K-Theory **25:** 299–311, 2002.

Classification of Abelian Track Categories

HANS-JOACHIM BAUES and MAMUKA JIBLADZE
Max-Planck-Institut für Mathematik, Vivatsgasse 17, 53111 Bonn, Germany

(Received: September 2000)

Abstract. We show that each category enriched in Abelian groupoids is a linear track extension and hence is determined up to weak equivalence by a characteristic chomology class. We also discuss compatibility with coproducts.

Mathematics Subject Classifications (2000): 18D99, 18G55.

Key words: Abelian groupoid, Abelian track category, natural system, chomology of categories.

Call a groupoid *Abelian* if all of its objects have Abelian automorphism groups. We show that any category $\mathcal{T}$ enriched in Abelian groupoids is linear in the sense that there is a canonical linear track extension

$$D \rightarrowtail \mathcal{T}_1 \rightrightarrows \mathcal{T}_0 \longrightarrow \mathcal{T}_{\simeq}$$

as defined by Baues and Dreckmann [GL]. Here $\mathcal{T}_{\simeq}$ is the homotopy category of $\mathcal{T}$ and D is the natural system associated with $\mathcal{T}$. If $\mathcal{T}$ is small, this implies that up to weak equivalence $\mathcal{T}$ is determined by a cohomology class $\langle\mathcal{T}\rangle \in H^3(\mathcal{T}_{\simeq}; D)$ called the *universal Toda bracket* of $\mathcal{T}$. Conversely, given any natural system D on a small category C and a cohomology class $\tau \in H^3(\mathcal{C}; D)$, there is a unique weak equivalence class of a category $\mathcal{T}$ enriched in Abelian groupoids such that $\tau = \langle\mathcal{T}\rangle$. This yields a classification of small categories enriched in Abelian groupoids. Important examples of Abelian track categories arise in the context of secondary cohomology operations. (Compare the book [B].)

1. Categories Enriched in Groupoids

Recall that a *groupoid* is a small category all of whose morphisms are invertible. We will use additive notation for groupoids; thus, the identity morphism of an object x of a groupoid $\mathbf{G}$ will be denoted 0_x, and for $\alpha\colon x \to y$, $\beta\colon y \to z$ their composite will be denoted $\beta + \alpha\colon x \to z$. For a groupoid $\mathbf{G}$, the set of its objects will be denoted by $\mathbf{G}_0$ and the set of morphisms by $\mathbf{G}_1$. We have the canonical source and target maps $\mathbf{G}_1 \rightrightarrows \mathbf{G}_0$. A groupoid is termed an *Abelian groupoid* if the automorphism group of each object is an Abelian group.

EXAMPLE 1.1. Given a topological space X one obtains the *fundamental groupoid* $\Pi(X)$. Its objects are the points of X and morphisms $x_0 \to x_1$ with $x_0, x_1 \in X$

are homotopy classes rel. ∂I of paths $\omega\colon I \to X$ with $\omega(0) = x_0$ and $\omega(1) = x_1$. Here $I = [0, 1]$ is the unit interval with boundary $\partial I = \{0, 1\}$. Composition in $\Pi(X)$ is given by addition of paths. It is well known that $\Pi(X)$ is an Abelian groupoid if X is a topological group or more generally if each path component of X has the homotopy type of an H-space.

A *category enriched in groupoids* $\mathcal{T}$, also termed *track category* for short, is the same as a 2-*category* all of whose 2-cells are invertible. It is thus a class of objects $\mathrm{Ob}(\mathcal{T})$, a collection of groupoids $\mathcal{T}(A, B)$ for $A, B \in \mathrm{Ob}\mathcal{T}$ called hom-*groupoids* of $\mathcal{T}$, identities $1_A \in \mathcal{T}(A, A)_0$ and composition functors $\mathcal{T}(B, C) \times \mathcal{T}(A, B) \to \mathcal{T}(A, C)$ satisfying the usual equations of associativity and identity morphisms. For generalities on enriched categories the reader may consult Kelly [EC]. Obviously $\mathcal{T}$ enriched in Abelian groupoids means that all hom-groupoids $\mathcal{T}(A, B)$ for $A, B \in \mathrm{Ob}(\mathcal{T})$ are Abelian; in this case we say that $\mathcal{T}$ is an *Abelian track category*. Objects of the hom-groupoids $f \in \mathcal{T}(A, B)_0$, called *maps* in $\mathcal{T}$, constitute morphisms of an ordinary category $\mathcal{T}_0$ having the same objects as $\mathcal{T}$. For $f, g \in \mathcal{T}(A, B)$ we shall write $f \simeq g$ (and say f is *homotopic* to g) if there exists a morphism $\alpha\colon f \to g$ in $\mathcal{T}(A, B)$. Occasionally this will be also denoted as $\alpha\colon f \simeq g$ or $\alpha\colon f \Rightarrow g$, α sometimes called a *homotopy* or a *track* from f to g. Homotopy is a natural equivalence relation on morphisms of $\mathcal{T}_0$ and determines the *homotopy category* $\mathcal{T}_{\simeq} = \mathcal{T}_0/\simeq$. Objects of $\mathcal{T}_{\simeq}$ are once again objects in $\mathrm{Ob}(\mathcal{T})$, while morphisms of $\mathcal{T}_{\simeq}$ are homotopy classes of morphisms in $\mathcal{T}_0$. Let $q\colon \mathcal{T}_0 \to \mathcal{T}_{\simeq}$ be the quotient functor. Moreover, let $\mathrm{Mor}(\mathcal{T}_1)$ be the disjoint union of all tracks in $\mathcal{T}$. One then has the source and target functions between sets

$$\mathrm{Mor}(\mathcal{T}_1) \underset{t}{\overset{s}{\rightrightarrows}} \mathrm{Mor}(\mathcal{T}_0)$$

with $qs = qt$. Here $\mathrm{Mor}(\mathcal{T}_0)$ denotes the set of morphisms in the category $\mathcal{T}_0$. The functions s and t are actually induced by functors $s, t\colon \mathcal{T}_1 \to \mathcal{T}_0$. Here the category $\mathcal{T}_1$ has the same objects as $\mathcal{T}_0$ and the functors s, t are the identity on objects. The morphisms $A \to B$ in $\mathcal{T}_1$ are triples (f, f_1, φ) where $f, f_1\colon A \to B$ are morphisms in $\mathcal{T}_0$ and $\varphi\colon f \Rightarrow f_1$ is a track. Identity of $A \in \mathrm{Ob}(\mathcal{T}_1)$ is $(1_A, 1_A, 0_{1_A})$, and composition in $\mathcal{T}_1$ is the $*$-*composition* defined by $(f, f_1, \varphi) * (g, g_1, \psi) = (fg, f_1g_1, \varphi * \psi)$, where

$$\varphi * \psi = \varphi g_1 + f\psi = f_1\psi + \varphi g. \tag{1.2}$$

Of course, we have $s(f, f_1, \varphi) = f$ and $t(f, f_1, \varphi) = f_1$.

Motivated by Example (1.3) below we borrow from topology the following notation in a track category $\mathcal{T}$. Let $[A, B] = \mathcal{T}(A, B)/\simeq$ be the set of homotopy classes of maps $A \to B$ and let $[\![A, B]\!] = \mathcal{T}(A, B)$ be the hom-groupoid of $\mathcal{T}$ so that $[A, B]$ is the set of path components of the groupoid $[\![A, B]\!]$. A map $f\colon A \to B$ is a *homotopy equivalence* if there exists a map $g\colon B \to A$ and tracks $fg \simeq 1$ and $gf \simeq 1$. This is the case if and only if the homotopy class of f is an isomorphism in the homotopy category $\mathcal{T}_{\simeq}$. In this case, A and B are called *homotopy equivalent* objects.

It is easy to see that $f: A \to B$ is a homotopy equivalence if and only if for all objects X the functor $f_*: [\![X, A]\!] \longrightarrow [\![X, B]\!]$ is an equivalence of categories.

Remark. The morphisms in $\mathcal{T}_0$ are also termed 1-cells and the tracks in $\mathcal{T}_1$ are 2-cells. In particular, the category $\mathfrak{Gpd}$ of groupoids is a track category. Objects are groupoids, morphisms are functors and tracks are natural transformations (since they are natural isomorphisms). Moreover, any category C can be considered to be a track category with only identity tracks.

EXAMPLE 1.3. The leading example is the track category $\mathbf{Top}^*$ of compactly generated Hausdorff spaces with basepoint $*$, given as follows. For pointed spaces A, B let $\mathbf{Top}^*(A, B)$ be the fundamental groupoid of the function space B^A of pointed maps $A \to B$ with the compactly generated compact-open topology. See 8.14 of Gray [HT]. Hence maps are pointed maps $f, g: A \to B$ between pointed spaces and tracks $\alpha: f \Rightarrow g$ are homotopy classes relative to $A \times \partial I$ of homotopies $H: A \times I / * \times I \to B$ with $H: f \simeq g$. In this case $\mathbf{Top}^*_{\simeq}$ is the usual homotopy category of pointed spaces. Let $\mathcal{X}$ be a class of pointed spaces. Then $\mathbf{Top}^*(\mathcal{X})$ is the track category consisting of all spaces A with $A \in \mathcal{X}$ which is a full subcategory of $\mathbf{Top}^*$.

An *H-group* G is a topological space with a homotopy associative multiplication, a homotopy identity $* \in G$, and a homotopy inverse in $\mathbf{Top}^*$. The dual of an H-group is a *co-H-group*. For example the suspension ΣX, resp. the loop space ΩX, of a pointed space X is a co-H-group, resp. an H-group. The next result yields many interesting examples of categories enriched in Abelian groupoids. For example we can choose $\mathcal{X}$ to be the set of one point unions of spheres S^n, $n \geqslant 1$, or the set of products of Eilenberg–Mac Lane spaces.

PROPOSITION 1.4. *Let $\mathcal{X}$ be a class of H-groups or let $\mathcal{X}$ be a class of co-H-groups. Then* $\mathbf{Top}^*(\mathcal{X})$ *is an Abelian track category.*

Proof. It is well known that the function space B^A is an H-group if B is an H-group or A is a co-H-group. Hence, the fundamental groupoid of B^A is Abelian. □

DEFINITIONS 1.5. A *track functor*, or else 2-functor $F: \mathcal{T} \to \mathcal{T}'$ between track categories is a groupoid enriched functor. Thus F assigns to each $A \in \mathrm{Ob}(\mathcal{T})$ an object $F(A) \in \mathrm{Ob}(\mathcal{T}')$, to each map $f: A \to B$ in $\mathcal{T}$ – a map $F(f): F(A) \to F(B)$, and to each track $\alpha: f \Rightarrow g$ for $f, g: A \to B$, a track $F(\alpha): F(f) \Rightarrow F(g)$ in a functorial way, i.e. so that one gets functors $F_{A,B}: [\![A, B]\!] \to [\![F(A), F(B)]\!]$. Moreover, these assignments are compatible with identities and composition, or equivalently induce a functor $\mathcal{T}_1 \to \mathcal{T}'_1$, that is, $F(1_A) = 1_{F(A)}$ for $A \in \mathrm{Ob}(\mathcal{T})$, $F(fg) = F(f)F(g)$, and $F(\varphi * \psi) = F(\varphi) * F(\psi)$ for any $\varphi: f \Rightarrow f_1$, $\psi: g \Rightarrow g_1$, $f, f_1: B \to C$, $g, g_1: A \to B$ in $\mathcal{T}$. It is evident from (1.2) that in presence of functoriality of the $F_{A,B}$ above, the latter condition is equivalent to requiring just $F(f\psi) = F(f)F(\psi)$ and $F(\varphi g) = F(\varphi)F(g)$.

A track functor $F\colon \mathcal{T} \to \mathcal{T}'$ is called a *weak equivalence* between track categories if the functors $\llbracket A, B\rrbracket \to \llbracket F(A), F(B)\rrbracket$ are equivalences of groupoids for all objects A, B of $\mathcal{T}$ and each object A' of $\mathcal{T}'$ is homotopy equivalent to some object of the form $F(A)$. Such a weak equivalence induces a functor $F\colon \mathcal{T}_{\simeq} \to \mathcal{T}'_{\simeq}$ between homotopy categories which is an equivalence of categories.

A 2-*transformation* $\Phi\colon F \Rightarrow G$ between track functors $F, G\colon \mathcal{T} \to \mathcal{T}'$ is a morphism of groupoid-enriched functors. So it is given by a collection of maps $\Phi_A\colon F(A) \to G(A)$ such that the resulting diagram in $\mathfrak{Gpd}$

$$\begin{array}{ccc} \llbracket F(A), F(B)\rrbracket & \xleftarrow{\llbracket \Phi_A, F(B)\rrbracket} & \llbracket G(A), F(B)\rrbracket \\ \Big\downarrow{\scriptstyle \llbracket F(A), \Phi_B\rrbracket} & & \Big\downarrow{\scriptstyle \llbracket G(A), \Phi_B\rrbracket} \\ \llbracket F(A), G(B)\rrbracket & \xleftarrow{\llbracket \Phi_A, G(B)\rrbracket} & \llbracket G(A), G(B)\rrbracket \end{array}$$

commutes.

For two 2-transformations $\Phi, \Psi\colon F \Rightarrow G$ between track functors $F, G\colon \mathcal{T} \to \mathcal{T}'$, a 2-*track* $\mathfrak{H}$ from Φ to Ψ is given by a collection of tracks $\mathfrak{H}_A\colon \Phi_A \Rightarrow \Psi_A$ such that for any $f, f_1\colon A \to B$ in $\mathcal{T}$ and any $\varphi\colon f \Rightarrow f_1$ one has $\mathfrak{H}_B * F(\varphi) = G(\varphi) * \mathfrak{H}_A$.

For any track categories $\mathcal{T}$, $\mathcal{T}'$ track functors $\mathcal{T} \to \mathcal{T}'$, 2-transformations and 2-tracks between them form a track category which we denote $\mathcal{T}'^{\mathcal{T}}$.

These definitions are all particular cases of standard 2-categorical machinery, see, e.g., [FC].

EXAMPLE 1.6. Any object A of a track category $\mathcal{T}$ gives rise to the *representable* track functor $\llbracket A, -\rrbracket\colon \mathcal{T} \to \mathfrak{Gpd}$ sending an object X to the groupoid $\llbracket A, X\rrbracket$. This 2-functor assigns to a map $f\colon X \to Y$ the functor $\llbracket A, f\rrbracket\colon \llbracket A, X\rrbracket \to \llbracket A, Y\rrbracket$ sending $g\colon A \to X$ to fg and $\gamma\colon g \Rightarrow g'$ to γf. And this 2-functor assigns to a track $\varphi\colon f \Rightarrow f'$ the natural transformation $\llbracket A, \varphi\rrbracket\colon \llbracket A, f\rrbracket \to \llbracket A, f'\rrbracket$ with components $\varphi g\colon fg \Rightarrow fg'$.

Any map $f\colon A \to B$ induces a 2-transformation $\llbracket f, -\rrbracket\colon \llbracket B, -\rrbracket \Rightarrow \llbracket A, -\rrbracket$, with components the functors $\llbracket f, -\rrbracket_X = \llbracket f, X\rrbracket\colon \llbracket B, X\rrbracket \to \llbracket A, X\rrbracket$; and any track $\varphi\colon f \Rightarrow f_1$ induces a 2-track $\llbracket \varphi, -\rrbracket$ from $\llbracket f, -\rrbracket$ to $\llbracket f_1, -\rrbracket$, with components the natural transformations $\llbracket \varphi, -\rrbracket_X = \llbracket \varphi, X\rrbracket\colon \llbracket f, X\rrbracket \to \llbracket f_1, X\rrbracket$ with components $\llbracket \varphi, X\rrbracket_g = g\varphi\colon gf \to gf_1$ for $g\colon B \to X$.

The standard 'Yoneda' argument then gives

LEMMA 1.7. *For any objects A, B in a track category $\mathcal{T}$ the assignments as in* (1.6) *above determine a (contravariant) track functor $\mathcal{T}^{\mathrm{op}} \to \mathfrak{Gpd}^{\mathcal{T}}$, where $\mathcal{T}^{\mathrm{op}}$ is the opposite track category of $\mathcal{T}$, defined in an obvious way. Moreover, this track functor has the property that the induced functors*

$$\llbracket A, B\rrbracket \to \mathfrak{Gpd}^{\mathcal{T}}(\llbracket B, -\rrbracket, \llbracket A, -\rrbracket)$$

are equivalences of groupoids. In particular, a map $f: A \to B$ *is a homotopy equivalence iff for any* $X \in \mathrm{Ob}(\mathcal{T})$ *the functor* $[\![f, X]\!]: [\![B, X]\!] \to [\![A, X]\!]$ *is an equivalence in* $\mathfrak{Gpd}$.

Proof. Most of this being standard, let us just sketch how to obtain an inverse homotopy equivalence $g: B \to A$ for $f: A \to B$, given that $[\![f, X]\!]$ is an equivalence for any X. For that, just take $X = A$. Then since $[\![f, A]\!]: [\![B, A]\!] \to [\![A, A]\!]$ is an equivalence, there is an object g of $[\![B, A]\!]$ whose image under $[\![f, A]\!]$ is isomorphic to 1_A, i.e. there is a track $\alpha: 1_A \simeq gf$. But $[\![f, B]\!]: [\![B, B]\!] \to [\![A, B]\!]$ is an equivalence too, so it induces a bijection between the set of tracks from 1_B to fg and the set of tracks from $1_B f$ to $(fg)f$. The latter set is nonempty as it contains $f\alpha: f \to fgf$, hence there must be a track $1_B \simeq fg$. □

2. Linearity

We recall from [GL] the notion of a linear track extension and we show that any category enriched in Abelian groupoids has canonically a structure of such an extension.

DEFINITION 2.1. Let C be a category. Then the category $F\mathcal{C}$ of factorizations in C is defined as follows. Objects of $F\mathcal{C}$ are morphisms $f: A \to B$ in C and morphisms $(\alpha, \beta): f \to g$ in $F\mathcal{C}$ are commutative diagrams

$$\begin{array}{ccc} B & \xrightarrow{\alpha} & B' \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle g} \\ A & \xleftarrow[\beta]{} & A' \end{array}$$

in the category C. A *natural system* on C with values in a category **G** is a functor $D: F\mathcal{C} \to \mathbf{G}$. We write $D(f) = D_f \in \mathbf{G}$ and $D(\alpha, \beta) = \alpha_* \beta^*$. In the situation $\xleftarrow[f]{}\xleftarrow[g]{}\xleftarrow[h]{}$ the induced homomorphisms f_* and h^* will be denoted by

$$f_*: D_g \to D_{fg}, \qquad \xi \mapsto f\xi = f_*(\xi),$$
$$h^*: D_g \to D_{gh}, \qquad \xi \mapsto \xi h = h^*(\xi).$$

A morphism of natural systems is just a natural transformation. For a functor $q: \mathcal{C}' \to \mathcal{C}$, any natural system D on C gives a natural system $D \circ (Fq)$ on $\mathcal{C}'$ which we will denote D_q.

EXAMPLE 2.2. For a track category $\mathcal{T}$ and for a map $f: A \to B$ in $\mathcal{T}$ denote by $\mathrm{Aut}(f) = \mathrm{hom}(f, f)$ the automorphism group of f in the groupoid $[\![A, B]\!] = \mathcal{T}(A, B)$. Then composition in $\mathcal{T}$ yields for $\xleftarrow{f}\xleftarrow{g}\xleftarrow{h}$ in $\mathcal{T}_0$ the homomorphisms

$$h^*: \mathrm{Aut}(g) \to \mathrm{Aut}(gh), \qquad \xi \mapsto \xi h,$$

$$f_*\colon \mathrm{Aut}(g) \to \mathrm{Aut}(fg), \qquad \xi \mapsto f\xi.$$

Evidently this defines a natural system $\mathrm{Aut}^{\mathcal{T}}$ on the category $\mathcal{T}_0$ with values in the category of groups.

Note that moreover any track $\eta\colon f \Rightarrow f_1$ induces a group homomorphism $(-)^\eta\colon \mathrm{Aut}(f_1) \to \mathrm{Aut}(f)$ which carries $\alpha \in \mathrm{Aut}(f_1)$ to $\alpha^\eta = -\eta + \alpha + \eta$.

DEFINITION 2.3. For a natural system D on a category C with values in Abelian groups, a *linear track extension of C by D* denoted by

$$D \overset{\sigma}{\rightarrowtail} \mathcal{T}_1 \rightrightarrows \mathcal{T}_0 \overset{q}{\twoheadrightarrow} \mathcal{C}$$

is a track category $\mathcal{T}$ equipped with a functor $q\colon \mathcal{T}_0 \to \mathcal{C}$ and an isomorphism $\sigma\colon D_q \to \mathrm{Aut}^{\mathcal{T}}$ of natural systems on $\mathcal{T}_0$.

Thus such a linear extension consists of a collection of isomorphisms of groups $\sigma_f\colon D_{q(f)} \to \mathrm{Aut}(f)$ for each map $f\colon A \to B$ in $\mathcal{T}_0$, which have the following properties.

(1) The functor q is full and the identity on objects, i.e. $\mathrm{Ob}(\mathcal{T}) = \mathrm{Ob}(\mathcal{C})$. In addition for $f, g\colon A \to B$ in $\mathcal{T}_0$ we have $q(f) = q(g)$ if and only if $f \simeq g$. In other words the functor q identifies C with $\mathcal{T}_{\simeq}$. We also write $q(f) = [f]$. Hence, for any $\varphi\colon f \Rightarrow g$ we have $[f] = [g]$.
(2) For $\varphi\colon f \Rightarrow g$ and $\xi \in D_{[f]} = D_{[g]}$ we have $\sigma_f(\xi) = \sigma_g(\xi)^\varphi$.
(3) For any three maps like $\xleftarrow{f}\xleftarrow{g}\xleftarrow{h}$ in $\mathcal{T}_0$ and any $\xi \in D_{[g]}$ one has

$$f\sigma_g(\xi) = \sigma_{fg}([f]\xi), \qquad \sigma_g(\xi)h = \sigma_{gh}(\xi[h]).$$

We say that a track category is *linear* if it occurs as a linear track extension – of its own homotopy category, necessarily – by some natural system D. Clearly a linear track category has Abelian hom-groupoids by the definition above. The following result shows that also the converse is true.

THEOREM 2.4. *Any Abelian track category $\mathcal{T}$ is linear.*

Proof. Given $\mathcal{T}$ we define the natural system D on $\mathcal{T}_{\simeq}$ as follows. Let $\mathbf{f}\colon A \to B$ be a morphism in $\mathcal{T}_{\simeq}$ so that $\mathbf{f}$ is a homotopy class of morphisms in $\mathcal{T}_0$. We define $D_{\mathbf{f}}$ to be the Abelian group generated by elements $[\alpha]$ for $\alpha\colon f \simeq f$ an automorphism of any 1-arrow f in the class $\mathbf{f}$, i.e. $f \in \mathbf{f}$ or $\mathbf{f} = [f]$. Defining relations of the group $D_{\mathbf{f}}$ are

$$[\alpha] + [\beta] = [\alpha + \beta] \quad \text{for } \alpha, \beta\colon f \Rightarrow f \quad \text{and}$$

$$[\varphi + \psi] = [\psi + \varphi] \quad \text{for any } \varphi\colon f \Rightarrow f', \qquad \psi\colon f' \Rightarrow f$$

with $f, f' \in \mathbf{f}$. Note that since all 2-arrows are isomorphisms, this second relation is equivalent to

$$[\alpha^\varphi] = [\alpha] \qquad (*)$$

for any $\alpha\colon f \Rightarrow f, \varphi\colon f' \Rightarrow f$ (indeed $[\alpha^\varphi] = [(-\varphi+\alpha)+\varphi] = [\varphi+(-\varphi+\alpha)]$ by definition).

Define $\tau_f\colon \mathrm{Aut}(f) \to D_{[f]}$ by $\alpha \mapsto [\alpha]$. Using (*) one easily sees that this map is onto. By the same reason, all relations in $D_{[f]}$ are consequences of relations involving only elements $[\alpha]$ for $\alpha\colon f \simeq f$; and these relations are as follows:

$$[(\varphi+\psi)^{\chi_1}] = [(\psi+\varphi)^{\chi_2}]$$

for any $\chi_1\colon f \Rightarrow f_1, \varphi\colon f_2 \to f_1, \psi\colon f_1 \to f_2, \chi_2\colon f \to f_2$; and

$$[(\alpha+\beta)^{\chi}] = [\alpha^{\chi'}] + [\beta^{\chi''}]$$

for any $\chi, \chi', \chi''\colon f' \Rightarrow f$ and any $\alpha, \beta\colon f \Rightarrow f$. Now the first of these relations can be rewritten as

$$\begin{aligned}&[(-\chi_1+\varphi+\chi_2)+(-\chi_2+\psi+\chi_1)]\\ &\quad= [(-\chi_2+\psi+\chi_1)+(-\chi_1+\varphi+\chi_2)]\end{aligned}$$

and the second as

$$\begin{aligned}[\alpha^\chi+\beta^\chi] \;=\; &[-(-\chi+\chi')+(-\chi+\alpha+\chi)+(-\chi+\chi')]+\\ &+[-(-\chi+\chi'')+(-\chi+\beta+\chi)+(-\chi+\chi'')].\end{aligned}$$

And both of these hold in $\mathrm{Aut}(f)$ since it is Abelian. One thus concludes that the τ_f are also one-to one, hence are isomorphisms. Let $\sigma_f = \tau_f^{-1}$. Then the collection of σ_f satisfies condition (2) for linear track extensions iff the τ_f satisfy $\tau_f(\alpha) = \tau_g(\beta)$ whenever $\beta = \alpha^\varphi$, for $\alpha\colon f \simeq f, \beta\colon g \simeq g, \varphi\colon g \Rightarrow f$. In other words, one must have $[\alpha^\varphi] = [\alpha]$, which is (*).

One now defines actions of the natural system by $[f][\alpha] = [f\alpha], [\alpha][h] = [\alpha h]$ for any $\xleftarrow{f}\xleftarrow{g}\xleftarrow{h}$ and $\alpha\colon g \Rightarrow g$. Here the right-hand-side does not depend on the choice of f since for any $\varphi\colon f' \Rightarrow f$ one has $\varphi g + f'\alpha = f\alpha + \varphi g$, i.e. $f'\alpha = (f\alpha)^{\varphi g}$, hence $[f\alpha] = [f'\alpha]$ in $D_{[fg]}$. And it also does not depend on the choice of α since $f(\varphi+\psi) = f\varphi + f\psi$ for any $\varphi\colon g' \Rightarrow g, \psi\colon g \Rightarrow g''$. Similarly for the right actions. Then one immediately has

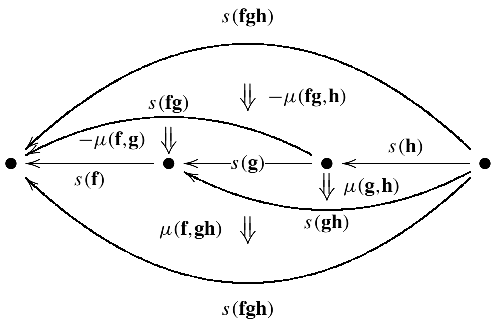
$$\begin{aligned}[f]\tau_g(\alpha) \;&=\; \tau_{fg}(f\alpha), \tau_g(\alpha)[h]\\ &=\; \tau_{gh}(\alpha h) \quad \text{for } \xleftarrow[f]{}\xleftarrow[g]{}\xleftarrow[h]{}, \quad \alpha\colon g \simeq g,\end{aligned}$$

which is clearly equivalent to condition (3) for the σ's. □

3. Classification

For each linear track extension $\mathcal{T}$ a certain characteristic cohomology class $\langle\mathcal{T}\rangle$ is defined which via Theorem (2.4) and the result in [GL] leads to a classification of categories enriched in Abelian groupoids. We recall from [GL] the definition of $\langle\mathcal{T}\rangle$ as follows. We assume that all categories used in this section are actually small. We use the cohomology of a category $\mathcal{C} = \mathcal{T}_{\simeq}$ with coefficients in a natural system as defined in [CC].

DEFINITION 3.1. For a linear track category $\mathcal{T}$, its *universal Toda bracket* $\langle\mathcal{T}\rangle$ is the element of $H^3(\mathcal{T}_{\simeq}; D)$ represented by the following cocycle: choose, for each morphism $\mathbf{f}$ in $\mathcal{T}_{\simeq}$, a representative 1-arrow of $\mathcal{T}$ denoted $s(\mathbf{f}) \in \mathbf{f}$. Furthermore, choose a track $\mu(\mathbf{f}, \mathbf{g})\colon s(\mathbf{f})s(\mathbf{g}) \Rightarrow s(\mathbf{fg})$. Then for each composable triple $\mathbf{f}, \mathbf{g}, \mathbf{h}$ the composite track in the diagram



determines an element in $\mathrm{Aut}(s(\mathbf{fgh}))$ and, hence, going back via σ, an element $c(\mathbf{f}, \mathbf{g}, \mathbf{h}) \in D_{\mathbf{fgh}}$. It can be checked that this determines a 3-cocycle of $\mathcal{T}_{\simeq}$ with coefficients in D, and that both choosing a different section s or another track category which is weakly equivalent to $\mathcal{T}$ leads to a cohomologous cocycle. One thus obtains a uniquely determined cohomology class $\langle\mathcal{T}\rangle \in H^3(\mathcal{T}_{\simeq}; D)$.

DEFINITION 3.2. A *classifying triple* is a triple $(\mathcal{C}, D, \tau)$ where C is a category, D is a natural system on C and τ is a cohomology class $\tau \in H^3(\mathcal{C}; D)$. Two such triples $(\mathcal{C}, D, \tau)$ and $(\mathcal{C}', D', \tau')$ are *equivalent* if there exists an equivalence of categories $F\colon \mathcal{C}' \to \mathcal{C}$ and a natural isomorphism $I\colon D' \cong F^*D$ such that $F^*\tau = I_*\tau'$. Compare [CC] for the definition of the induced maps I_* and F^* in cohomology. One readily checks that this is a well-defined equivalence relation.

We say that a function is a one-to-one correspondence if the function is injective and surjective.

CLASSIFICATION 3.3. *There is a one-to-one correspondence between weak equivalence classes of Abelian track categories and equivalence classes of classifying triples.*

Proof. The one-to-one correspondence carries the weak equivalence class of a track category $\mathcal{T}$ to the equivalence class of the triple $(\mathcal{T}_{\simeq}, D, \langle\mathcal{T}\rangle)$, where the natural system is defined as in the proof of (2.4) and where $\langle\mathcal{T}\rangle$ is the universal

Toda bracket (3.1) of the linear track extension given by the proof of (2.4). Now we apply Theorem 4.6 from [GL] or VI.3.15 of [CH] to obtain the result. □

For the classification above compare also [MC], [MH], [CA], and [BC].

4. Compatibility with Sums

Sums in a track category $\mathcal{T}$ can be understood either in the strong or in the weak sense. In any case, the sum of objects A, B is the object $A \vee B$ equipped with the maps $i_A\colon A \to A \vee B$, $i_B\colon B \to A \vee B$ such that the induced functor

$$[\![A \vee B, X]\!] \to [\![A, X]\!] \times [\![B, X]\!] \tag{4.1}$$

given by

$$f \mapsto (fi_A, fi_B), \quad \eta \mapsto (\eta i_A, \eta i_B)$$

is an *isomorphism* of groupoids in the strong case, or an *equivalence* in the weak case, for all objects X.

Evidently, a strong sum is a weak sum, too. Moreover, if $i_A\colon A \to A \vee B \leftarrow B\colon i_B$ is a sum in any sense and $f\colon A \vee B \to S$ is an equivalence, then S with the maps fi_A, fi_B can serve as a weak sum. Note however that it might be that strong sums do not exist, whereas weak ones do.

Explicitly, the universal property of weak sums means that for any maps $a\colon A \to X$, $b\colon B \to X$ there is a map $\binom{a}{b}\colon A \vee B \to X$ and tracks $\binom{a}{b}i_A \simeq a$, $\binom{a}{b}i_B \simeq b$. Furthermore, these are unique in the sense that for any maps $h, h'\colon A \vee B \to X$ and tracks $\alpha\colon hi_A \Rightarrow h'i_A$, $\beta\colon hi_B \Rightarrow h'i_B$ there is a unique track $\binom{\alpha}{\beta}\colon h \Rightarrow h'$ with $\alpha = \binom{\alpha}{\beta}i_A$, $\beta = \binom{\alpha}{\beta}i_B$. This uniqueness condition can be equivalently stated in two parts:

(a) for any $h, h'\colon A \vee B \to X$, if there are tracks $hi_A \simeq h'i_A$ and $hi_B \simeq h'i_B$ then there is a track $h \simeq h'$;
(b) for any $h\colon A \vee B \to X$, the group homomorphism $\mathrm{Aut}(h) \to \mathrm{Aut}(hi_A) \times \mathrm{Aut}(hi_B)$ given by $\alpha \mapsto (\alpha i_A, \alpha i_B)$ is an isomorphism.

Thus although for $f\colon A \to X$, $g\colon B \to X$, the map $\binom{f}{g}\colon A \vee B \to X$ is not uniquely defined, any two candidates are isomorphic – in particular they have isomorphic automorphism groups. We will assume that one such is chosen; as usual, this implies functoriality of coproducts. So, for $f\colon A \to X$, $g\colon B \to Y$, the map $\binom{fi_X}{gi_Y}\colon A \vee B \to X \vee Y$ will be denoted $f \vee g$; and this also extends to group homomorphisms $\mathrm{Aut}(f) \times \mathrm{Aut}(g) \to \mathrm{Aut}(f \vee g)$ similarly given by

$$(\alpha, \beta) \mapsto \alpha \vee \beta = \begin{pmatrix} \alpha i_X \\ \beta i_Y \end{pmatrix}.$$

Similarly, an initial object $*$ of $\mathcal{T}$ is *strong* if $[\![*, X]\!]$ is a groupoid with a unique morphism. Whereas a *weak* initial object is one for which $[\![*, X]\!]$ is *equivalent* to such a groupoid, i.e. for any maps $f, f'\colon * \to X$ there is a unique track $f \simeq f'$.

Remark 4.2. Sums as above are also termed coproducts in a track category $\mathcal{T}$. In a dual way one defines as usual *products* $A \times B$ in $\mathcal{T}$ in the strong sense and the weak sense, respectively. Also a *final object* 1 in $\mathcal{T}$ in the strong (resp. weak) sense is defined dually to the corresponding initial object above. The results below formulated for coproducts in $\mathcal{T}$ have an obvious dualization for products in $\mathcal{T}$.

DEFINITION 4.3. We say that a category $\mathbf{T}$ is a *theory* if $\mathbf{T}$ has an initial object and finite sums, denoted by $A \vee B$. Similarly we say that a track category $\mathcal{T}$ is a *track theory* if weak sums exist in $\mathcal{T}$. This is an *Abelian* track theory if $\mathcal{T}$ is Abelian; i.e. if all hom-groupoids of $\mathcal{T}$ are Abelian.

One readily checks that the homotopy category $\mathbf{T} = \mathcal{T}_{\simeq}$ of a track theory $\mathcal{T}$ is a theory. We now want to classify track theories with Abelian hom-groupoids. For this we recall the following notion; compare [SN] and [CT].

DEFINITION 4.4. A natural system D on a category $\mathbf{T}$ is said to be *compatible with sums* if for any sum diagram $i_k\colon X_k \to X_1 \vee \ldots \vee X_n$, $k = 1, \ldots, n$, and any morphism $f\colon X_1 \vee \ldots \vee X_n \to Y$ the homomorphism

$$D_f \xrightarrow{\xi \mapsto (\xi i_1, \ldots, \xi i_n)} D_{fi_1} \times \cdots \times D_{fi_n}$$

is an isomorphism.

PROPOSITION 4.5. *For any Abelian track theory $\mathcal{T}$ the associated natural system D in* (2.4) *is compatible with sums.*

Proof. Obviously the quotient functor $\mathcal{T} \to \mathcal{T}_{\simeq}$ preserves sums. Now given any $f\colon X_1 \vee \ldots \vee X_n \to Y$ consider the diagram

$$\begin{array}{ccc} D_{[f]} & \xrightarrow{\xi \mapsto (\xi i_1, \ldots, \xi i_n)} & D_{[fi_1]} \times \cdots \times D_{[fi_n]} \\ \cong \big\downarrow \sigma_f & & \cong \big\downarrow \sigma_{fi_1} \times \cdots \times \sigma_{fi_n} \\ \mathrm{Aut}(f) & \xrightarrow{\cong} & \mathrm{Aut}(fi_1) \times \cdots \times \mathrm{Aut}(fi_n), \end{array}$$

where the lower horizontal isomorphism is obtained from the equivalence of groupoids

$$[\![X_1 \vee \ldots \vee X_n, Y]\!] \xrightarrow{([\![i_1, Y]\!], \ldots, [\![i_n, Y]\!])} [\![X_1, Y]\!] \times \cdots \times [\![X_n, Y]\!]. \qquad \square$$

The converse of (4.5) also holds:

PROPOSITION 4.6. *Let $\mathcal{T}$ be an Abelian track category and let D be the associated natural system on the homotopy category $\mathcal{T}_{\simeq}$. If $\mathcal{T}_{\simeq}$ is a theory and D is compatible with sums then $\mathcal{T}$ is an Abelian track theory.*

Proof. We can suppose given canonical sum diagrams $i_k\colon X_k \to X_1 \vee \ldots \vee X_n$, $k = 1, \ldots, n$, $n \geqslant 0$, for each finite family $(X_1, \ldots, X_n)$ of objects of $\mathcal{T}_{\simeq}$. Choose arbitrarily an 1-arrow i'_k in $\mathcal{T}$ from each isomorphism class i_k (it will also depend on all of the $i_1, \dots, i_n$ in the family). Then for any such family and any other X, the functors

$$[\![X_1 \vee \ldots \vee X_n, X]\!] \xrightarrow{([\![i'_1, X]\!], \dots, [\![i'_n, X]\!])} [\![X_1, X]\!] \times \cdots \times [\![X_n, X]\!]$$

induce bijections on isomorphism classes of objects since $\mathcal{T}_{\simeq}$ is a theory; and since the natural system D is compatible with sums, they also induce bijections of automorphism groups of objects. Hence, they are equivalences. □

DEFINITION 4.7. A *weak theory-equivalence* between track theories is a weak equivalence between track categories which preserves weak sums; see (1.5). Accordingly a *theory-equivalence* between theories is an equivalence of categories which preserves sums. Two classifying triples $(\mathbf{T}, D, \tau)$ and $(\mathbf{T}', D', \tau')$ for which $\mathbf{T}$ and $\mathbf{T}'$ are theories and D and D' are compatible with sums are *theory-equivalent* if there exist F and I as in (3.2) where F is a theory-equivalence and I is compatible with sums.

Using (4.5) and (4.6) the classification in (3.3) yields the next result.

CLASSIFICATION 4.8. *There is a one-to-one correspondence between weak theory-equivalence classes of Abelian track theories and theory-equivalence classes of classifying triples* $(\mathbf{T}, D, \tau)$ *for which* $\mathbf{T}$ *is a theory and* D *is compatible with sums.* □

Just as theories are used to describe various algebraic structures, track theories determine structures in track categories that satisfy identities up to specified tracks.

DEFINITION 4.9. A *model* of a theory $\mathbf{T}$ in a category $\mathbf{C}$ is a contravariant functor $\mathbf{T}^{op} \to \mathbf{C}$ carrying sums to products. The category of all such functors and their natural transformations is denoted $\mathbf{T}$-$\mathbf{mod}(\mathbf{C})$, or, if $\mathbf{C}$ is the category of sets, simply $\mathbf{T}$-$\mathbf{mod}$.

Similarly, for a track theory $\mathcal{T}$ a *model* of $\mathcal{T}$ in a track category $\mathcal{C}$ is a contravariant track functor from $\mathcal{T}$ to $\mathcal{C}$ which carries weak sums to weak products. Such track functors, track transformations between them, and 2-tracks between the transformations (see (1.5)) form a track category denoted $\mathcal{T}$-$\mathfrak{mod}(\mathcal{C})$, or just $\mathcal{T}$-$\mathfrak{mod}$ when $\mathcal{C} = \mathfrak{Gpd}$.

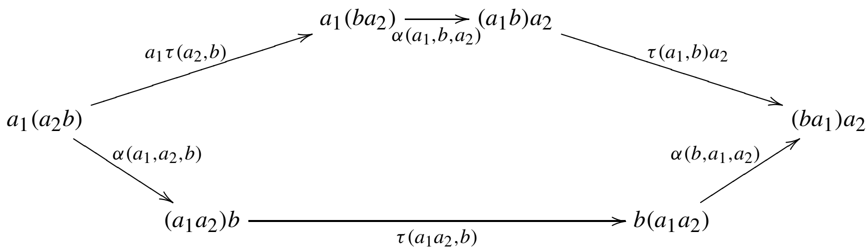
EXAMPLE 4.10. For any object X of a theory $\mathbf{T}$, the representable functor $\hom(-, X)$ from $\mathbf{T}^{op}$ to the category of sets carries sums to products, by the very definition of sums; hence, it is a model of $\mathbf{T}$. It is a consequence of the Yoneda lemma in category theory that in this way one can identify $\mathbf{T}$ with a full subcategory of the category of its models. The models from this subcategory are called *free*.

Similarly, for any object X of a track theory $\mathcal{T}$, the track functor $[\![-, X]\!]\colon \mathcal{T}^{op} \to \mathfrak{Gpd}$ is a model of $\mathcal{T}$, and such *free* models form a full track subcategory of $\mathcal{T}$-$\mathfrak{mod}$. There is an analog of the Yoneda lemma in enriched category theory (see [EC]); in particular for categories enriched in groupoids it gives a weak equivalence of $\mathcal{T}$ with the track category of free $\mathcal{T}$-models.

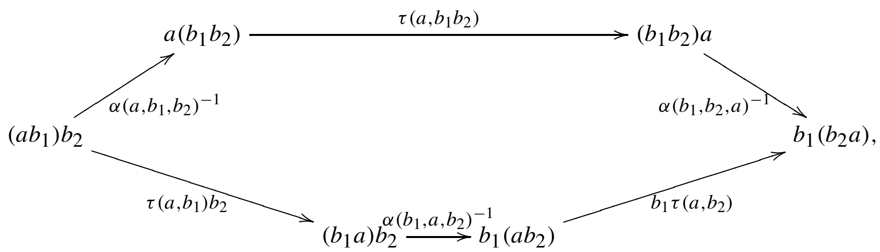
Take, for example, the category of groups. Its smallest subcategory containing the group of integers and closed under finite sums is a theory. Clearly, it is equivalent to the category of free finitely generated groups. It is well known that the category of models of this theory in any category **C** is equivalent to the category of internal groups in **C**. Similarly for Abelian groups, and, in fact, any kind of equational universal algebras.

For similar examples with track theories, we will need the corresponding notions.

DEFINITIONS 4.11. A monoidal category **G** (see [EC]) – with operation written as juxtaposition and the neutral object i – is termed a *bigroup* if it is a groupoid and, moreover, for any object a of **G** there is an object x and a morphism $i \to ax$ in **G**. A bigroup **G** is called *braided* if there are natural isomorphisms $\tau(a, b)\colon ab \to ba$ for all objects a, b in **G** which fit into commutative diagrams



and



where $\alpha(x, y, z)\colon x(yz) \to (xy)z$ are the associativity isomorphisms included in the monoidal structure of **G**. A braided bigroup is called *symmetric* if it is symmetric as a monoidal category, that is, $\tau(a, b)^{-1} = \tau(b, a)$ for any pair of objects. Bigroups, monoidal functors between them, and natural isomorphisms form a track category which we denote $\mathfrak{Big}$. It has track subcategories $\mathfrak{Sym} \subset \mathfrak{Br}$

whose objects are symmetric, resp. braided bigroups, with morphisms those monoidal functors which preserve the isomorphisms τ, and tracks – all isomorphisms between such functors.

References

[B] Baues, H.-J.: The algebra of secondary cohomology operations, Preprint.

[CH] Baues, H.-J.: *Combinatorial Homotopy and 4-Dimensional Complexes*, Expositions in Math. 2, De Gruyter, Berlin, 1991.

[GL] Baues, H.-J. and Dreckmann, W.: The cohomology of homotopy categories and the general linear group, *K-Theory* **3**(4) (1989), 307–338.

[SN] Baues, H.-J. and Tonks, A.: On sum-normalized cohomology of categories, twisted homotopy pairs and universal Toda brackets, *Quart. J. Math. Oxford Ser. (2)* **47**(188) (1996) 405–433.

[HT] Gray, B.: *Homotopy Theory*, Academic Press, New York, 1975.

[FC] Gray, J. W.: *Formal Category Theory: Adjointness for 2-Categories*, Lecture Notes in Math. 391, Springer-Verlag, New York, 1974.

[BC] Jibladze, M.: Bicategorical interpretation of the third cohomology group, *Trudy Tbiliss. Mat. Inst. Razmadze Akad. Nauk Gruzii* **97**(3) (1992), 3–9.

[CT] Jibladze, M. and Pirashvili, T.: Cohomology of algebraic theories, *J. Algebra* **137**(2) 1991, 253–296.

[EC] Kelly, G. M.: *Basic Concepts of Enriched Category Theory*, London Math. Soc. Lecture Note Ser. 64, Cambridge Univ. Press, 1982.

[MC] Pirashvili, T.: Models for the homotopy theory and cohomology of small categories, *Soobshch. Akad. Nauk Gruz. SSR* **129**(2) (1988), 261–264.

[MH] Pirashvili, T.: H^3 and models for the homotopy theory, *Trudy Tbiliss. Mat. Inst. Razmadze Akad. Nauk Gruzii* **94** (1991), 73–85.

[CA] Pirashvili, T.: Cohomology of small categories in homotopical algebra, In: *K-Theory and Homological Algebra* (*Tbilisi, 1987–88*), Lecture Notes in Math. 1437, Springer-Verlag, Berlin, 1990, pp. 268–302.