 5)(or 6)) and 7). If f_1 annihilates $\bar{x_2}^{2^{s-1}}$ and $\bar{y_2}^{2^{s-1}}$ then by (14) g is decomposable into the monomials 1), 2), 3) and 7) of Lemma 5.2.

By Lemma 5.2 i) the elements 1)-7) are independent, therefore they form a basis ($16^s/2 + 4^s/2$ elements in total) for the invariants.

iii) Consider now the decomposition $[K(s)^*(H)]^{\mathcal{C}_2} = (\mathcal{F})^{\mathcal{C}_2} + \mathcal{T}$, corresponding to the decomposition of $K(s)^*(H)$ into free and trivial $\mathcal{C}_2$-modules.
Clearly the composition $\rho^*Tr^* = 1 + t$ is onto $(\mathcal{F})^{\mathcal{C}_2}$. Also i) and the Fröbenius

reciprocity of the transfer implies $ImTr^* \subset A$, where A is the subalgebra in $K(s)^*(BG)$ generated by c,a,b,x_2,y_2,T. Here we use the formulas (2), (3) for $Tr^*(u)$, $Tr^*(v)$ respectively and the decompositions of x_1 and y_1 of Theorem 1.1. We have to check whether the invariants in $\mathcal{T}$ are also covered by ρ^*. Let $m = \chi_s(\mathcal{F}^{C_2})$ and $n = \chi_s(\mathcal{T})$ be the $K(s)^*$-Euler characteristics. Then

$$\chi_s(H) = 2m+n,\ \chi_s(H)^{C_2} = m+n.$$

Clearly $2m+n = 16^s$ and by ii) $m+n = 16^s/2+4^s/2$. It follows that $m = 16^s/2-4^s/2$ and $n = 4^s$.

Let us consider the invariants modulo $Im(1+t)$ and denote by S_k, $k = 1,\cdots,7$, the set of invariants of type k) of Lemma 5.2 ii) modulo $Im(1+t)$. First note that

$$S_7 \subset S_8, \text{where}\, S_8 = \{\bar{x_2}^i\bar{y_2}^j, i < 2^s, j < 2^{s-1}\}, \tag{15}$$

that is except $\{\bar{x_2}^i\bar{y_2}^j, i < 2^s, j < 2^{s-1}\}$ all monomials of Lemma 5.2 of type 7) $\equiv 0$. This is because by definition $\bar{x}_1,\bar{y}_1 \equiv 0$; By (9) $\bar{y_2}^{2^{s-1}} \equiv \bar{y}_1$; The decomposition of $\bar{x}_1$ after (9) implies $\bar{x_2}^{2^s} \equiv \bar{b}^2 \equiv 0$ as by Theorem 1.1 $\bar{b}^2 \equiv \bar{y_2}^{2^s} \equiv \bar{y_1}^2 \equiv 0$.

Then by (12) one has

$$\bar{y_2}^{2^{2s-1}-2^{s-1}}v \equiv \bar{x_2}^{2^{s-1}},\ \text{hence}\ S_3 \subset S_8. \tag{16}$$

Also it follows all monomials of type 1) $\equiv 0$, that is $S_1 = \{0\}$ as $\bar{x}_2^{2^{2s-1}} = 0$.

(13) implies

$$\bar{y_2}^{2^{s-1}}u = \bar{y_2}^{2^{s-1}}\sum_{i=1}^{s}\bar{y_2}^{2^{2s-1}-2^{2s-i}+2^{s-i}-2^{s-1}}v, modulo\, \bar{x_2}^{2^{s-1}},$$

hence $S_2 \subset S_3$. Thus by (16) $S_2 \subset S_8$.

Recall by definition $\bar{T} \in Im(1+t)$. Also the first line in the decomposition (13) $\in Im(1+t)$ as it has factor $\bar{y_2}^{2^{s-1}} = \bar{y}_1$. Therefore if one denotes by $\breve{T}$ the sum of second and third lines in (13) one has

$$0 \equiv \breve{T} = (\bar{x_2}^{2^{s-1}} + \bar{y_2}^{2^{s-1}}\sum_{i=1}^{s}\bar{y_2}^{2^{2s-1}-2^{2s-i}+2^{s-i}-2^{s-1}} + \bar{x_2}^{2^{2s-2}}\bar{y_2}^{2^{2s-2}})v + \bar{y_2}^{2^{s-1}}u.$$

Then the elements of S_4 are $\bar{x_2}^i\bar{y_2}^{j-2^{s-1}}\breve{T} \equiv 0$; The same argument implies $S_6 = \{0\}$. Now only the elements of S_5 remain. It suffices to prove that the element of S_5 obtained for $i = 2^{2s-1}-2^{s-1}$ and $j = 0$ corresponds to $\bar{a}$, that is

$$\bar{a} \equiv \bar{x_2}^{2^{2s-1}-2^{s-1}}u + \bar{x_2}^{2^{2s-1}-2^{s-1}}(\sum_{k=1}^{s}\bar{y_2}^{2^{2s-1}-2^{2s-k}+2^{s-k}-2^{s-1}})v$$
$$+ \bar{x_2}^{2^{2s-1}-2^s}\bar{y_2}^{2^{2s-1}-2^{s-1}}uv.$$

First note

$$\bar{a} \equiv u\bar{x_1}^{2^s-1}: \tag{17}$$

Recall $\bar{a} = u^{2^s} + v^{2^s} + (uv)^{2^{2s-1}}$. Then (10) implies $u^{2^s} \equiv u\bar{x_1}^{2^s-1} + \bar{x_2}^{2^{s-1}}$. Recall $\bar{b} = v^{2^s}$ and by (11) and (12) $\bar{x_2}^{2^{s-1}} + b \equiv 0$. It suffices to see $(uv)^{2^{2s-1}} \equiv 0$. (10) also implies $u^{2^{2s-1}} = \bar{x_2}^{2^{2s-2}}$. Similarly $v^{2^{2s-1}} = \bar{y_2}^{2^{2s-2}}$. Then recall by (9) $\bar{y_2}^{2^{s-1}} \equiv 0$ hence $(uv)^{2^{2s-1}} = \bar{x_2}^{2^{2s-2}}\bar{y_2}^{2^{2s-2}} \equiv 0$.

Now it follows (see the proof of (12)) from $\bar{x_1} = \bar{b} + \bar{x_2}^{2^{s-1}} + \bar{b}^{2^{s-1}}\bar{x_2}^{2^{2s-2}}$ and $\bar{b}^2 = \bar{y_2}^{2^s}$ that

$$\begin{aligned} u\bar{x_1}^{2^s-1} =& u\bar{x_1}\bar{x_1}^{2^s-2} = ux_1(\bar{x_2}^{2^s} + \bar{y_2}^{2^s})^{2^{s-1}-1} \\ =& (u\bar{b} + u\bar{x_2}^{2^{s-1}})(\bar{x_2}^{2^s} + \bar{y_2}^{2^s})^{2^{s-1}-1} + u(\bar{x_2}\bar{y_2})^{2^{2s-2}}(\bar{x_2}^{2^s} + \bar{y_2}^{2^s})^{2^{s-1}-1} \\ =& (u\bar{b} + u\bar{x_2}^{2^{s-1}})(\bar{x_2}^{2^s} + \bar{y_2}^{2^s})^{2^{s-1}-1}. \end{aligned}$$

Now let

$$(u\bar{b} + u\bar{x_2}^{2^{s-1}})(\bar{x_2}^{2^s} + \bar{y_2}^{2^s})^{2^{s-1}-1} = f_0 + f_1u + f_2v + f_3uv,$$

where f_i are polynomials in $\bar{x_2}, \bar{y_2}$. Let us prove that $f_3 = \bar{x_2}^{2^{2s-1}-2^s}\bar{y_2}^{2^{2s-1}-2^{s-1}}$. Then we will restore f_1 and f_2 as above and complete the proof.

To get f_3 note that by (11) only the summand $v\bar{y_2}^{2^{2s-1}-2^{s-1}}$ in the decomposition of $\bar{b}$ is relevant. This gives

$$uv\bar{y_2}^{2^{2s-1}-2^{s-1}}(\bar{x_2}^{2^s} + \bar{y_2}^{2^s})^{2^{s-1}-1} = uv\bar{x_2}^{2^{2s-1}-2^s}\bar{y_2}^{2^{2s-1}-2^{s-1}}$$

and proves our claim for f_3. Then by (14)

$$f_1 \equiv \bar{x_2}^{2^{2s-1}-2^{s-1}}, \; f_2 \equiv \bar{x_2}^{2^{2s-1}-2^{s-1}}\sum_{j=1}^{s}\bar{y_2}^{2^{2s-1}-2^{2s-j}+2^{s-j}-2^{s-1}}$$

modulo $\bar{x_2}^k\bar{y_2}^l$, $l \geq 2^{s-1}$, which are elements of $Im(1+t)$. Hence $a \equiv f_1u + f_2v + f_3uv \equiv$ the element of S_5 with $i = 2^{2s-1} - 2^{s-1}$ and $j = 0$. After multiplying by $\bar{x_2}^i\bar{y_2}^j$, $i < 2^s$, $j < 2^{s-1}$ we get $a\bar{x_2}^i\bar{y_2}^j$ modulo $Im(1+t)$ (still elements of S_5).

iv) Consider the Serre spectral sequence for the extension $1 \to H \to G \to G/H \to 1$. Then as $G/H = \mathcal{C}_2$

$$E_2 = H^*(\mathcal{C}_2, K(s)^*(BH)) \cong \mathcal{F}^{\mathcal{C}_2} \oplus \mathcal{T} \otimes H^*(\mathcal{C}_2, \mathbb{F}_2),$$

where $K(s)^*(BH) = \mathcal{F} \oplus \mathcal{T}$ is the decomposition into free and trivial modules. As in iii) let A be the subalgebra of $K(s)^*(BG)$ generated by c, a, b, x_2, y_2, T.

Then iii) says that the restriction $\rho^* : A \to [K(s)^*(BH)]^{\mathcal{C}_2}$ is onto. $\rho^*(c) = 0$ by definition. Hence all invariants are permanent cycles and there is only one differential $d_{2^{s+1}-1}(t) = v_s t^{2^{s+1}}$. Since t^2 is represented by c one obtains that c,a,b,x_2,y_2,T are $K(s)^*$-algebra generators of $K(s)^*(BG)$ and its $K(s)^*$-rank equals $\chi_s(\mathcal{F}^{\mathcal{C}_2} \oplus \mathcal{T} \otimes \mathbb{F}_2[c]/c^{2^s}) = (16^s - 4^s)/2 + 4^s 2^s = 16^s/2 + 8^s - 4^s/2$ as it is already known from [19].

This completes the proof. □

Lemma 5.3 *Let $G = G_{38}, G_{41}$. Then*
$t(u) = u + v^{2^s} + u^{2^{s-1}} v^{2^{2s-1}}$;
$t(v) = v + \bar{b} + v^{2^{s-1}} \bar{b}^{2^{s-1}}$.

Proof: Recall the action of the involution on bundles in Section 3. For G_{38}

$$t(\lambda) = \lambda\mu,\, t(\nu) = \lambda^2\nu$$

and for G_{41}

$$t(\lambda) = \lambda\rho^*(\alpha\beta),\, t(\nu) = \lambda^2\nu.$$

Since $\lambda^4 = \mu^2 = (\alpha\beta)^2 = 1$ again we need only the initial segment of the formal group law. □

Recall that H is isomorphic to $C_4 \times C_2 \times C_2$ for $G = G_{38}$ and to $C_4 \times C_4$ for $G = G_{41}$. Again by the Künneth isomorphism we have that as a $K(s)^*$-algebra

$$K(s)^*(BG_{41}) = K(s)^*[u,v]/(u^{4^s}, v^{4^s})$$

and

$$K(s)^*(BG_{38}) = K(s)^*[u,v,w]/(u^{4^s}, v^{2^s}, w^{2^s}),$$

where $w = c_1(\mu)$ is invariant under action of $G/H = \mathcal{C}_2$.

Then we have by definition

$\bar{x_1} = u + t(u) = v^{2^s} + u^{2^{s-1}} v^{2^{2s-1}}$;
$\bar{x_2} = ut(u) = u(u + v^{2^s} + u^{2^{s-1}} v^{2^{2s-1}})$;
$\bar{y_1} = v + t(v) = \bar{b} + v^{2^{s-1}} \bar{b}^{2^{s-1}}$;
$\bar{y_2} = vt(v) = v(v + \bar{b} + v^{2^{s-1}} \bar{b}^{2^{s-1}})$;
and as $\bar{T} = uv + t(uv) = uv + (\bar{x_1} + u)(\bar{y_1} + v)$ we have

$$\bar{T} = \bar{x_1}\bar{y_1} + \bar{x_1} v + \bar{y_1} u. \quad (18)$$

Now to describe all invariants and see that $K(s)^*(BG)$ restricts onto $K(s)^*(BH)^{\mathcal{C}_2}$, we turn to the following

Lemma 5.4 *i) Let* $G = G_{38}, G_{41}$ *and let* $x^\omega = \bar{x_1}^i \bar{y_1}^j \bar{x_2}^k \bar{y_2}^l$, $i,j < 2^s, k,l < 2^{s-1}$. *Then the set* $x^\omega, x^\omega u, x^\omega v, x^\omega uv$ *is a* $K(s)^*$ *basis in* $K(s)^*(BH)$.

ii) $K(s)^*$*-rank of* $K(s)^*(BH)^{C_2}$ *is* $16^s/2 + 4^s/2$ *and a basis is*

1) $\bar{x_1}^i \bar{y_1}^j \bar{x_2}^k \bar{y_2}^l$, $i,j < 2^s, k,l < 2^{s-1}$;

2) $\bar{x_1}^{2^s-1} \bar{y_1}^{2^s-1} \bar{x_2}^k \bar{y_2}^l uv$, $k,l < 2^{s-1}$;

3) $\bar{x_1}^{2^s-1} \bar{y_1}^i \bar{x_2}^k \bar{y_2}^l u$, $\bar{x_1}^i \bar{y_1}^{2^s-1} \bar{x_2}^k \bar{y_2}^l v$, $i < 2^s$, $k,l < 2^{s-1}$;

4) $\bar{x_1}^i \bar{y_1}^j \bar{x_2}^k \bar{y_2}^l u + \bar{x_1}^{i+1} \bar{y_1}^{j-1} \bar{x_2}^k \bar{y_2}^l v$, $i, j-1 < 2^s - 1, k,l < 2^{s-1}$.

iii) The set $x_2^i y_2^j$, $a x_2^i y_2^j$, $b x_2^i y_2^j$, $ab x_2^i y_2^j$, $i,j < 2^{s-1}$ *restricts to a* $K(s)^*$ *basis in the trivial summand* $\mathcal{T}$ *of the* C_2 *module* $K(s)^*(H)$.

iv) $K(s)^*(BG)$ *is generated by* c,a,b,x_2,y_2,T *as a* $K(s)^*$*-algebra.*

Proof: i) As in Lemma 5.2.i) any polynomial $g(u,v)$ can be uniquely written as $g_0 + g_1 u + g_2 v + g_3 uv$ where $g_i = g_i(\bar{x_1}, \bar{y_1}, \bar{x_2}, \bar{y_2})$. Then by Theorem 1.1 $\bar{x_2}^{2^{s-1}}$, $\bar{y_2}^{2^{s-1}}$ can be expressed by $\bar{x_1}, \bar{y_1}, \bar{a}, \bar{b}$. Now

$$\bar{x_1} = u + F(u,\bar{a}) = u + u + \bar{a} + u^{2^{s-1}} \bar{a}^{2^{s-1}} = \bar{a} + u^{2^{s-1}} \bar{x_1}^{2^{s-1}}$$

and we get

$$\bar{a} = \bar{x_1} + \bar{x_1}^{2^s-1} u + \sum_{i=1}^{s-1} \bar{x_1}^{2^s-2^i} \bar{x_2}^{2^{i-1}}. \tag{19}$$

Similarly $\bar{y_1} = \bar{b} + v^{2^{s-1}} \bar{y_1}^{2^{s-1}}$ implies

$$\bar{b} = \bar{y_1} + \bar{y_1}^{2^s-1} v + \sum_{i=1}^{s-1} \bar{y_1}^{2^s-2^i} \bar{y_2}^{2^{i-1}}. \tag{20}$$

Because of nilpotence of $\bar{x_1}$ and $\bar{y_1}$ substituting $\bar{a}$ and $\bar{b}$ in $g(u,v)$ we arrive at i) after finite number of steps.

ii) Let

$$g = f_0 + f_1 u + f_2 v + f_3 uv, \ f_i = f_i(\bar{x_1}, \bar{y_1} \bar{x_2}, \bar{y_2})$$

be an invariant, that is, $g \in Ker(1+t)$. Then by (18)

$$f_1 \bar{x_1} + f_2 \bar{y_1} + f_3 \bar{T} = f_1 \bar{x_1} + f_2 \bar{y_1} + f_3(\bar{x_1}\bar{y_1} + \bar{x_1} v + \bar{y_1} u) = 0$$

and we get

$$f_3 \bar{x_1} = f_3 \bar{y_1} = 0; \ f_1 \bar{x_1} = f_2 \bar{y_1}. \tag{21}$$

Now i) implies ii).

iii) Let us look at invariants modulo $1+t$: The invariants of Lemma 5.4 1) with $i,j>0$ are zero as $\bar{x_1},\bar{y_1}\in Im(1+t)$; Invariants of 4) are all zero as by (18)

$$\bar{x_1}v+\bar{y_1}u\in Im(1+t). \tag{22}$$

By the same argument invariants of 3) are zero except $\bar{x_1}$ or $\bar{y_1}$ are omitted, that is except $\bar{x_1}^{2^s-1}\bar{x_2}^k\bar{y_2}^l u$, $\bar{y_1}^{2^s-1}\bar{x_2}^k\bar{y_2}^l v$. This completes the basis invariants of $\mathcal{F}^{C_2}$ because of the total number $16^s/2-4^s/2$ (see the proof of Lemma 5.2. iii).

Thus the invariants of 2) are all nonzero. Then we have that the basis invariants corresponding to $\mathcal{T}$ are as follows:
$\bar{x_2}^i\bar{y_2}^j$ is resticted $x_2^iy_2^j$, $i,j<2^{s-1}$; and modulo $1+t$
$\bar{x_1}^{2^s-1}\bar{x_2}^i\bar{y_2}^j u$ is restricted $ax_2^iy_2^j$ by (19);
$\bar{y_1}^{2^s-1}\bar{x_2}^i\bar{y_2}^j v$ is restricted $bx_2^iy_2^j$ by (20);
$\bar{x_1}^{2^s-1}\bar{y_1}^{2^s-1}\bar{x_2}^i\bar{y_2}^j uv$ is restricted $abx_2^iy_2^j$ by (19), (20) and (22).

iv) This is the consequence of the arguments similar to that of Lemma 5.2 iv).

□

Remark 5.5. Of course there are alternative bases for $K(s)^*(BH)$. For instance Lemma 5.2.i) is also true for $G=G_{41}$; For $G=G_{38},G_{41}$ an alternative $K^*(s)$-basis is $X^\omega, X^\omega u, X^\omega v, X^\omega uv$, where $X^\omega=\bar{x_1}^i\bar{x_2}^j\bar{y_2}^k$, $i<2^s$, $j<2^{2s-1}$, $k<2^{s-1}$.

End of the proof

By Lemma 5.2 iv) and Lemma 5.4 iv) we have that c,a,b,x_2,y_2,T is a complete set of $K(s)^*$ algebra generators of $K(s)^*(BG)$. Now we want to verify that for all our groups the defining relations of Theorem 1.1 give us a ring of Euler characteristic already computed in [19] $\chi_{s,2}=16^s/2+8^s-4^s/2$. For each of our groups, one can choose a basis for $K(s)^*(BG)$. Lemma 5.2 ii) suggests the following

Lemma 5.6 *A basis for $K^*(s)(BG)$, $G=G_{39},G_{40}$ is*
$\{x_2^iy_2^j\,|\,i,j<2^{2s-1}\}$;
$\{ax_2^iy_2^j\,|\,i<2^s, j<2^{s-1}\}$;
$\{bx_2^iy_2^j\,|\,i<2^{2s-1}, j<2^{s-1}\}$;
$\{Tx_2^iy_2^j\,|\,i<2^{2s-1}, j<2^{s-1}(2^s-1)\}$;
$\{c^ix_2^jy_2^k,\ c^iax_2^jy_2^k\,|\,0<i<2^s, j<2^s, k<2^{s-1}\}$.

Proof: One can work modulo c and check that first four lines give a basis for $K^*(BG)/ker\rho^*$, and then the last line forms a basis for $ker\rho^*$, where $\rho: BH\to BG$.

Choose the lexicographic monomial ordering (lp) corresponding to the variables (a,T,b,y_2,x_2,c) in that order. Then the first four lines constitute a Gröbner basis of $K(s)^*(BG)/ker\rho^*$. The last line, a Gröbner basis of $ker\rho^*$, is the union of $\{c^i x_2^j y_2^k, c^i a x_2^j y_2^k, |0 < i < 2^s - 1\}$, a basis of $ker\rho^* \bigcap K(s)^*(BG)/ImTr^*$, and $\{c^{2^s-1} x_2^j y_2^k, c^{2^s-1} a x_2^j y_2^k\}$, a basis of $Tr^*(\mathcal{T})$, the image of the trivial module $\mathcal{T}$ of Lemma 5.2 iii) under the transfer homomorphism. For the last sentence recall $Tr^*(1) = v_s c^{2^s-1}$.

Let us give the proof in the following steps and in this way explain the range restrictions for indices.

Step 1. Any monomial of $ker\rho^*$ is decomposable into a sum of elements from the last line of Lemma 5.6.

cb : (as $cb^2 = c^2 b$ the decomposition of cb^i will follow).
Multiply the decomposition of x_1 by c and take into account the relation $c(c + x_1 + \sum c^{2s-2^i} x_2^{s-1}) = 0$ (note also $bc^2 = b^2 c$ implies $c(bc)^{2^{s-1}} = 0$). This gives the decomposition of cb into the $c^i x_2^j$, $0 < i < 2^s, j \le 2^{s-1}$. Namely

$$cb = c(x_2^{2^{s-1}} + \sum_{i=1}^{s-1} c^{2^s-2^i} x_2^{2^{i-1}}). \tag{23}$$

$cx_2^{2^s}$: Multiply the decomposition of $x_2^{2^s}$ by c . As $c^{2^s} = 0$, $a^2 c = ac^2$ and $b^2 c = bc^2$ we have $cx_2^{2^s} = a^2 c + b^2 c + ac^2 = b^2 c = bc^2$. Then by (23) we get

$$cx_2^{2^s} = c^2(x_2^{2^{s-1}} + \sum_{i=2}^{s-1} c^{2^s-2^i} x_2^{2^{i-1}}). \tag{24}$$

$cy_2^{2^{s-1}}$: One has

$$cy_2^{2^{s-1}} = c\sum_{i=1}^{s-1} c^{2^s-2^i} y_2^{2^{i-1}} + \begin{cases} 0, & G = G_{39} \\ c^2, & G = G_{40} \end{cases} \tag{25}$$

For this multiply the decomposition of y_1 of Theorem 1.1 by c and apply the relation $c(c + y_1 + v_s \sum_{i=1}^{s-1} c^{2^s-2^i} y_2^{2^{i-1}}) = 0$.

Now as $ca^2 = c^2 a$ and $cT = 0$ we have proper decomposition for any monomial having factor c.

Note one has

$$x_2^{2^{2s-1}} = ac^{2^s-1}, \; y_2^{2^{2s-1}} = c^{2^s-1} x_2^{2^{s-1}}. \tag{26}$$

This explains the range restrictions for the first line of the basis of Lemma 5.6. For this we need the decompositions of $x_2^{2^s}$ and $y_2^{2^s}$ of Theorem 1.1.

$$x_2^{2^{2s-1}} = (ac)^{2^{s-1}} = ac^{2^s-1}.$$

Similarly

$$y_2^{2^{2s-1}} = (bc)^{2^{s-1}} = c^{2^s-1}b = c^{2^s-1}x_2^{2^{s-1}}.$$

For the last two equalities apply $cb^2 = c^2 b$ and (23).

$bx_2^{2^{2s-1}}$: Multiply the first equation of (26) by b and apply (23). This gives

$$bx_2^{2^{2s-1}} = ac^{2^s-1}x_2^{2^{s-1}}. \tag{27}$$

Step 2.

b^2 : Rewrite the decomposition of $y_2^{2^s}$ of Theorem 1.1 and apply (23) to get the proper decomposition

$$b^2 = \begin{cases} y_2^{2^s} + c(x_2^{2^{s-1}} + \sum_{i=1}^{s-1} c^{2^s-2^i} x_2^{2^{i-1}}), & G = G_{39} \\ y_2^{2^s} + c^2 + c(x_2^{2^{s-1}} + \sum_{i=1}^{s-1} c^{2^s-2^i} x_2^{2^{i-1}}), & G = G_{40} \end{cases} \tag{28}$$

a^2 : By Theorem 1.1 $a^2 = x_2^{2^s} + b^2 + ac + abc^{2^s-1}$. By (23) $abc^{2^s-1} = ac^{2^s-1}x_2^{2^{s-1}}$. Taking into account (28) we get the proper decomposition

$$a^2 = x_2^{2^s} + ac + ac^{2^s-1}x_2^{2^{s-1}} + \begin{cases} y_2^{2^s} + c(x_2^{2^{s-1}} + \sum_{i=1}^{s-1} c^{2^s-2^i} x_2^{2^{i-1}}), & G = G_{39} \\ y_2^{2^s} + c^2 + c(x_2^{2^{s-1}} + \sum_{i=1}^{s-1} c^{2^s-2^i} x_2^{2^{i-1}}), & G = G_{40}. \end{cases} \tag{29}$$

In the following we will work modulo c in the ring R with lexicographic ordering determined by variables $(y_1, x_1, a, T, b, y_2, x_2, c)$ in that order and give the decompositions in the above Gröbner basis of $K(s)^*(BG)/ker\rho^*$.

$by_2^{2^{s-1}}$: Clearly we need the ideal I_1, generated by the following relations of Theorem 1.1: b^{2^s}, $b(b + y_1 + \sum_{i=1}^{s-1} b^{2^s-2^i} y_2^{2^{i-1}})$, the decomposition of y_1 and (28) modulo c. Then in the quotient ring R/I_1 $by_2^{2^{s-1}}$ is decomposable into the elements $\{y_2^j\}$.

ab : Let I_2 be the ideal generated by the relations of I_1, (29), $a(a + x_1 + \sum_{i=1}^{s-1} a^{2^s-2^i} x_2^{2^{i-1}})$ and decomposition of x_1 multiplied by a. This gives the decomposition of ab in R/I_2 into the elements of the first three lines of our basis.

bT : Let I_3 be generated by the relations of I_2 and $T(b + y_1 + v_s \sum_{i=1}^{s-1} b^{2^s-2^i} y_2^{2^{i-1}}) + b^{2^s-1}y_2(c + x_1)$. Then bT is decomposable in the quotient ring R/I_3 into the elements of the first and fourth lines of our basis.

T^2 : Consider the ideal

$I_4 = (I_3, T^2 + Tx_1y_1 + x_2y_1(c + y_1 + \sum_{i=1}^{s-1} c^{2^s-2^i} y_2^{2^{i-1}}) + x_1y_2(c + x_1 + \sum_{i=1}^{s-1} c^{2^s-2^i} x_2^{2^{i-1}}))$.

This gives a decomposition of T^2 which is not yet the proper decomposition, one has terms Ty^i, $i \geq (2^s-1)2^{s-1}$. We will need the decomposition of $Ty_2^{2^{2s-1}-2^{s-1}}$ below to get the proper decomposition of T^2 into elements of the first, third and fourth lines of our basis.

aT : We need the ideal
$I_5 = (I_4, T(a + x_1 + \sum_{i=1}^{s-1} a^{2^s-2^i} x_2^{2^{i-1}}) + a^{2^s-1}x_2(c+y_1))$.
This gives the proper decomposition of aT into the first and fourth lines of our basis.

$ay_2^{2^{s-1}}$: Take the ideal $I_6 = I_5$+the decomposition of y_1 multiplied by a and the relation $(c + y_1 + \sum_{i=1}^{s-1} c^{2^s-2^i} y_2^{i-1})(a + x_1 + \sum_{i=1}^{s-1} a^{2^s-2^i} a_2^{i-1}) + a^{2^s-1}T$. Again this gives a decomposition with terms Ty^i, $i \geq (2^s-1)2^{s-1}$. To get the proper decomposition into the first, third and fourth lines of our theorem we need the following decomposition of

$Ty_2^{2^{2s}-2^{s-1}}$: Finally put
$I_7 = (I_6, (c + x_1 + \sum_{i=1}^{s-1} c^{2^s-2^i} x_2^{2^{i-1}})(b + y_1 + \sum_{i=1}^{s-1} b^{2^s-2^i} y_2^{2^{i-1}}) + b^{2^s-1}T)$.
By the above decomposition of bT one has $b^{2^s-1}T = Ty_2^{2^{2s}-2^{s-1}}$ for the last summand above. This is what we need for the decomposition of $Ty_2^{2^{2s}-2^{s-1}}$ into the third and fourth lines of our basis.

Finally note that the decomposition of $ax_2^{2^s}$ already follows from the decomposition of $x_2^{2^s}$ of Theorem 1.1 and decompositions of ab and a^2.

□

Similarly Lemma 5.4 ii) suggests the following

Lemma 5.7 *A basis for $K(s)^*(BG)$, $G = G_{38}, G_{41}$ is*
$\{x_1^i y_1^j x_2^k y_2^l | i,j < 2^s, k,l < 2^{s-1}\}$;
$\{abx_2^k y_2^l | k,l < 2^{s-1}\}$;
$\{y_1^i a x_2^k y_2^l,\ x_1^i b x_2^k y_2^l | i < 2^s, k,l < 2^{s-1}\}$;
$\{Tx_1^i y_1^j x_2^k y_2^l | i,j < 2^s-1, k,l < 2^{s-1}\}$;
$\{c^i x_2^k y_2^l,\ c^i a x_2^k y_2^l,\ c^i b x_2^k y_2^l,\ c^i ab x_2^k y_2^l | 0 < i < 2^s, k,l < 2^{s-1}\}$.

Let us give a sketch of the proof. Choose the lexicographic ordering corresponding to (T, a, b, y_2, x_2, c) in that order. This eliminates a and b (by decompositions of x_1 and y_1 of Theorem 1.1). Then again we have to apply the relations of Theorem 1.1 and extract the following Gröbner basis of $K(s)^*(BG)/ker\rho^*$:

$\{x_1^i y_1^j x_2^k y_2^l | i,j < 2^s, k,l < 2^{s-1}\}$;
$\{x_2^{2^{s-1}+2^k} y_2^{2^{s-1}+2^l} | k,l < 2^{s-1}\}$;
$\{x_2^{2^{s-1}+2^k} y_1^i y_2^l,\ x_1^i x_2^k y_2^{2^{s-1}+2^l} | i < 2^s, k,l < 2^{s-1}\}$;
$\{Tx_1^i y_1^j x_2^k y_2^l | i,j < 2^s-1, k,l < 2^{s-1}\}$;

(here a is replaced by $x_2^{2^{s-1}}$ and b by $y_2^{2^{s-1}}$ in the first four lines of Lemma 5.7) and a Gröbner basis of $ker\rho^*$

$$\{c^i x_2^k y_2^l, c^i x_2^{2^{s-1}+2^k} y_2^l, c^i x_2^k y_2^{2^{s-1}+2^l}, c^i x_2^{2^{s-1}+2^k} y_2^{2^{s-1}+2^l} | 0 < i < 2^s, k, l < 2^{s-1}\}$$

corresponding to the last line of Lemma 5.7.

6. Remarks

The families of non-abelian p-groups whose Morava K-theory is known to be good in the sense of Hopkins-Kuhn-Ravenel is listed in [19]. In particular, if G belongs to any of the following families of p-groups, then $K(n)^{odd}(BG) = 0$.

(a) wreath products of the form $H \wr C_p$ with H good [12], [13];

(b) metacyclic p-groups [25];

(c) minimal non-abelian p-groups, i.e., groups all of whose maximal subgroups are abelian [26];

(d) groups of p-rank 2 [27];

(e) elementary abelian by cyclic groups, i.e., the extensions $V \to G \to C$ with V elementary abelian and C cyclic [28], [17];

(f) central product of the form $H \circ C_{p^m}$ with H good [19].

(g) H is a normal subgroup in G of index p, H is good and the integral Morava K-theory $\tilde{K}(s)(BH)$ is a permutation module for the action of G/H [17].

For these families the ring structure of $K(s)^*(BG)$ is either studied in the works mentioned above or can be read off from previously performed computations modulo some definite indeterminacy. Namely, Yagita and Tezuka determined the multiplicative structure modulo the transfer formula (1). On the other hand, our main aim here is to check (at least for the groups with maximal abelian subgroup of index 2) whether the transfer formula is sufficient to get the ring structure in combination with the methods of characteristic classes and transfer (double coset formula, etc.) The papers [5, 4, 6, 9] treat the same problem.

Schuster suggested an alternative way to obtain explicit relations by choosing some artificial generators in the spectral sequence, not equal to Chern classes [19, 23].

There are 51 groups of order 32. The first 7 groups are abelian and the next 8 have an abelian factor, hence the task of computing the ring structure is reduced to the smaller nonabelian groups. We refer the reader to [21, 22] for some details. In this paper, we carry out the complete details for the groups G in the Hall-Senior list with the numbers $39, \cdots, 41$.

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MALKHAZ BAKURADZE malkhaz.bakuradze@tsu.ge

Iv. Javakhishvili Tbilisi State University
Faculty of Exact and Natural Sciences
Georgia

MAMUKA JIBLADZE jib@rmi.ge

Iv. Javakhishvili Tbilisi State University
Faculty of Exact and Natural Sciences
Georgia

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