

UNIMODULARITY OF POINCARÉ POLYNOMIALS OF LIE ALGEBRAS FOR SEMISIMPLE SINGULARITIES

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ABSTRACT. We single out a large class of semisimple singularities with the property that all roots of the Poincaré polynomial of the Lie algebra of derivations of the corresponding suitably (not necessarily quasihomogeneously) graded moduli algebra lie on the unit circle; for a still larger class there might occur exactly four roots outside the unit circle. This is a corrected version of a theorem by Elashvili and Khimshiashvili.

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1. FORMULATIONS

1.1. Tensor product of graded algebras and their gradings. An important invariant of an isolated singularity S given by a holomorphic germ $f: (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ is the algebra $\mathbb{C}[[\mathbf{x}]]/(\frac{\partial f}{\partial \mathbf{x}})$. We will denote this algebra by $A(S)$.

For complex isolated singularities S_j given by germs $f_j \in \mathbb{C}[[\mathbf{x}_j]]$, $\mathbf{x}_j \in \mathbb{C}^{n_j}$, $j = 1, \dots, k$, their direct sum $S = \bigoplus_{j=1}^k S_j$ is the singularity given by the germ of $f(\mathbf{x}) = \sum_{j=1}^k f_j(\mathbf{x}_j)$ in $\mathbb{C}^{\sum_{j=1}^k n_j}$, see [1]. The algebra $A(S)$ of such a direct sum can be naturally identified with the tensor product of algebras $A(S_j)$ of the summands.

Simple singularities are those without moduli, see [1] for precise definition. They are classified by the Coxeter groups A_k , D_k and the exceptional ones E_6 , E_7 and E_8 , and are denoted by A_k , D_k , E_6 , E_7 and E_8 correspondingly. A semisimple singularity is a direct sum of simple singularities.

Suppose that each algebra $A(S_j)$ is endowed with a \mathbb{Z} -grading. Then their tensor product $A(S)$ acquires a natural \mathbb{Z}^k -grading.

Definition 1. For a finite-dimensional \mathbb{Z}^k -graded algebra $A = \bigoplus_{\alpha \in \mathbb{Z}^k} A_\alpha$ we define its Poincaré polynomial P_A by the equality

$$P_A(t_1, \dots, t_k) = \sum_{\alpha} \dim(A_\alpha) t_1^{\alpha_1} \dots t_k^{\alpha_k}.$$

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The Poincaré polynomial $P(S)$ of $A(S)$ is equal to the product of the Poincaré polynomials $P(S_j)(t_j)$ of $A(S_j)$:

$$P(S)(\mathbf{t}) = \prod_{j=1}^k P(S_j)(t_j), \quad \mathbf{t} = (t_1, \dots, t_k) \in \mathbb{C}^k.$$

Lie algebras $L(S_j)$ of derivations of the graded algebras $A(S_j)$ inherit \mathbb{Z} -gradings following the convention $\deg(\partial/\partial x) = -\deg(x)$ for all involved variables x . Similarly, the \mathbb{Z}^k -grading of the algebra $A(S)$ induces a \mathbb{Z}^k -grading on the Lie algebra $L(S)$ of its derivations. Let $P_L(S_j)$ denote the Poincaré polynomials of $L(S_j)$ corresponding to these gradings. A theorem of Block [2] implies that the Poincaré polynomial of $L(S)$ is

$$P_L(S)(\mathbf{t}) = \left[\sum_{j=1}^k \frac{P_L(S_j)(t_j)}{P(S_j)(t_j)} \right] \prod_{j=1}^k P(S_j)(t_j).$$

The above \mathbb{Z}^k -gradings can produce various \mathbb{Z} -gradings of $A(S)$ and $L(S)$ via linear functionals $\phi: \mathbb{Z}^k \rightarrow \mathbb{Z}$. In other words, one can define a \mathbb{Z} -grading on $A(S)$ and on $L(S)$ as a linear combination with integer weights w_j of the \mathbb{Z} -gradings for S_j . The Poincaré polynomial $P_L^\phi(S)(t)$ of $L(S)$ with respect to the resulting grading will be just $P_L(S)(t^{w_1}, \dots, t^{w_k})$.

For the semisimple singularities the most natural grading is the quasihomogeneous one. Indeed, since the functions f_j defining simple singularities are quasihomogeneous, their sum $f = \sum f_j(\mathbf{x}_j)$ is quasihomogeneous as well.

That said, in this paper we follow a different choice of weights considered in [3]. It corresponds to the linear functional alluded to above given by $\phi(n_1, \dots, n_k) = n_1 + \dots + n_k$, or, in terms of the weights, $w_1 = \dots = w_k = 1$. This choice of weights, together with a suitable choice of quasihomogeneous weights for simple singularities, leads to our main result — Theorem 1 below.

This theorem can be viewed as an alternative version of Theorem 4.5 from [3]. There it is stated that, in our terms, polynomials $P_L^+(S)$ are *unimodular*, i. e. have all their roots on the unit circle, for certain two classes of semisimple singularities, i. e., the sums of simple ones.

As it turns out, this is actually not true for the semisimple singularities from the second named class. The first counterexample occurs for the singularity $D_{17} \oplus E_7$, as shown in our table below (note that, although some of our gradings are different, those for D_{2k+1} and E_7 are identical with those from [3]).

The problem lies in the proof of 4.5 in [3]. That proof involves an argument from the proof of Proposition 4.2 of the same paper. Although the proposition itself is correct, its proof, belonging to the first author of the present paper, contains an erroneous claim, as pointed out by the second author (namely, it was falsely assumed that for any two polynomials of the same degree with positive leading coefficients which are both odd or both even and all of their roots are real and lie in the segment $[-2, 2]$, all roots of their sum are also real and lie in the same segment).

Thus our aim is to describe certain class of semisimple singularities for which the statement of Theorem 4.5 from [3] holds true. This is done in our Theorem 1. As already mentioned, we choose different quasihomogeneous gradings for some

simple singularities — namely, for A_k and D_{2k} . This gives unimodularity of Poincaré polynomials for Lie algebras of a class of semisimple singularities that is strictly larger than the one corresponding to gradings considered in [3].

1.2. Two types of singularities and unimodularity of their Poincaré polynomials. The quasihomogeneous gradings of algebras $A(S)$ (and, correspondingly, their Lie algebras $L(S)$) of simple singularities $S = A_k, D_m$ and E_j ($j = 6, 7, 8$) are defined by the following quasihomogeneous weights ω of variables:

$$\begin{array}{llll}
 A_k, \quad k \geq 1 & x^{k+1} & \omega(x) = 2; & \\
 D_m, \quad m \geq 4 & x^2y + y^{m-1} & \omega(x) = m - 2, \quad \omega(y) = 2; & \\
 E_6 & x^3 + y^4 & \omega(x) = 4, \quad \omega(y) = 3; & (1) \\
 E_7 & x^3 + xy^3 & \omega(x) = 3, \quad \omega(y) = 2; & \\
 E_8 & x^3 + y^5 & \omega(x) = 5, \quad \omega(y) = 3. &
 \end{array}$$

Proposition 1 (cf. [3]). *Poincaré polynomials $P(S)$ and $P_L(S)$ for simple singularities S with respect to the above weights are:*

$$\begin{array}{ll}
 P(A_k)(t) = \frac{1 - t^{2k}}{1 - t^2}, & P_L(A_k)(t) = \frac{1 - t^{2k-2}}{1 - t^2}; \\
 P(D_m)(t) = \frac{(1 + t^{m-2})(1 - t^m)}{1 - t^2}, & P_L(D_m)(t) = \frac{(1 + t^{m-4})(1 - t^m)}{1 - t^2}; \\
 P(E_6)(t) = \frac{(1 + t^4)(1 - t^9)}{1 - t^3}, & P_L(E_6)(t) = \frac{(1 + t^4)(1 - t^6) + 1 - t^9}{1 - t^3}; \\
 P(E_7)(t) = \frac{(1 + t^3)(1 - t^7)}{1 - t^2}, & P_L(E_7)(t) = \frac{(1 + t^3)(1 + t)(1 - t^4)}{1 - t^2}; \\
 P(E_8)(t) = \frac{(1 + t^5)(1 - t^{12})}{1 - t^3}, & P_L(E_8)(t) = \frac{(1 + t^5)(1 - t^9) + 1 - t^{12}}{1 - t^3}.
 \end{array}$$

These formulæ conform with those from [3] except for E_6 and E_8 , as for the latter the non-quasihomogeneous gradings are used in [3]. In these two remaining cases, the polynomials are easily obtained by direct calculation.

We define semisimple singularities of type $A \oplus D$ to be the direct sums of any number of A_k 's and D_m 's.

Theorem 1. *For the above choice of weights the Poincaré polynomial $P_L^+(S)$ of the Lie algebra of a semisimple singularity S of type $A \oplus D$ is unimodular, i. e. has all roots on the unit circle $\{|t| = 1\}$.*

For the same choice of weights the Poincaré polynomial $P_L^+(S \oplus E_7^{\oplus l})$ of the Lie algebra of the direct sum of a semisimple singularity S of type $A \oplus D$ and any number l of copies of E_7 is either unimodular or has exactly four roots outside the unit circle $\{|t| = 1\}$.

Note that $E_6 = A_2 \oplus A_3$ and $E_8 = A_2 \oplus A_4$. However, the gradings of E_6 and E_8 in (1) differ from gradings of $A_2 \oplus A_3$ and $A_2 \oplus A_4$ in Theorem 1. Indeed, for the direct sums of A_k the weights of all variables in Theorem 1 are chosen to be equal. For the quasihomogeneous weights the Poincaré polynomials $P_L(E_6)$ and $P_L(E_8)$ are not palindromic, therefore not unimodular (see Definitions 2, 3 below).

2. PROOFS

2.1. Palindromic polynomials

Definition 2. A polynomial $\sum_{k=0}^n a_k t^k$ of degree n is called *palindromic* if $a_k = a_{n-k}$ for all k .

Definition 3. A polynomial is called *unimodular* if all its roots lie on the unit circle.

A real unimodular polynomial $P(t)$ is either palindromic or becomes palindromic after division by $1 - t$.

Remarkably, Poincaré polynomials $P(S)$ and $P_L(S)$ for singularities $S = A_k, D_k$ and E_7 are palindromic (this is a consequence of the duality existing on the moduli algebras) and moreover unimodular, see [3].

Any product of two palindromic polynomials and a sum of palindromic polynomials of the same degree is again palindromic. Therefore the Poincaré polynomials considered in Theorem 1 are palindromic.

Lemma 1. *The function $t^{-d}P(t)$, where P is a palindromic polynomial of degree $2d$ with real coefficients, takes real values on the unit circle $\{|t| = 1\}$.*

Proof. Indeed, $t^{-d}P(t) = a_d + \sum_{k=1}^d a_{d-k}(t^k + t^{-k})$. Since $t^k + t^{-k}$ is real on the unit circle and a_k are real by assumption, the result follows. \square

Any palindromic polynomial $P(t)$ of odd degree is a product of $1 + t$ and a palindromic polynomial of even degree. Therefore the following is a corollary of the previous lemma.

Lemma 2. *Let $Q = \sum \frac{P_j}{\tilde{P}_j}$ be a rational function, where P_j, \tilde{P}_j are palindromic polynomials with real coefficients, and assume that the differences $\deg P_j - \deg \tilde{P}_j$ are all equal to some number $2d$. Then $t^{-d}Q(t)$ takes real values on the unit circle $\{|t| = 1\}$.*

For such rational functions Q the function $\varphi(x) = e^{-2dix}Q(e^{2ix})$ is a real π -periodic function on \mathbb{R} . Moreover, since Q is real, the function $\varphi(x)$ is even. In fact, $\varphi(x)$ is a rational function of the $\cos(kx)$, $k \in \mathbb{Z}$.

Lemma 3. *Let φ be as above and assume that φ has only simple poles on some interval $[a, b]$. Denote by n_+ (resp. n_-) the number of poles of φ on an interval $[a, b]$ with positive (resp. negative) residue. Then the number of different zeros of φ on $[a, b]$ is at least $|n_+ - n_-| - 1$.*

Proof. Call an interval (x_1, x_2) “good” if x_1, x_2 are two poles of φ with residues of the same sign, and φ is continuous on (x_1, x_2) . Evidently, φ has a zero in any good interval: φ tends to $+\infty$ at one endpoint, and to $-\infty$ at another.

The number of good intervals is at least $|n_+ - n_-| - 1$. \square

In fact, one can prove a stronger result.

Lemma 4. *Let φ, n_+ and n_- be as above and assume that a, b are neither zeros nor poles of φ . Let c be 1 if $\varphi(a)\varphi(b) < 0$ and 0 otherwise.*

Then the number of zeros of φ on $[a, b]$ counted with multiplicities is at least $|n_+ - n_-| - c$ and differs from the latter expression by an even number.

Proof. Choose a smooth extension Φ of φ to $S = \mathbb{R}/(|a - b| + 1)\mathbb{Z} \cong S^1$ having exactly c additional simple zeros and without additional poles. It defines a smooth mapping $\Phi : S \rightarrow \mathbb{R}P^1 = \mathbb{R} \cup \{\infty\}$. Taking standard orientations and looking on $\Phi^{-1}(\infty)$ we conclude that $\deg \Phi = n_- - n_+$.

Zeros of φ fall into three classes: zeros of odd multiplicity where φ increases (denote their number by m_+), zeros of odd multiplicity where φ decreases (denote their number by m_-), and zeros of even multiplicity. Evidently $\deg \Phi = m_+ - m_- \pm c$, and the claim easily follows. \square

2.2. Proof of Theorem 1. The Poincaré polynomial $P_L(S)$ for S of type $A \oplus D$ has the following form:

$$P(t) = \left[\prod_{j=1}^k P(A_{k_j})(t) \prod_{j=1}^M P(D_{m_j})(t) \right] Q(t),$$

where $Q(t) = \sum_{j=1}^k \frac{P_L(A_{k_j})(t)}{P(A_{k_j})(t)} + \sum_{j=1}^M \frac{P_L(D_{m_j})(t)}{P(D_{m_j})(t)}$. All roots of the first two factors of P lie on the unit circle, so one has to prove that all roots of Q lie on the unit circle as well.

Note that $Q(t)$ is a sum of ratios of palindromic polynomials and one has $\deg P_L(A_{k_j}) - \deg P(A_{k_j}) = \deg P_L(D_{m_j}) - \deg P(D_{m_j}) = -2$.

Taking the table from Proposition 1 into account, we can compute $\varphi(x) = e^{2ix}Q(e^{2ix})$:

$$\varphi(x) = \sum_{j=1}^k \frac{\sin((2k_j - 2)x)}{\sin(2k_j x)} + \sum_{j=1}^M \frac{\cos((m_j - 4)x)}{\cos((m_j - 2)x)}.$$

Note that φ is an even π -periodic trigonometric function.

Lemma 5. Residues of poles of the functions $f_{k_j}^A = \frac{\sin((2k_j - 2)x)}{\sin(2k_j x)}$ and $f_{m_j}^D = \frac{\cos((m_j - 4)x)}{\cos((m_j - 2)x)}$ are negative for $x \in (0, \pi/2)$.

Proof. Consider first $f_{k_j}^A$. Let x be a pole of $f_{k_j}^A$, i. e. $2k_j x = n\pi$ for some $n \in \mathbb{Z}$. Then the residue of $f_{k_j}^A$ at x is equal to $\frac{\sin((2k_j - 1)x)}{2k_j \cos(2k_j x)}$.

If n is odd, then $\cos(2k_j x) = -1$, and $\sin((2k_j - 2)x) = \sin(n\pi - 2x) = \sin 2x$, which is positive for $x \in (0, \pi/2)$.

Similarly, if n is even, then $\cos(2k_j x) = 1$, and $\sin((2k_j - 2)x) = \sin(n\pi - 2x) = -\sin 2x$, which is negative for $x \in (0, \pi/2)$.

For $f_{m_j}^D$ the argument is similar. Poles of $f_{m_j}^D$ are at $x = (\pi/2 + n\pi)/(m_j - 2)$. The residue is equal to $-\frac{\cos((m_j - 4)x)}{(m_j - 2) \sin((m_j - 2)x)}$.

For even n we have $\sin((m_j - 2)x) = 1$, and $\cos((m_j - 4)x) = \cos(\pi/2 - 2x)$, which is positive for $x \in (0, \pi/2)$.

For odd n we have $\sin((m_j - 2)x) = -1$, and $\cos((m_j - 4)x) = \cos(-\pi/2 - 2x)$, which is negative for $x \in (0, \pi/2)$. \square

Therefore by Lemma 3 the number of zeros of φ on $[0, \pi/2]$ is at least the number of poles of φ on $[0, \pi/2]$ minus 1. Since φ is even, the same holds for the interval $[-\pi/2, 0]$. We conclude that the number of different zeros of $\varphi(x)$ on $(-\pi/2, \pi/2)$ is at least the number of its poles on $(-\pi/2, \pi/2)$ minus 2 (note that $\varphi(x)$ has no poles at 0 and $\pm\pi/2$).

Equivalently, the number of zeros of Q on the unit circle is at least the number of its poles on the unit circle minus 2.

The rational function Q has a double zero at infinity. Therefore the number of all finite zeros counted with multiplicities is equal to $N - 2$, where N is the number of poles of Q counted with multiplicities. But the poles of Q are simple and lie on the unit circle. Therefore the number of zeros of Q on the unit circle is already at least $N - 2$, i. e. the maximum possible. We conclude that all zeros of Q are on the unit circle (and they are all simple).

The case of $S \oplus E_7^{\oplus l}$: The Poincaré polynomial $P_L(S \oplus E_7^{\oplus l})$ in Theorem 1 has the following form:

$$P = \left[(P(E_7)(t))^l \prod_{j=1}^k P(A_{k_j})(t) \prod_{j=1}^M P(D_{m_j})(t) \right] Q(t),$$

where $Q = l \frac{P_L(E_7)(t)}{P(E_7)(t)} + \sum_{j=1}^k \frac{P_L(A_{k_j})(t)}{P(A_{k_j})(t)} + \sum_{j=1}^M \frac{P_L(D_{m_j})(t)}{P(D_{m_j})(t)}$.

Taking Proposition 1 into account, we see that the real function $\varphi(x) = Q(e^{2ix})$ has the form

$$\varphi(x) = l \frac{\sin 4x \cos x}{\sin 7x} + \sum_{j=1}^k \frac{\sin((2k_j - 2)x)}{\sin(2k_j x)} + \sum_{j=1}^M \frac{\cos((m_j - 4)x)}{\cos((m_j - 2)x)}.$$

Again, the function φ is even and π -periodic.

The function $f^{E_7} = \frac{\sin 4x \cos x}{\sin 7x}$ has 3 poles in the interval $[0, \pi/2]$ at points $k\pi/7$, $k = 1, 2, 3$. The residues are negative at $\pi/7, 2\pi/7$ and positive at $3\pi/7$.

Therefore by Lemma 3 the number of zeros of φ on the interval $(0, \pi/2)$ is at least the number of poles of φ on this interval minus 3.

Let us apply Lemma 4 to φ and $[0, \pi/2]$. First, $\varphi(0) = l \cdot 4/7 + \sum \frac{k_j - 1}{k_j} + M > 0$. Also, $\varphi(\pi/2) = -\sum \frac{k_j - 1}{k_j} + \sum \lim_{x \rightarrow \pi/2} \frac{\cos((m_j - 4)x)}{\cos((m_j - 2)x)} < 0$, since $\lim_{x \rightarrow \pi/2} \frac{\cos((m_j - 4)x)}{\cos((m_j - 2)x)}$ is either -1 or $-\frac{m-4}{m-2}$ depending on whether m_j is even or odd. Therefore the number c in Lemma 4 is equal to 1.

We conclude that the number of zeros of φ on the interval $(0, \pi/2)$ differs from the number of poles of φ on this interval by an odd number.

Again, since φ is even, the number of different roots of Q on the unit circle is at least the number of its poles on the unit circle minus twice an odd number.

As before, Q has a double zero at infinity, and does not have poles outside the unit circle. Therefore the total number of zeros of Q counted with multiplicities is equal to the number of poles of Q lying on the unit circle minus 2. Subtracting the zeros lying on the unit circle we get that the number of zeros not lying on the unit circle is either 4 or 0.

2.3. Final notes. Of course one would like to have more conceptual proof of the above theorem, e. g. by relating to a semisimple singularity some naturally defined unitary operator whose characteristic polynomial would coincide with the Poincaré polynomial of the corresponding Lie algebra.

Results of some numerical computations reproduced in the table below show that for the Poincaré polynomials of Lie algebras corresponding to $S \oplus E_7^{\oplus l}$ both possibilities mentioned in Theorem 1 are indeed realised. That is, there might occur 4 non-unimodular roots, and there might be none too.

k	Number of roots outside the unit circle for		
	$A_k \oplus E_7$	$D_{2k} \oplus E_7$	$D_{2k+1} \oplus E_7$
2			0
3		0	0
4	0	0	0
5	0	4	0
6	0	4	0
7	0	4	0
8	4	0	4
9	4	0	4
10	4	0	4
11	4	0	4
12	0	4	0
13	0	4	0
14	0	4	0

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