An Axiomatization of the d-logic of Planar Polygons

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Abstract. We introduce the modal logic of planar polygonal subsets of the plane, with the modality interpreted as the Cantor-Bendixson derivative operator. We prove the finite model property of this logic and provide a finite axiomatization for it.

1 Introduction

There is a separate direction in modal logic, dealing with the specific phenomena related to logical reasoning about various objects of geometric nature with the aid of modal operators. In the literature there are quite a few alternative approaches to the way one interprets modalities in the context of space, shape, dimension, contiguity, etc. When applying logical calculi to reason about planar or spatial regions one often chooses some particular properties one wants to express, and restricts the kind of regions considered to those for which it makes sense to ask whether they have these properties. To point to an example, in a certain context one may find useful to investigate mereological relationships between *regular* subsets of a space—roughly, those which feature "filled up" areas without cracks, hairs or punctures. There also are many other, entirely different approaches. Let us limit ourselves to naming some sources—[1,2,9,10]; Let us also mention a recent paper [4], whose approach is most similar in spirit to ours.

In a recent paper [8] we introduced one more version of such an approach: instead of restricting topologically invariant properties of regions, we have severely restricted their shapes, namely we considered *polygonal regions*—subsets of the plane obtainable as boolean combinations of (either open or closed) halfplanes—equivalently, these are subsets that may be determined by (either strict or non-strict) linear inequalities. In that paper we interpreted modalities as topological closure/interior operators acting on such polygonal regions.

In this paper, our target is another interpretation of modal operators frequently studied in context of topological semantics. Namely, here we interpret $\Diamond$

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as the *Cantor-Bendixson derivative* operator. For a polygonal region this roughly means to take its closure but also throw out any isolated points that the region might have.

We thus obtain a certain modal logic $\mathbf{PL}_2^d$: all formulæ that hold true under this interpretation about arbitrary polygonal subsets in the Euclidean plane. Algebraically, we study the variety of modal algebras generated by the Boolean algebra P_2 of polygonal subsets of the plane equipped with the derivative operator $\Diamond$.

We are going to prove that the logic $\mathbf{PL}_2^d$ has the finite model property, and provide five axioms that axiomatize it.

Our main approach is to employ a link to Kripke semantics with the aid of certain maps, called (partial, polygonal) *d-morphisms*, from the plane to various finite Kripke frames. Essentially these are exactly the maps that preserve validity of modal formulas. Algebraically, a d-morphism f is a map with the property that the induced Boolean algebra homomorphism f^{-1} from the powerset of the frame to the powerset of the plane (a) lands in the subalgebra consisting of polygonal subsets and (b) is a modal homomorphism with respect to the standard ("R^{-1}") interpretation of the modality $\Diamond$ on the Kripke side, and its above derivative interpretation on the polygonal side.

Using such morphisms helps us in applying a mixture of geometric and relational intuitions to find various finite modal algebras among subquotient algebras of P_2, or, on the contrary, prove that some other finite modal algebras cannot occur as such subquotients. Specifically, we introduce a sequence of finite Kripke frames, called *ir-crown frames*, well suited to be "test objects" for P_2—namely, any point in any polygonal configuration on the plane admits a (partial) polygonal d-morphism, in the above sense, onto some ir-crown frame. This allows us to prove that $\mathbf{PL}_2^d$ is the logic of all ir-crown frames, which gives as a result the finite model property for it.

On the other hand, we gather a sufficient but finite supply of certain other "forbidden" finite Kripke frames—those not admitting any partial polygonal d-morphisms onto them from the plane. Then using techniques similar to that of Jankov-De Jongh formulas we manage to express the fact of "forbiddenness" of these frames in the modal language, thus providing an axiom system for $\mathbf{PL}_2^d$.

2 Preliminaries

2.1 Syntax and Semantics

Our aim is to set up the reasoning paradigm where we use the basic modal language as our formalism and interpret its formulas as polyhedra in the Euclidean space of fixed dimension, while interpreting the modal diamond as the topological derivative operator and the boolean connectives as their set-theoretic counterparts. In this section we provide the necessary definitions of the relevant notions.

Syntax. We consider the basic modal language $\mathcal{ML}$. The alphabet of $\mathcal{ML}$ consists of a countable set PROP of letters for propositional variables and the symbols $\bot$, $\vee$, $\neg$, $\Diamond$. Formulas of $\mathcal{ML}$ are given by the recursive definition:

$$\varphi ::= \bot \mid \ p \mid \neg\varphi \mid \varphi \vee \psi \mid \Diamond\varphi.$$

The other connectives, such as $\wedge$, $\to$, $\leftrightarrow$, $\top$ and $\Box$, will be used as standard shorthand notations. A *normal modal logic* Λ in the modal language $\mathcal{ML}$ is a set of formulas of $\mathcal{ML}$ that contains all propositional tautologies, the formula $\Box(p \to q) \to (\Box p \to \Box q)$, and is closed under the inference rules of *modus ponens* (i.e. if $\varphi \in \Lambda$ and $\varphi \to \psi \in \Lambda$, then $\psi \in \Lambda$), *uniform substitution* (i.e if φ is in Λ, then so are all of its substitution instances) and *necessitation* (if $\varphi \in \Lambda$, then $\Box\varphi \in \Lambda$). All the logics we consider will be normal modal logics.

Kripke Semantics. We recall the basic notions from the Kripke semantics of modal logic.

A *Kripke frame* $\mathfrak{F}$ consists of a nonempty set W together with a binary relation $R \subseteq W \times W$. Such a pair is denoted by $\mathfrak{F} = (W, R)$ with the set W called the *underlying set* of the frame, and the relation R called the *accessibility relation* on W. To indicate that $(x, y) \in R$ holds we often write xRy (and say that x sees y by R); in such a case the element y is called a *successor* of the element x, and x a *predecessor* of y.

A relation is said to be *transitive* if for any three points $x, y, z \in W$, whenever xRy and yRz, then xRz holds as well. All frames we will consider are transitive.

We say that a point $r \in W$ is a *root* of a transitive Kripke frame $\mathfrak{F} = (W, R)$ if any point y of W distinct from r is a successor of r. A Kripke frame is said to be *rooted* if it has a root. A point in a Kripke frame is said to be *irreflexive* if $(x, x) \notin R$.

If R is a relation on W, and $A \subseteq W$, then the set $\{y \in W \mid \exists\, x \in A(xRy)\}$ of all the successors of elements of A is denoted by $R(A)$; the set $\{y \in W \mid \exists\, x \in A(yRx)\}$ of all the predecessors of elements of A is denoted by $R^{-1}(A)$.

A subset $U \subseteq W$ is called *upwards closed* (or simply an *up-set*) if it contains all successors of all of its elements, i.e. if $R(U) \subseteq U$ holds. Dually, a *down-set* or a *downwards closed* set D is defined as a set containing all predecessors of its elements, i.e. satisfying $R^{-1}(D) \subseteq D$. It is easy to see that the complement of an up-set is a down-set and vice versa.

A subset $A \subseteq W$ is called a *cluster* iff for any distinct $w, v \in A$ both wRv and vRw hold. In words, any two distinct points of a cluster are successors of each other.

Note that with any accessibility relation R on a Kripke frame $\mathfrak{F}$ we can associate a partially ordered set $S_{\mathfrak{F}}$ with elements the equivalence classes with respect to the equivalence relation $\sim_R$ defined by

$$x \sim_R y \ \Leftrightarrow\ xR^*y \text{ and } yR^*x,$$

where R^* is the transitive-reflexive closure of R. On these equivalence classes we define the partial order $\leq_R$ via

$$[x] \leq_R [y] \ \Leftrightarrow\ xR^*y.$$

A partial order S has *finite height* h if the maximum of the cardinalities of chains in S is equal to h. Then the *height of a Kripke frame* $\mathfrak{F}$ is the height of the partial order $S_{\mathfrak{F}}$ corresponding to $\mathfrak{F}$ as above.

Kripke frames provide semantics for modal logic in the following way. A *valuation* of propositional letters on a Kripke frame $\mathfrak{F} = (W, R)$ is a map $\nu : \textsc{Prop} \to \mathcal{P}(W)$ assigning a subset of W to each propositional letter. Such valuation is then extended to the valuation of all well-formed formulas of the language $\mathcal{ML}$,

$$\begin{array}{lll} x \not\models \bot & & \forall x \in W; \\ x \models p & \text{iff} & x \in \nu(p); \\ x \models \neg\varphi & \text{iff} & x \not\models \varphi; \\ x \models \varphi \vee \psi & \text{iff} & x \models \varphi \text{ or } x \models \psi; \\ x \models \Diamond\varphi & \text{iff} & x \in R^{-1}(\nu(\varphi)). \end{array}$$

The pair $\mathcal{M} = (\mathfrak{F}, \nu)$ is called a *Kripke model*, where $\mathfrak{F} = (W, R)$ is a Kripke frame and ν is a valuation as above. We write $\mathcal{M}, x \models \phi$ if a formula φ holds at the point x of a model $\mathcal{M}$.

For a subset $A \subseteq$ we write $\mathcal{M}, A \models \phi$ if $\mathcal{M}, x \models \phi$ holds for all $x \in A$. Further, $\mathcal{M} \models \phi$ (ϕ is valid in $\mathcal{M}$) means that $\mathcal{M}, x \models \phi$ for all $x \in W$. We write $\mathfrak{F} \models \phi$ (ϕ is valid on $\mathfrak{F}$) whenever $(\mathfrak{F}, \nu) \models \phi$ for an arbitrary valuation ν on $\mathfrak{F}$. If $\mathbf{K}$ is a class of Kripke frames we write $\mathbf{K} \models \phi$ when $\mathfrak{F} \models \phi$ for each $\mathfrak{F} \in \mathbf{K}$. By $\mathsf{Log}(\mathbf{K})$ we denote the set of all modal formulas valid in all members $\mathfrak{F} \in \mathbf{K}$ of $\mathbf{K}$. It is a basic fact of Kripke semantics for modal logic that $\mathsf{Log}(\mathbf{K})$ is always a normal modal logic.

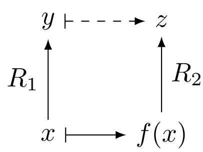
Certain operations on frames preserving the modal validity will be useful in our considerations. Notably those of taking a generated subframe of a frame, taking a p-morphic image of a frame and the combination of these two called *an up-reduction* of one frame to another. We proceed to define these.

Definition 1. *Let $\mathfrak{F}_1 = (W_1, R_1)$ and $\mathfrak{F}_2 = (W_2, R_2)$ be Kripke frames. We say that $\mathfrak{F}_1$ is a* subframe *of $\mathfrak{F}_2$ if $W_1 \subseteq W_2$ and $R_1 = R_2 \cap (W_1 \times W_1)$. If in addition W_1 is an up-set in W_2, then we say that W_1 is a* generated subframe *of W_2.*

Proposition 1. *Let $\mathfrak{F}_1$ be a generated subframe of $\mathfrak{F}_2$. Then $\mathfrak{F}_2 \models \phi$ implies $\mathfrak{F}_1 \models \phi$ for any modal formula ϕ.*

The proof can be seen in e.g. [5].

Let $\mathfrak{F}_1 = (W_1, R_1)$ and $\mathfrak{F}_2 = (W_2, R_2)$ be Kripke frames. A map $f : \mathfrak{F}_1 \to \mathfrak{F}_2$ is said to be *monotone* if whenever $(x, y) \in R_1$, then $\big(f(x), f(y)\big) \in R_2$. A map $f : \mathfrak{F}_1 \to \mathfrak{F}_2$ between Kripke frames is said to be a *p-morphism* if it is monotone, and whenever $\big(f(x), z\big) \in R_2$, there exists $y \in W_1$ such that $(x, y) \in R_1$ and $f(y) = z$.



The onto p-morphisms also preserve validity of formulas, i.e. the following holds (see [5]):

Proposition 2. *Let $\mathfrak{F}_1$ and $\mathfrak{F}_2$ be Kripke frames and $f : \mathfrak{F}_1 \to \mathfrak{F}_2$ be an onto p-morphism. Then $\mathfrak{F}_1 \models \phi$ implies $\mathfrak{F}_2 \models \phi$ for any modal formula ϕ.*

Hence, $\mathsf{Log}(\mathfrak{F}_1) \subseteq \mathsf{Log}(\mathfrak{F}_2)$ whenever $\mathfrak{F}_2$ is a p-morphic image of $\mathfrak{F}_1$.

Combining the above two constructions, we say that $\mathfrak{F}_2$ is an *up-reduction*[1] of $\mathfrak{F}_1$, if there exists a generated subframe (i.e. an up-set) $\mathfrak{G}_1$ of $\mathfrak{F}_1$, such that $\mathfrak{F}_2$ is a p-morphic image of $\mathfrak{G}_1$. We immediately infer:

Proposition 3. *Let $\mathfrak{F}_1$ and $\mathfrak{F}_2$ be Kripke frames and let $\mathfrak{F}_2$ be an up-reduction of $\mathfrak{F}_1$. Then $\mathfrak{F}_1 \models \phi$ implies $\mathfrak{F}_2 \models \phi$ for any modal formula ϕ.*

Topological Semantics. There are two standard topological semantics for modal logic, depending on whether the modal diamond is interpreted as the closure operator or as the derivative operator. In this paper we focus on the latter. For the corresponding definitions in the closure semantics see [8].

A *topological space* is a pair $\mathfrak{X} = (X, \tau)$, where $X \neq \emptyset$ is a set and $\tau \subseteq \mathcal{P}(X)$ is a family of subsets of X, such that $\emptyset \in \tau$, $X \in \tau$, and τ is closed under finite intersections and arbitrary unions. X is called the underlying set of $\mathfrak{X}$, and τ a topology on X. When there is no danger of ambiguity, we often write just X instead of (X, τ). The elements of τ are called *open* subsets of $\mathfrak{X}$, or simply opens. Set-theoretic complements of opens are called *closed* subsets. Clearly finite unions and arbitrary intersections of closed sets are closed. For a set $A \subseteq X$ the *closure of* A is the intersection of all closed sets containing A,

$$\mathbb{C}(A) = \bigcap \{F \mid A \subseteq F \subseteq X \text{ and } F \text{ is closed}\}.$$

A point x of a topological space $\mathfrak{X}$ is called a *limit point* of a subset $A \subseteq X$ if $x \in \mathbb{C}(A - \{x\})$. The set of all limit points of A is called the *derived set* of A and is denoted by $d(A)$. It is easy to see that $x \in d(A)$ if and only if for any open neighborhood U of x the intersection $U \cap A$ contains at least one point distinct from x, i.e. $U \cap A - \{x\} \neq \emptyset$ (see e.g. [7]). The derived set operator has the following properties:

(i) $\mathbb{C}(A) = A \cup d(A)$;
(ii) If $A \subseteq B$, then $d(A) \subseteq d(B)$;
(iii) $d(A \cup B) = d(A) \cup d(B)$;
(iv) $\bigcup_{i \in I} d(A_i) \subseteq d(\bigcup_{i \in I} A_i)$.

Elements of the set $A - d(A)$ are called *isolated points* of A. A set $A \subseteq X$ is called *dense in itself* (dii for short) if $A \subseteq d(A)$. Thus a subset is dii iff it has no isolated points.

[1] We introduce this terminology extending the terminology of [6] where *reduction* means taking a p-morphic image and *subreduction* means taking a p-morphic image of a subframe of the frame. Thus, *up-reductions* are special cases of subreduction, where the subframe under question is an up-set.

At the outset, we slightly generalize the derivative semantics by the following definition.

Definition 2 (General derivative topological space). *A general derivative topological space is a tuple* $\mathfrak{X} = (X, \tau, d, \mathcal{D})$*, where* (X, τ) *is a topological space,* d *is the derived set operator on* (X, τ)*, and* $\mathcal{D} \subseteq \mathcal{P}(X)$*, such that* $X \in \mathcal{D}$ *and* $\mathcal{D}$ *is closed under the set-theoretic operations (e.g. taking unions and taking complements), as well as under the derivative operation. Subsets from* $\mathcal{D}$ *will be called the* admissible sets *for* $\mathfrak{X}$*.*

For further use we mention that a collection $\mathcal{D}$ of subsets in a topological space which is closed under the boolean set-theoretic operations and the derivative operation is said to be a *derivative algebra* of subsets. If the derivative algebra is not specified in a general derivative topological space, we assume that it is equal to the collection of all subsets.

Suppose $\mathfrak{X}$ is a general derivative topological space. A valuation on $\mathfrak{X}$ is a map $\nu : \textsc{Prop} \to \mathcal{D}(X)$. Each such valuation extends uniquely from propositional variables to all well-formed formulas of the language $\mathcal{ML}$ in the following way:

$$
\begin{array}{lll}
x \not\models \bot & & \forall x \in X; \\
x \models p & \text{iff} & x \in \nu(p); \\
x \models \neg\varphi & \text{iff} & x \not\models \varphi; \\
x \models \varphi \vee \psi & \text{iff} & x \models \varphi \text{ or } x \models \psi; \\
x \models \Diamond\varphi & \text{iff} & x \in d(\nu(\varphi)).
\end{array}
$$

The notions of truth in a subset, validity in a model and validity on a space are defined like in the case of Kripke semantics. We just point out that according to the above definition, for a modal formula of the form $\varphi = \Diamond\psi$ we have that $\nu(\varphi) = \nu(\Diamond\psi) = d\nu(\psi)$, i.e. the truth set of $\Diamond\psi$ is the derivative of the truth-set of ψ.

If $\mathbf{K}$ is a class of general derivative spaces, by $\mathsf{Log}(\mathbf{K})$ we denote the set of all modal formulas valid in all members $\mathfrak{X} \in \mathbf{K}$. In case $\mathbf{K}$ consists of a single member $\mathfrak{X}$ we write $\mathsf{Log}(\mathfrak{X})$ to denote the modal logic of $\mathfrak{X}$. It is well-known that the modal logic of T_1 topological spaces (in which each singleton subspace is closed) is the modal logic $K4$, the logic of all transitive Kripke frames. We will only be dealing with T_1 spaces in this paper.

To connect the two semantics described above we need a definition of maps which preserve the validity of modal formulas when the domain of a map is a topological space and the target a Kripke frame (see [3]).

Definition 3. *A map* $f : \mathfrak{X} \to \mathfrak{F}$ *where* $\mathfrak{X} = (X, \tau)$ *is a topological space and* $\mathfrak{F} = (W, R)$ *is a transitive Kripke frame, is called a d-morphism if the following properties are satisfied:*

(i) For each open $U \in \tau$ *it holds that* $R(f(U)) \subseteq f(U)$*, i.e. images of opens are up-sets;*

(ii) For each $V \subseteq W$ such that $R(V) \subseteq V$, it holds that $f^{-1}(V) \in \tau$, i.e. the pre-images of up-sets are open;
(iii) For each irreflexive point $w \in W$ it holds that $f^{-1}(w)$ is a discrete space w.r.t. subspace topology;
(iv) For each reflexive point $w \in W$ it holds that $f^{-1}(w) \subseteq d(f^{-1}(w))$, i.e. $f^{-1}(w)$ is dense in itself (dii).

To synchronize the terminology, notice that the up-sets of a transitive Kripke frame always form a topology. Thus we may occasionally refer to up-sets as open sets and down-sets as closed sets. In this terminology, the first two conditions of a d-morphism amount to the map being *open* and *continuous* (such maps are often called *interior maps*).

It is well-known that d-morphisms preserve modal validity [3]. It is also known that taking open subspaces preserves modal validity in derivative semantics. These facts motivate the following definition, which is similar to that of up-reduction for frames, and serves as a bridge between the Kripke semantics and the derivative semantics.

Let $f : \mathfrak{X} \twoheadrightarrow \mathfrak{F}$ denote a *partial* map from the topological space $\mathfrak{X}$ to the Kripke frame $\mathfrak{F}$. In case the domain of f is an open subset of $\mathfrak{X}$, and f satisfies the conditions of being a d-morphism, we say that f is a *partial d-morphism*.

Proposition 4. *Let $\mathfrak{X}$ be a topological space, $\mathfrak{F}$ be a Kripke frame and $f : \mathfrak{X} \twoheadrightarrow \mathfrak{F}$ be a partial onto d-morphism. Then for an arbitrary modal formula ϕ we have $\mathfrak{F} \models \phi$ whenever $\mathfrak{X} \models \phi$.*

2.2 The Polygonal Plane

We are interested in specific general models defined over Euclidean spaces $\mathbb{R}^n$. Let P_n be the boolean algebra of the n-dimensional *polyhedra* in $\mathbb{R}^n$. *Basic*, or *elementary* polyhedra can be described as sets that are intersections of finitely many halfspaces in $\mathbb{R}^n$, and are known to be those polyhedra which are convex subsets of $\mathbb{R}^n$. General polyhedra are then the finite unions of the latter.

To be more precise, a *polyhedron* is any subset of $\mathbb{R}^n$ of the form $P = \{\bar{x} \mid \bigvee \bigwedge(\ell_i(\bar{x}) \bowtie a_i)\}$ where ℓ_i are linear forms on $\mathbb{R}^n$ and a_i are real numbers, $\bowtie$ denotes any of the inequality symbols $\geq, >, \leq, <$, while $\bigvee$ and $\bigwedge$ denote finite disjunction and finite conjunction. The sets of the form $P = \{\bar{x} \mid \bigwedge(\ell_i(\bar{x}) \bowtie a_i)\}$ we call *basic* or *elementary* polyhedra.

Then it is clear from definitions that the set P_n of all the n-dimensional polyhedra forms a boolean subalgebra of the powerset $\mathscr{P}(\mathbb{R}^n)$ (note that the negation of an inequality is again an inequality). Moreover, the following holds:

Proposition 5. *The boolean algebra P_n is a modal subalgebra of the powerset derivative algebra $\mathscr{P}(\mathbb{R}^n)$ equipped with the derivative operator for the Euclidean topology on $\mathbb{R}^n$.*

Proof. To show that P_n is closed under the derivative operator, first note that the derivative operator distributes over finite unions. Since every polyhedron is a finite union of basic polyhedra, it suffices to point out that given a nonempty basic polyhedron P defined by a finite conjunction of inequalities, its derivative is either the closed basic polyhedron obtained by turning all strict inequalities into non-strict ones, or is the empty set in case P happens to consist of a single point.

Definition 4. *Let $\mathfrak{P}_n = (\mathbb{R}^n, P_n, d)$ be the general derivative space defined by means of the derivative algebra of polyhedra in the Euclidean space $\mathbb{R}^n$. We call such a space the* n-dimensional Euclidean polyhedral derivative space*. The modal logic $\mathbf{PL}_n^d$ of the n-dimensional Euclidean polyhedral derivative space is defined to be the set of all modal formulas which are valid on $\mathfrak{P}_\mathfrak{n}$.*

$$\mathbf{PL}_n^d ::= \mathsf{Log}^{\mathsf{d}}(\mathfrak{P}_n)$$

In this paper we concentrate on the 2-dimensional polyhedral modal logic $\mathbf{PL}_2^d$. We call this logic the polygonal modal logic for simplicity and the corresponding general space $\mathfrak{P}_2 = (\mathbb{R}^2, P_2, d)$ is called the *derivative polygonal plane*.

The admissible sets in $\mathfrak{P}_2$ are finite unions of *generalized planar polygons*, where under a generalized planar polygon we understand a (possibly unbounded) region in the plane which is an intersection of finitely many (closed or open) half-planes. It is clear that any point, line, ray or segment also falls under this definition, as do triangles, pentagons and n-gons in general.

We consider a class **IC** of finite frames which are called *crown frames with irreflexive root*, ir-crown frames for short (not to be confused with the crown graphs); this class will play crucial role of finite models for $\mathbf{PL}_2^d$ as we will see later.

Definition 5. *A* crown frame with irreflexive root*, or* ir-crown frame *is a frame $\mathfrak{S}_n = (S_n, Q_n)$ such that $S_n = \{r, s_1, \cdots, s_{2n}\}$ and Q_n is defined as follows:*

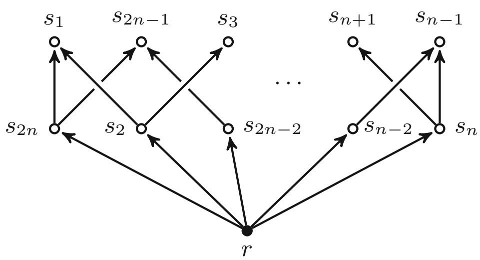


Fig. 1. An ir-crown frame. Here and in further pictures we depict irreflexive points by black circles and the reflexive ones by circles with white interior.

$(r, r) \notin Q_n$;
$(r, s_i) \in Q_n$ *for all* $s_i \in S_n$;
$(s_i, s_i) \in Q_n$ *for all* $s_i \in S_n$;
$(s_i, s_j) \in Q_n$ *when* $i < 2n$ *is even and* $j = i - 1, i + 1$;
$(s_{2n}, s_1) \in Q_n$;
$(s_{2n}, s_{2n-1}) \in Q_n$.

We are going to show that the modal logic of ir-crown frames and the modal logic of derivative polygonal plane coincide. To show that the formulas valid on the derivative polygonal plane are valid on ir-crown frames as well, first we prove the following theorem:

Theorem 1. *For an arbitrary ir-crown frame* $\mathfrak{F} = (W, R)$ *there is an onto d-morphism* $f : \mathfrak{P}_2 \to \mathfrak{F}$ *from the derivative polygonal plane to* $\mathfrak{F}$ *such that for any point* $w \in W$ *the inverse image* $f^{-1}(w)$ *belongs to* P_2, *i.e. is an element of the derivative algebra* P_2 *of planar polygons.*

Proof. Let $\mathfrak{F} = (W, R)$ be an ir-crown frame. According to the construction of ir-crown frames, $\mathfrak{F}$ is a finite frame with the equal numbers of points on the second (middle) and the third (maximum) layer; let us denote by n the number of maximal points of $\mathfrak{F}$. The construction for obtaining an onto mapping from the polygonal plane to a crown frame introduced in Proposition 3.2 of [8] works here too. Namely, consider any point x and arbitrary distinct rays $l_1, \ldots, l_n$ emanating from x and enumerated in the counter-clockwise direction (Fig. 2).

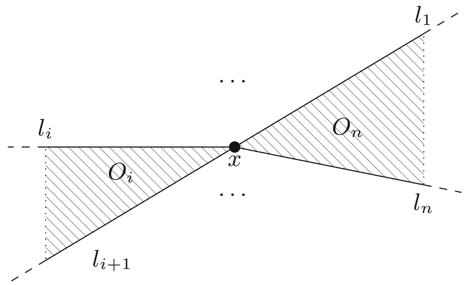


Fig. 2. Polygonal partition of the Euclidean plane corresponding to a d-morphism onto an ir-crown frame.

For $i = 1, ..., n-1$ let O_i be the open regions between l_i and l_{i+1}, and let O_n be the open region between l_n and l_1.
Define the map $f : \mathfrak{P}_2 \to \mathfrak{F}$ by putting $f(x) = r$, $f(l_i) = s_{2i}$ and $f(O_i) = s_{2i-1}$ where r is the irreflexive root of the ir-crown frame while s_{2i} and s_{2i-1} are as in Fig. 1. We only need to check that this f satisfies conditions from Definition 3. Conditions (i) and (ii) are easily checked. By the definition of ir-crown frames, the only irreflexive point is the root. Hence we see that condition (iii) holds as well since the preimage of r is the single point x which obviously is a discrete subspace of $\mathbb{R}^2$. Now since by construction of f for any $i \leq n$ the pre-image $f^{-1}(s_i)$ has no isolated points, the condition (iv) holds as well.

Thus by Proposition 4, if $\mathfrak{F}$ is an ir-crown frame with root r and ϕ is a modal formula, it holds that $\mathfrak{F}, r \models \phi$ whenever $\mathfrak{P}_2, x \models \phi$ for some point $x \in \mathbb{R}^2$. Hence $\mathbf{PL}_2^d \subseteq \mathsf{Log}(\mathbf{IC})$.

Let us now prove the converse inclusion.

Theorem 2. *If a formula φ is satisfiable on the derivative polygonal plane $\mathfrak{P}_2$ then it is satisfiable on some ir-crown frame $\mathfrak{F}$.*

Proof. Let ν be a valuation and x be a point such that $\mathfrak{P}_2, \nu, x \models \phi$, where ϕ depends on propositional variables $p_1, \ldots, p_k$. Our strategy is to find a small enough open neighborhood U around x such that the partial d-morphism could be built from U onto one of the ir-crown frames, in such a way that the pre-images of points from the ir-crown frame have constant valuation for propositional variables occurring in ϕ. We follow the construction introduced in Theorem 3.1 of [8].

Suppose ϕ depends on propositional variables $p_1, \ldots, p_k$. It is clear that the truth of ϕ will not be affected if we assume that all the other propositional variables are mapped to the empty set. Let $A_i = \nu(p_i)$ for $i \in \{1, \ldots, k\}$. Then each A_i is a finite union of simple polygons (two-dimensional cases of basic, or elementary polyhedra, as described above towards the beginning of Sect. 2.2). Let S be the collection of all the simple polygons occurring in the A_is. Let E be the collection of all lines or line segments occurring as an edge of one of the simple polygons in S. It is obvious that E is finite. Furthermore, we observe the following:

Key observation: For any segment I on the plane, if the valuation of a propositional letter p_i changes along this segment, then I must intersect with a member of E, namely with the one that is represented as a border of A_i which I must cross in order to change valuation from one point to the other.

Now, for each line in E, calculate the distance from x to that line and to its endpoints (if it has such). This will produce a finite number of non-negative real numbers. Let α be the least *positive* number thus obtained and let $B = B(x, \frac{\alpha}{2})$ be the open ball of radius $\frac{\alpha}{2}$ centered at x. It is straightforward that only those lines from E that pass through x (or have it as an endpoint) will intersect with B. Let us label the intersection points of lines from E with the boundary of B

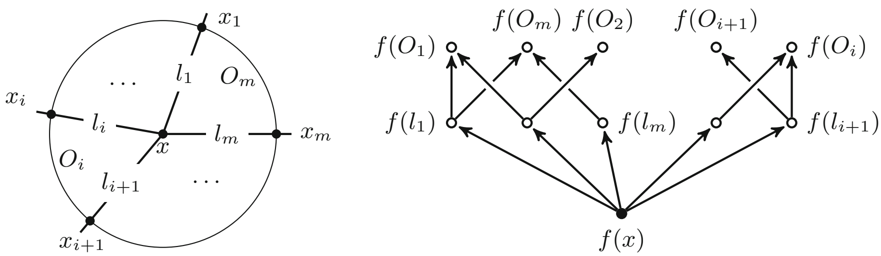


Fig. 3. The open n-gon and its partition corresponding to an ir-crown frame.

in the clockwise direction as $x_1, x_2, \ldots, x_m$, with $m \leq k$. Let l_i denote the open segments (x, x_i) and let O_i denote the open triangles $x_i x x_{i+1}$ inside B bounded by l_i and l_{i+1} for $i \in \{1, \ldots, m-1\}$. Let O_m be the remaining open triangle $x_1 x x_m$ confined between l_m and l_1. Then the open n-gon $x_1 \ldots x_m$ inside B breaks down into the sets $\{x\}$, l_i and O_i (Fig. 3).

The desired d-morphism is built in the same way as in Theorem 1. Then the valuation μ on F defined by putting $\mu(p) = f(\nu(p))$ for each p is such that

$$f(x) \in \mu(p) \text{ iff } x \in \nu(p);$$

and hence $\mathfrak{F}, \mu, f(x) \models \phi$.

Thus we have proved that the two logics $\mathbf{PL}_2^d$ and $\mathsf{Log}(\mathbf{IC})$ coincide.

Corollary 1. *The logic* $\mathbf{PL}_2^d$ *is determined by the class* $\mathbf{IC}$. *Hence this logic has the finite model property* (fmp).

3 Axiomatization

In this section we give a complete axiomatization of the logic $\mathsf{Log}(\mathbf{IC}) = \mathbf{PL}_2^d$.

Let Λ_2^d be the modal logic axiomatized by the following formulas:

$$\begin{aligned}
\theta_1 &= \Diamond\top \\
\theta_2 &= \Diamond\Diamond p \leftrightarrow \Diamond p \\
\theta_3 &= (\Diamond p \wedge \Diamond\neg p) \rightarrow \Diamond\big((p \wedge \Diamond\neg p) \vee (\neg p \wedge \Diamond p)\big) \\
\theta_4 &= \Box\big(p \rightarrow \Box(\neg p \rightarrow \Box\neg p)\big) \\
\theta_5 &= \big[r \wedge \gamma \wedge \Box(r \rightarrow \gamma)\big] \rightarrow \Diamond(\neg r \wedge \Diamond\Box p \wedge \Diamond\Box\neg p)
\end{aligned}$$

where γ is the formula $\Diamond\Box(p \wedge q) \wedge \Diamond\Box(p \wedge \neg q) \wedge \Diamond\Box(\neg p \wedge q)$.

In this section we aim to show that $\Lambda_2^d = \mathsf{Log}(\mathbf{IC}) = \mathbf{PL}_2^d$. To this end, we (a) associate a semantic condition to each of the axioms; (b) demonstrate that each of the five axioms is valid on any ir-crown frame; and (c) show that any rooted frame validating all of these axioms is an up-reduction of some ir-crown frame.

First we associate semantical conditions to each of these formulas. Some of them are well-known - e.g. validating θ_1 is equivalent to the frame being *serial*, i.e. each point having a successor, while validating θ_2 is equivalent to the frame being both *transitive* (successor of a successor is a successor) and *dense* (any successor is a successor of a successor).

In our description we will employ the notions of a subframe, generated subframe and *convex subframe*. We say that $\mathfrak{F} = (W, R)$ is a convex subframe of $\mathfrak{G} = (V, S)$, if $\mathfrak{F}$ is a subframe of $\mathfrak{G}$ and additionally, for arbitrary $w, u \in W$ and $v \in V$, if $wSvSu$ holds, then $v \in W$. In words, a convex subframe must contain whatever is "in between" its two points, akin to the geometric meaning of the term "convex". Clearly, generated subframes are always convex (Fig. 4).

The following picture lists some of the frames that will be useful in the next theorem. Here, as well as in subsequent pictures, hollow circles denote reflexive points, filled circles denote irreflexive points, while the symbol $\times$ is used to denote a point that is either reflexive or irreflexive, so in effect e.g. $\mathfrak{B}_3$ below is a generic name denoting any of the four distinct frames obtained from the corresponding picture by substituting reflexive or irreflexive points in place of $\times$:

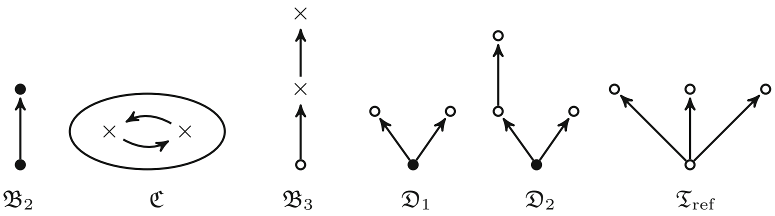


Fig. 4. Frames

Theorem 3. *Let $\mathfrak{F} = (W, R)$ be an arbitrary Kripke frame. Then:*

(i) $\mathfrak{F} \models \theta_1$ *iff* $\mathfrak{F}$ *is serial, i.e.* $\forall w \in W\ \exists v \in W(wRv)$*. Moreover,* $\mathfrak{F}$ *contains no generated subframe consisting of a single irreflexive point.*
(ii) $\mathfrak{F} \models \theta_2$ *iff* $\mathfrak{F}$ *is transitive and dense, i.e.* $\forall w, u, v \in W(wRv \wedge vRu \to wRv)$ *and* $\forall w, v \in W(wRv \to \exists u \in W(wRu \wedge uRv))$*. Moreover,* $\mathfrak{F} \models \Diamond p \to \Diamond\Diamond p$ *iff* $\mathfrak{F}$ *contains no convex subframe isomorphic to* $\mathfrak{B}_2$*.*
(iii) If $\mathfrak{F} \models \theta_3$ *then* $\mathfrak{F}$ *cannot be up-reduced to any of the frames* $\mathfrak{D}_1, \mathfrak{D}_2$*.*
(iv) If $\mathfrak{F} \models \theta_4$*, then* $\mathfrak{F}$ *is of height* ≤ 3 *and does not contain subframes isomorphic to either* $\mathfrak{C}$ *or* $\mathfrak{B}_3$*. If in addition* $\mathfrak{F}$ *is transitive, the converse holds as well.*
(v) If $\mathfrak{F} \models \theta_5$ *then* $\mathfrak{F}$ *has no generated subframes isomorphic to* $\mathfrak{T}_{\mathrm{ref}}$*.*

Proof. (i) This is well-known and easy to check. We note in addition that for finite transitive frames seriality means that all maximal points are reflexive.

(ii) That $\Diamond\Diamond p \to \Diamond p$ corresponds to transitivity and $\Diamond p \to \Diamond\Diamond p$ corresponds to density is well-known and easy to check. We only show the second part of the claim.

Suppose $\mathfrak{F}$ has a convex subframe consisting of two irreflexive points xRy as in $\mathfrak{B}_2$. Consider a valuation ν such that $\nu(p) = \{y\}$. It is obvious that $\mathfrak{F}, x \models \Diamond p$ and $\mathfrak{F}, x \not\models \Diamond\Diamond p$, hence $\mathfrak{F}, x \not\models \Diamond p \to \Diamond\Diamond p$.

To show the converse, suppose $\mathfrak{F} \not\models \Diamond p \to \Diamond\Diamond p$. Then there exists a point x of $\mathfrak{F}$ such that $\mathfrak{F}, x \models \Diamond p$ and $\mathfrak{F}, x \models \neg\Diamond\Diamond p$. Thus there exists a further point y with xRy and for any point z of $\mathfrak{F}$, either $(x, z) \notin R$ or $(z, y) \notin R$. It clearly follows that both x and y are irreflexive. Hence there is a copy of $\mathfrak{B}_2$ as a convex subframe of $\mathfrak{F}$.

(iii) It is immediate that θ_3 is refuted on the frames $\mathfrak{D}_1$ and $\mathfrak{D}_2$. Just take the valuation shown on the picture below (Fig. 5).

Since $\mathfrak{F}$ validates θ_3 by assumption, and validity is preserved by up-reductions and taking generated subframes, the claim follows.

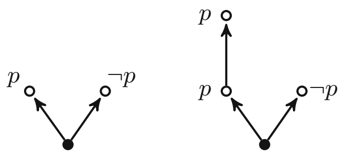


Fig. 5. Refutation of θ_3

(iv) Suppose $\mathfrak{F} \models \theta_4$. Assume $\mathfrak{F}$ contains $\mathfrak{C}$ as a subframe. The frame $\mathfrak{F}$ contains at least two points. Take any two points u and v from $\mathfrak{C}$. Take on it a valuation ν such that $\nu(p) = \{u\}$. Since u and v are interrelated, clearly $v \not\models \Box\neg p$, which means that $v \not\models \neg p \to \Box\neg p$. Hence $u \not\models \Box(\neg p \to \Box\neg p)$ and equally $u \not\models p \to \Box(\neg p \to \Box\neg p)$. The latter implies that $v \not\models \Box(p \to \Box(\neg p \to \Box\neg p))$ which contradicts the assumption. Now assume that $\mathfrak{F}$ contains $\mathfrak{B}_3$ as a subframe. The same reasoning as in the previous case goes through. Again let us show that $\Box\big(p \to \Box(\neg p \to \Box\neg p)\big)$ is refuted on $\mathfrak{F}$. Clearly $\mathfrak{F}$ contains at least three distinct points u, v and w with uRv and vRw, where R is the accessibility relation of $\mathfrak{F}$. Take a valuation ν such that $\nu(p) = \{u, w\}$. Then $v \not\models \neg p \to \Box\neg p$. Hence $u \not\models \Box(\neg p \to \Box\neg p)$ and $u \not\models p \to \Box(\neg p \to \Box\neg p)$. Since uRu we conclude that $u \not\models \Box(p \to \Box(\neg p \to \Box\neg p))$ which is again in contradiction with the assumption. Now in case $\mathfrak{F}$ contains a frame of strict height more than 3 as a subframe, which means that there are at least four distinct points u, v, w and z with uRv, vRw and wRz, then we choose a valuation V as follows: $V(p) = \{v, z\}$, and the formula is refuted at u. We therefore obtain that if $\mathfrak{F} \models \theta_4$ then $\mathfrak{F}$ is of height ≤ 3 and does not contain subframes isomorphic to $\mathfrak{C}$ and $\mathfrak{B}_3$. To show the converse, assume $\mathfrak{F}$ is transitive and suppose $\mathfrak{F} \not\models \theta_4$, i.e. there is a point x with $x \models \neg\Box\big(p \to \Box(\neg p \to \Box\neg p)\big)$. This is the same as $x \models \Diamond\big(p \wedge \Diamond(\neg p \wedge \Diamond p)\big)$, i.e. there exist points y, z, u with $xRyRzRu$ and $y \models p$, $z \models \neg p$, $u \models p$. To wrap up, if $y = u$ or uRx, then $\mathfrak{F}$ contains a copy of $\mathfrak{C}$ as a subframe; if $y \neq u$ and $\neg(uRx)$, then either $\mathfrak{F}$ contains subframes isomorphic to $\mathfrak{B}_3$ in case $x = y$, or is of height greater than 3 in case $x \neq y$.

(v) It is easy to see that the frame $\mathfrak{T}_{\text{ref}}$ refutes the formula θ_5. Indeed, consider the valuation on $\mathfrak{T}_{\text{ref}}$ as follows (Fig. 6):

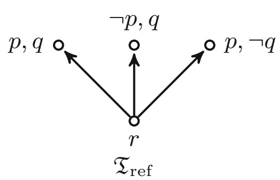


Fig. 6. Refutation of θ_5

Consider the root of $\mathfrak{T}_{\text{ref}}$. We see that r is true only at the root, which models γ as well. Hence the root models $(r \wedge \gamma \wedge \Box(r \to \gamma))$. It is also clear that the root refutes $\Diamond(\neg r \wedge \Diamond\Box p \wedge \Diamond\Box\neg p)$. Hence θ_5 is refuted at the root.

Since $\mathfrak{F}$ validates θ_5 by assumption, and validity is preserved by up-reductions and taking generated subframes, (v) follows.

Next we show that ir-crown frames validate all of the axioms $\theta_1 - \theta_5$.

Theorem 4. *Let $\mathfrak{S}_n = (S_n, Q_n)$ be an arbitrary ir-crown frame. Then for each $i \leq 5$ we have $\mathfrak{S}_n \models \theta_i$.*

Proof. We rely partially on the semantic characterizations afforded by Theorem 3. That ir-crown frames are serial (have reflexive maximum), transitive and dense is trivial to check. Thus $\mathfrak{S}_n \models \theta_1 \wedge \theta_2$. It is also clear that ir-crown frames contain no non-trivial clusters, have height ≤ 3 and contain no subframe isomorphic to $\mathfrak{B}_3$. Thus $\mathfrak{S}_n \models \theta_4$. We show in detail below that $\mathfrak{S}_n \models \theta_3$ and $\mathfrak{S}_n \models \theta_5$.

To show that $\mathfrak{S}_n \models \theta_3$, take an arbitrary ir-crown frame $\mathfrak{S}_n = (S_n, Q_n)$ where $S_n = \{r, s_1, \cdots, s_{2n}\}$ and an arbitrary valuation ν. Take an arbitrary point $u \in S_n$.

Let us distinguish three cases:

Case 1: $u = s_{2k-1}$ for some $1 \leq k \leq n$. This by definition means that u belongs to the maximal layer. Then $u \not\models \Diamond p \wedge \Diamond\neg p$, which implies that $u \models \theta_3$.

Case 2: $u = s_{2k}$ where $0 \leq k \leq n$. This by definition means that u belongs to the middle layer. Without loss of generality we can assume that $u \models p$. Assume that $u \models \Diamond p \wedge \Diamond\neg p$. Then either $s_{2k-1} \models \neg p$ or $s_{2k+1} \models \neg p$. Hence $u \models p \wedge \Diamond\neg p$. We thus conclude that $u \models \theta_3$ since s_{2k} is reflexive.

Case 3: $u = r$. Assume that $u \models \Diamond p \wedge \Diamond\neg p$. Then there exist w and v with $w \models p$ and $v \models \neg p$.

Let us consider the case when both w and v belong to the middle layer; other cases follow in a similar way. Assume $w = s_{2k}$ and $v = s_{2k+2l}$ (Fig. 7).

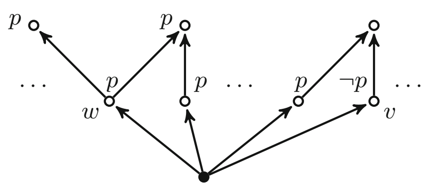


Fig. 7. Case 3.

In case $s_{2k+1} \not\models p$ we are done since rQ_nw and wQ_ns_{2k+1} which yields $r \models \Diamond(p \wedge \Diamond\neg p)$. In case $s_{2k+1} \models p$ we proceed by looking at an immediate (distinct from s_{2k}) predeccessor of s_{2k+1} which is s_{2k+2}. If $s_{2k+2} \not\models p$ we are done since rQ_ns_{2k+2} and $s_{2k+2}Q_ns_{2k+1}$ which means that $r \models \Diamond(\neg p \wedge \Diamond p)$. If

$s_{2k+2} \models p$ we proceed by the same reasoning and either we arrive at an s_m satisfying $s_m \not\models p$ for some m with $2k < m \leq 2k + 2l - 1$, or $s_{2k+2l-1}$ also models p. In the last case since $rQ_n s_{2k+2l-1}$ and $s_{2k+2l-1}Q_n v$ we have that $r \models \Diamond(p \wedge \Diamond\neg p)$. We omit the details for the other cases.

To show that $\mathfrak{S}_n \models \theta_5$, suppose the antecedent of the formula is true at a point w in an ir-crown frame $\mathfrak{S}_n = (S_n, Q_n)$ for some valuation ν. Note that making γ true forces w to have at least three distinct successors. It follows that w is the irreflexive root of $\mathfrak{S}_n$, and is the only point making r true. Moreover, $w \models \gamma$ also implies that both p and $\neg p$ are true at some maximal points of $\mathfrak{S}_n$. Then there exists an $i < n$ such that the maximal points s_{2i-1} and s_{2i+1} differ on the value of p (otherwise all maximal points would agree on the value of p). Since $wQ_n s_{2i}$, $s_{2i}Q_n s_{2i-1}$ and $s_{2i}Q_n s_{2i+1}$, it follows that w makes the consequent of the formula true as well.

Theorem 5. *The logic Λ_2^d has the finite model property.*

Proof. Note that since frames from $B_{\geq 3}$ are not admitted by Λ_2^d, the logic is of finite depth. By Segerberg's Theorem (see e.g. Theorem 8.85 of [6]) any logic of finite depth is characterized by its finite frames. It follows that Λ_2^d has the finite model property.

Since each ir-crown frame validates all the axioms of Λ_2^d, we have the inclusion $\mathbf{IC} \subseteq \mathbf{Fr}(\Lambda_2^d)$. The other inclusion follows from the following theorem.

Let $\mathfrak{F}$ be a rooted transitive frame of height 3 with the root irreflexive, and let $\mathfrak{F}^*$ be the frame obtained from $\mathfrak{F}$ by making the root reflexive. Then the following holds:

Lemma 1. *Let $\mathfrak{S}_n$ be an ir-crown frame, let $\mathfrak{F}$ be a rooted frame with irreflexive root and with height equal to 3. Then $\mathfrak{S}_n$ up-reduces to $\mathfrak{F}$ if and only if $\mathfrak{S}_n^*$ up-reduces to $\mathfrak{F}^*$.*

Proof. Assume that $\mathfrak{S}_n$ up-reduces to $\mathfrak{F}$. Then $\mathfrak{F}$ is obtainable as a p-morphic image of some generated subframe of $\mathfrak{S}_n$. Let us fix U and f to be the mentioned generated subframe and p-morphism. If U is a strict subframe, i.e. $U \subset \mathfrak{S}_n$, then U is a generated subframe of $\mathfrak{S}_n^*$ as well and we take the same function f which is a p-morphism from U to $\mathfrak{F}^*$. In case $U = \mathfrak{S}_n$ it is clear that the irreflexive root is mapped to the irreflexive root since f cannot map reflexive points to irreflexive points due to the monotonicity condition. It follows that the restriction of f to $U \setminus \{r\}$ is a p-morphism, so that f is a p-morphism from $\mathfrak{S}_n^*$ to $\mathfrak{F}^*$.

Conversely, assume that $\mathfrak{S}_n^*$ up-reduces to $\mathfrak{F}^*$. Again let us fix U and f to be the generated subframe and the p-morphism doing the up-reduction. The case $U \subset \mathfrak{S}_n^*$ is exactly the same as for the previous direction. Now assume that $U = \mathfrak{S}_n^*$ and let us show that the preimage of the root $w \in \mathfrak{F}^*$ is exactly the root r of $\mathfrak{S}_n^*$. Clearly $r \in f^{-1}(w)$. Assume that some other point $x \in f^{-1}(w)$. Since the height of $\mathfrak{F}^*$ is three, there are at least two distinct points $v, v' \in \mathfrak{F}^*$ with $v \neq w \neq v'$ and $wRvRv'$. Since f is a p-morphism, there exist u and u' in U with $xQ_n uQ_n u'$ and $f(u) = v$, $f(u') = v'$. But this is a contradiction, since ir-crown frames do not contain a chain of four distinct points such as $rQ_n xQ_n uQ_n u'$.

Theorem 6. *If a finite rooted frame validates all five axioms of Λ_2^d, then it is an up-reduction of some ir-crown frame.*

Proof. Assume that $\mathfrak{F} = (W, R)$ is a finite rooted frame validating the logic Λ_2^d. By Theorem 3 (iv) we know that $\mathfrak{F}$ does not contain clusters and is of height ≤ 3. Let us distinguish three cases.

Case 1 (height = 1). This means that there are no arrows between distinct points in $\mathfrak{F}$. Since $\mathfrak{F}$ is a rooted serial frame, it can only consist of a single reflexive point. Clearly we can obtain one reflexive point by taking a generated subframe of the ir-crown frame $\mathfrak{S}_1$. Hence $\mathfrak{F}$ is an up-reduction of an ir-crown frame (Fig. 8).

Case 2 (height = 2). By Theorem 3 (v), $\mathfrak{F}$ cannot have width > 2. Additionally, by seriality, we know that the maximum of $\mathfrak{F}$ must be reflexive. Let us picture all frames with no clusters, with reflexive maximal points, having width < 3 and height 2. These are:

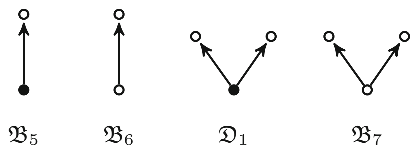


Fig. 8. Frames with height 2, width < 3 and reflexive maximum.

By Theorem 3 (iii), $\mathfrak{F}$ cannot be $\mathfrak{D}_1$. The other three frames are up-reductions of ir-crown frames. $\mathfrak{B}_5$ is a p-morphic image of the ir-crown frame $\mathfrak{S}_1$ below; $\mathfrak{B}_6$ is a generated subframe of $\mathfrak{S}_1$ and $\mathfrak{B}_7$ is a generated subframe of $\mathfrak{S}_2$, as pictured below (Fig. 9).

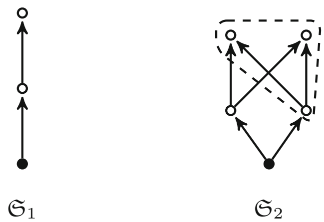


Fig. 9. Frames $\mathfrak{G}_1$ and $\mathfrak{G}_2$.

Case 3 (height = 3). Let us define a relation $\overline{R}$ by $(x, y) \in \overline{R}$ iff $xRy \wedge x \neq y$. Denote the root of $\mathfrak{F}$ by r_1. We define a partition of W as follows:

(i) $G_0 = \{r_1\}$
(ii) $G_1 = \{x \in W \mid r_1 \overline{R} x \wedge \neg \exists y (r_1 \overline{R} y \wedge y \overline{R} x)\}$
(iii) $G_2 = \{x \in W \mid \exists y \in G_1, y \overline{R} x\}$

Since height $= 3$, we know that $G_2 \neq \emptyset$. We investigate the structure of $\mathfrak{F}$ in a series of claims below.

Claim 1. Every point of G_2 is reflexive. Clearly every point of G_2 is a maximal point of $\mathfrak{F}$. The claim then follows from seriality of $\mathfrak{F}$.

Claim 2. The root r_1 is irreflexive. Since the height of $\mathfrak{F}$ is exactly 3, there exist distinct points $u, v \in W$ with $r_1 R u R v$. Recall that by Theorem 3 (iv) the frame $\mathfrak{B}_3$ cannot be a subframe of $\mathfrak{F}$. It follows that r_1 is irreflexive.

Claim 3. Every point of G_1 is reflexive. Indeed, as r_1 is irreflexive and $\mathfrak{B}_2$ is not a convex subframe of $\mathfrak{F}$, we deduce that all points in G_1 are reflexive.

Claim 4. Each point in G_1 sees at most 2 points from G_2. Indeed, otherwise the frame $\mathfrak{T}_{\text{ref}}$ would be a generated subframe of $\mathfrak{F}$, which contradicts Theorem 3 (v).

Claim 5. G_1 does not contain maximal points. Assume it does. Since the height of $\mathfrak{F}$ is 3 it can be divided into three nonempty parts—the root r_1, $A = \{w \in G_1 \mid w \text{ is maximal}\}$ and $B = W \setminus (\{r_1\} \cup A)$. We construct a p-morphism $f : \mathfrak{F} \to \mathfrak{D}_1$ by sending the root to the root, A to the left successor of the root and B to the right successor of the root.

Given the properties of $\mathfrak{F}$ revealed by these claims, we further claim that $\mathfrak{F}^*$ cannot be up-reduced to any of the following five frames taken from the paper [8] (Fig. 10).

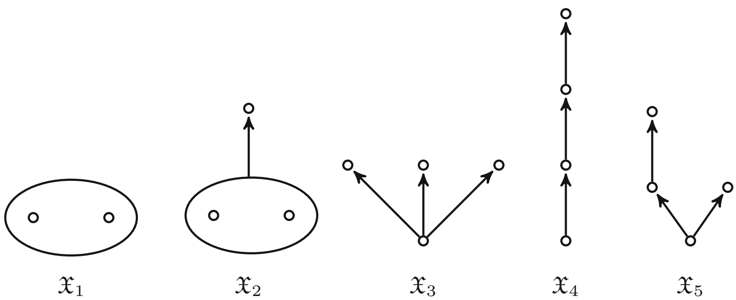


Fig. 10. Posets $\mathfrak{X}_1 - \mathfrak{X}_5$.

Indeed, that $\mathfrak{F}^*$ is reflexive, transitive, without non-trivial clusters and of height 3, means precisely that it cannot be up-reduced to any of $\mathfrak{X}_1, \mathfrak{X}_2$ and $\mathfrak{X}_4$.

Suppose that $\mathfrak{F}^*$ is up-reduced to $\mathfrak{X}_3$ using a generated subframe U and a p-morphism f. If $U \subsetneq W$, then U is also a generated subframe of $\mathfrak{F}$ and the same up-reduction works for reducing $\mathfrak{F}$ to $\mathfrak{X}_3$, which is isomorphic to $\mathfrak{T}_{\text{ref}}$, and this is forbidden by Theorem 3 (v). Suppose then $U = W$. It is clear, that $f(r_1)$ is the root of $\mathfrak{X}_3$. If any other point from W maps to the root of $\mathfrak{X}_3$, we can reason as for the case $U \subsetneq W$. Otherwise, the same f works as a p-morphism from $\mathfrak{F}$ onto $\mathfrak{T}_{\text{irr}}$ below, which clearly maps p-morphically onto $\mathfrak{D}_1$ thus contradicting Theorem 3 (iv) (Fig. 11).

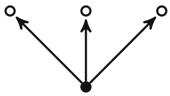


Fig. 11. The frame $\mathfrak{T}_{\text{irr}}$.

Therefore, $\mathfrak{F}^*$ cannot be up-reduced to $\mathfrak{X}_3$.

Suppose, finally, that $\mathfrak{F}^*$ can be up-reduced to $\mathfrak{X}_5$. A reasoning similar to that in Lemma 1 suffices to deduce that $\mathfrak{F}$ can be up-reduced to $\mathfrak{D}_2$, in contradiction to Theorem 3 (iv).

It follows that none of the frames $\mathfrak{X}_1 - \mathfrak{X}_5$ is an up-reduction of $\mathfrak{F}^*$. By Lemma 4.2 in [8] we can now claim that $\mathfrak{F}^*$ is an up-reduction of some crown frame (i.e. a frame similar to an ir-crown frame, but with reflexive root). Using Lemma 1 above, we can conclude that $\mathfrak{F}$ is an up-reduction of an ir-crown frame, as required.

Theorem 7. *The d-logic of the polygonal plane is axiomatized by the axioms* $\theta_1 - \theta_5$*, i.e.* $\mathbf{PL}_2^d = \Lambda_2^d$*.*

Proof. By Theorem 4 any ir-crown validates Λ_2^d, hence $\Lambda_2^d \subseteq \mathsf{Log}(\mathbf{IC})$. By Theorem 5 the logic Λ_2^d is determined by its finite frames and by Theorem 6 each such rooted frame is an up-reduction of an ir-crown frame. Since up-reductions preserve validity, it follows that $\mathsf{Log}(\mathbf{IC}) \subseteq \Lambda_2^d$. Since by Corollary 1 we have $\mathbf{PL}_2^d = \mathsf{Log}(\mathbf{IC})$, the proof is completed.

4 Conclusion

We have axiomatized the modal logic of the Euclidean plane when propositional letters denote planar polygons, while the modal diamond is interpreted as the standard derivative operator on the plane. The obvious question is how these results can be generalized to higher dimensions. The research is under way to determine the C-logic of the polyhedra in the Euclidean space of dimension 3. We are convinced that the approach taken in this paper to determine the d-logic of planar polygons given the knowledge of their C-logic can be lifted to higher dimensions as well. That for each $n < \omega$ the C-logic and the d-logic of n-dimensional polyhedra are Kripke complete can be proved using the Segerberg's theorem on transitive logics of finite height, since a formula like θ_4 can be written in each dimension utilizing the fact that the border $\mathbb{C}A - A$ of a polyhedron is a polyhedron of strictly lower dimension. The link between Kripke semantics for the C-logic and the d-logic seems to be that admissible rooted frames of maximal possible height for these logics are very similar, with the only difference of having an irreflexive root in the case of d-logic and the reflexive one in the case of C-logic. Some further general observations can be made for such frames, like all of them necessarily being without trivial clusters, but the precise details have to be postponed until a more in-depth investigation.

Some final words about possible applications of the formalism and semantic interpretation studied in this paper. Modal language is often praised for its fine balance between simplicity and expressivity. Thus it is desirable to find ways of interpreting it on mathematical structures modeling phenomena of particular interest. Many spatial phenomena and their interrelations can be modelled with arbitrary precision using polyhedra in the Euclidean space. Our approach in this paper interprets the modal language on such structures and studies the emerging basic reasoning mechanisms. We believe this prepares the ground for fruitful applications in the area of spatial knowledge representation and reasoning.

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