Chapter 10
Topological Interpretations of Provability Logic

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In memory of Leo Esakia

Abstract Provability logic concerns the study of modality $\Box$ as provability in formal systems such as Peano Arithmetic. A natural, albeit quite surprising, topological interpretation of provability logic has been found in the 1970s by Harold Simmons and Leo Esakia. They have observed that the dual $\Diamond$ modality, corresponding to consistency in the context of formal arithmetic, has all the basic properties of the topological derivative operator acting on a scattered space. The topic has become a long-term project for the Georgian school of logic led by Esakia, with occasional contributions from elsewhere. More recently, a new impetus came from the study of polymodal provability logic **GLP** that was known to be Kripke incomplete and, in general, to have a more complicated behavior than its unimodal counterpart. Topological semantics provided a better alternative to Kripke models in the sense that **GLP** was shown to be topologically complete. At the same time, new fascinating connections with set theory and large cardinals have emerged. We give a survey of the results on topological semantics of provability logic starting from first contributions by Esakia. However, a special emphasis is put on the recent work on topological models of polymodal provability logic. We also include a few results that have not been published so far, most notably the results of Sect. 10.4 (due to the second author) and Sects. 10.7, 10.8 (due to the first author).

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10.1 Provability Logics and Magari Algebras

Provability logics and algebras emerge from, respectively, a modal logical and an algebraic point of view on the proof-theoretic phenomena around Gödel's incompleteness theorems. These theorems are usually perceived as putting fundamental restrictions on what can be formally proved in a given axiomatic system (satisfying modest natural requirements). For the sake of a discussion, we call a formal theory T *gödelian* if

- T is a first order theory in which the natural numbers along with the operations $+$ and $\cdot$ are interpretable;
- T proves some basic properties of these operations and a modicum of induction (it is sufficient to assume that T contains Elementary Arithmetic EA, see [7]);
- T has a recursively enumerable (r.e.) set of axioms.

The Second Incompleteness Theorem of Kurt Gödel (G2) states that a gödelian theory T cannot prove its own consistency provided it is indeed consistent. More accurately, for any r.e. presentation of such a theory T, Gödel has shown how to write down an arithmetical formula $\mathsf{Prov}_T(x)$ expressing that *x is (a natural number coding) a formula provable in T*. Then the statement $\mathsf{Con}(T) := \neg\mathsf{Prov}_T(\ulcorner\bot\urcorner)$ naturally expresses that the theory T is consistent. G2 states that $T \nvdash \mathsf{Con}(T)$ provided T is consistent.

Provability logic emerged from the question of what properties of formal provability Prov_T can be verified in T, even if the consistency of T cannot. Several such properties have been stated by Gödel himself [33]. Hilbert and Bernays [36] and then Löb [44] stated them in the form of conditions any adequate formalization of a provability predicate in T must satisfy. After Gödel's and Löb's work it was clear that the formal provability predicate calls for a treatment as a modality. It led to the formulation of the Gödel–Löb provability logic **GL** and eventually to the celebrated arithmetical completeness theorem due to Solovay [55].

Independently, Macintyre and Simmons [45] and Magari [46] took a very natural algebraic perspective on the phenomenon of formal provability which led to the concept of *diagonalizable algebra*. Such algebras are now more commonly called *Magari algebras*. This point of view is more convenient for our present purposes.

Recall that the Lindenbaum–Tarski algebra of a theory T is the set of all T-sentences Sent_T modulo provable equivalence in T, that is, the structure $\mathscr{L}_T = \mathrm{Sent}_T/{\sim_T}$ where, for all $\varphi, \psi \in \mathrm{Sent}_T$,

$$\varphi \sim_T \psi \iff T \vdash (\varphi \leftrightarrow \psi).$$

Since we assume T to be based on classical propositional logic, $\mathscr{L}_T$ is a boolean algebra with operations $\wedge$, $\vee$, $\neg$. Constants $\bot$ and $\top$ are identified with the sets of refutable and provable sentences of T, respectively. The standard ordering on $\mathscr{L}_T$ is defined by

$$[\varphi] \leq [\psi] \iff T \vdash \varphi \to \psi \iff [\varphi \wedge \psi] = [\varphi],$$

where $[\varphi]$ denotes the equivalence class of φ.

It is well known that for consistent gödelian theories T all such algebras are isomorphic to the unique countable atomless boolean algebra. (This is a consequence of a strengthening of Gödel's First Incompleteness Theorem due to Rosser.) We obtain more interesting algebras by enriching the structure of the boolean algebra $\mathscr{L}_T$ by additional operation(s).

Gödel's consistency formula induces a unary operator $\Diamond_T$ acting on $\mathscr{L}_T$:

$$\Diamond_T : [\varphi] \longmapsto [\mathsf{Con}(T + \varphi)].$$

The sentence $\mathsf{Con}(T + \varphi)$ expressing the consistency of T extended by φ can be defined as $\neg\mathsf{Prov}_T(\ulcorner\neg\varphi\urcorner)$. The dual operator is $\Box_T : [\varphi] \longmapsto [\mathsf{Prov}_T(\ulcorner\varphi\urcorner)]$, thus $\Box_T x = \neg\Diamond_T\neg x$ for all $x \in \mathscr{L}_T$.

Hilbert–Bernays–Löb derivability conditions ensure that $\Diamond_T$ is correctly defined on the equivalence classes of the Lindenbaum–Tarski algebra of T. Moreover, it satisfies the following identities (where we write $\Diamond_T$ simply as $\Diamond$ and the variables range over arbitrary elements of $\mathscr{L}_T$):

M1. $\Diamond\bot = \bot$; $\quad \Diamond(x \vee y) = \Diamond x \vee \Diamond y$;
M2. $\Diamond x = \Diamond(x \wedge \neg\Diamond x)$.

Notice that Axiom M2 is a formalization of G2 stated for the theory $T' = T + \varphi$, where $[\varphi] = x$. In fact, the left hand side states that T' is consistent, whereas the right hand side states that $T' + \neg\mathsf{Con}(T')$ is consistent, that is, $T' \nvdash \mathsf{Con}(T')$. The dual form of Axiom M2, $\Box(\Box x \to x) = \Box x$, expresses the formalization of Löb's theorem [44].

A Boolean algebra with an operator $\mathscr{M} = (M, \Diamond)$ satisfying M1, M2 is called *Magari algebra*. Thus, the main example of a Magari algebra is the structure $(\mathscr{L}_T, \Diamond_T)$ for any consistent gödelian theory T.

Notice that M1 induces $\Diamond$ to be monotone: if $x \leq y$ then $\Diamond x \leq \Diamond y$. The *transitivity* inequality $\Diamond\Diamond x \leq \Diamond x$ is often postulated as an additional axiom of Magari algebras, however, as discovered independently by de Jongh, Kripke and Sambin in the 1970s, it follows from M1 and M2.

Proposition 1. *In any Magari algebra $\mathscr{M}$ it holds that $\Diamond\Diamond x \leq \Diamond x$ for all $x \in M$.*

Proof Given any $x \in M$, consider $y := x \vee \Diamond x$. On the one hand, we have

$$\Diamond\Diamond x \leq (\Diamond x \vee \Diamond\Diamond x) = \Diamond y.$$

On the other hand, since $\Diamond x \wedge \neg\Diamond y = \bot$ we obtain

$$\Diamond y \leq \Diamond(y \wedge \neg\Diamond y) \leq \Diamond((x \vee \Diamond x) \wedge \neg\Diamond y) = \Diamond(x \wedge \neg\Diamond y) \vee \Diamond\bot \leq \Diamond x.$$

Hence, $\Diamond\Diamond x \leq \Diamond x$. □

In general, we call an *identity* of an algebraic structure $\mathscr{M}$ a formula of the form $t(\mathbf{x}) = u(\mathbf{x})$, where t, u are terms, such that $\mathscr{M} \vDash \forall\mathbf{x}\,(t(\mathbf{x}) = u(\mathbf{x}))$. Identities of Maragi algebras can be described in terms of modal logic as follows. Any term (built from the variables using boolean operations and $\Diamond$) is naturally identified with a formula in the language of propositional logic with a new unary connective $\Diamond$. If $\varphi(\mathbf{x})$ is such a formula and $\mathscr{M}$ a Magari algebra, we write $\mathscr{M} \vDash \varphi$ iff $\forall\mathbf{x}\,(t_\varphi(\mathbf{x}) = \top)$ is valid in $\mathscr{M}$, where t_φ is the term corresponding to φ. Since any identity in Magari algebras can be equivalently written in the form $t = \top$ for some term t, the axiomatization of identities of $\mathscr{M}$ amounts to axiomatizing modal formulas valid in $\mathscr{M}$. The *logic of* $\mathscr{M}$, $\mathrm{Log}(\mathscr{M})$, is the set of all modal formulas valid in $\mathscr{M}$, that is, $\mathrm{Log}(\mathscr{M}) := \{\varphi : \mathscr{M} \vDash \varphi\}$, and the logic of a class of modal algebras is defined similarly.

One of the main parameters of a Magari algebra $\mathscr{M}$ is its *characteristic* $\mathrm{ch}(\mathscr{M}) := \min\{k \in \omega : \Diamond^k\top = \bot\}$ and $\mathrm{ch}(\mathscr{M}) := \infty$ if no such k exists. If T is arithmetically sound, that is, if the arithmetical consequences of T are valid in the standard model, then $\mathrm{ch}(\mathscr{L}_T) = \infty$. Theories (whose algebras are) of finite characteristics are, in a sense, close to being inconsistent and may be considered a pathology.

Solovay [55] proved that any identity valid in the structure $(\mathscr{L}_T, \Diamond_T)$ follows from the boolean identities together with M1–M2, provided T is arithmetically sound. This has been generalized by Visser [58] to arbitrary theories of infinite characteristic.

Theorem 1. (Solovay, Visser) *Suppose* $\mathrm{ch}(\mathscr{L}_T, \Diamond_T) = \infty$. *An identity holds in* $(\mathscr{L}_T, \Diamond_T)$ *iff it holds in all Magari algebras.*

Apart from the equational characterization by M1, M2 above, the identities of Magari algebras can be axiomatized modal-logically. In fact, the logic of all Magari algebras, and by the Solovay theorem the logic $\mathrm{Log}(\mathscr{L}_T, \Diamond_T)$ of the Magari algebra of T, for any fixed theory T of infinite characteristic, coincides with the familiar Gödel–Löb logic **GL**. Abusing the language we will often identify **GL** with the set of identities of Magari algebras.[1]

A Hilbert-style axiomatization of **GL** is usually given in the modal language where $\Box$ rather than $\Diamond$ is taken as basic and the latter is treated as an abbreviation for $\neg\Box\neg$. The axioms and inference rules of **GL** are as follows.

Axiom schemata:

L1. All instances of propositional tautologies;
L2. $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$;

[1] For normal modal logics, going from an equational to a Hilbert-style axiomatization and back is automatic, as they are known to be strongly finitely algebraizable (see [19, 31]). We do not assume the reader's familiarity with algebraic logic and prefer to give explicit axiomatizations for the systems at hand.

L3. $\Box(\Box\varphi \to \varphi) \to \Box\varphi$.

Rules: $\varphi,\ \varphi \to \psi/\psi$ (modus ponens), $\varphi/\Box\varphi$ (necessitation).

By a well-known result of Segerberg [51], **GL** is sound and complete w.r.t. the class of all transitive and upwards well-founded Kripke frames. In fact, it is sufficient to restrict the attention to frames that are finite irreflexive trees. Thus, summarizing various characterizations above, we have

Theorem 2. *Let T be a gödelian theory of infinite characteristic. For any modal formula φ, the following statements are equivalent:*

(i) $\mathbf{GL} \vdash \varphi$;
(ii) *φ is valid in all Magari algebras;*
(iii) $(\mathscr{L}_T, \Diamond_T) \vDash \varphi$;
(iv) *φ is valid in all finite irreflexive tree-like Kripke frames.*

10.2 Topological Interpretation

A natural, albeit quite surprising, topological interpretation of provability logic was found by Simmons [53]. He observed that the topological derivative operator acting on a scattered topological space satisfies all the identities of Magari algebras. Esakia [28], working independently, considered a more general problem of set-theoretic interpretations of Magari algebras.

Let X be a nonempty set and let $\mathscr{P}(X)$ the boolean algebra of subsets of X. Consider any operator $\delta : \mathscr{P}(X) \to \mathscr{P}(X)$ and the structure $(\mathscr{P}(X), \delta)$. Can $(\mathscr{P}(X), \delta)$ be a Magari algebra and, if yes, when? Esakia [28] found what may be called a canonical answer to this question (Theorem 4 below).

Let (X, τ) be a topological space, where τ denotes the set of open subsets of X, and let $A \subseteq X$. Topological *derivative* $d_\tau(A)$ *of* A is the set of limit points of A:

$$x \in d_\tau(A) \iff \forall U \in \tau\ (x \in U \Rightarrow \exists y \neq x\ (y \in U \cap A)).$$

Notice that $c_\tau(A) := A \cup d_\tau(A)$ is the closure of A and $iso_\tau(A) := A \setminus d_\tau(A)$ is the set of isolated points of A.

The classical notion of a scattered topological space is due to Georg Cantor. (X, τ) is called *scattered* if every nonempty subspace $A \subseteq X$ has an isolated point.

Theorem 3. (Simmons, Esakia) *The following statements are equivalent:*

(i) *(X, τ) is scattered;*
(ii) *$(\mathscr{P}(X), d_\tau)$ is a Magari algebra, that is, for all $A \subseteq X$, $d_\tau(A) = d_\tau(A \setminus d_\tau(A))$.*

Notice that $d_\tau(A) = d_\tau(A \setminus d_\tau(A))$ means that each limit point of A is a limit point of its isolated points. The algebra of the form $(\mathscr{P}(X), d_\tau)$ associated with a topological space (X, τ) will be called *the derivative algebra of* X. Thus, this theorem states that the derivative algebra of (X, τ) is Magari iff (X, τ) is scattered.

Proof Suppose (X, τ) is scattered, $A \subseteq X$ and $x \in d_\tau(A)$. Consider any open neighborhood U of x. Since $(U \cap A) \setminus \{x\}$ is nonempty, it has an isolated point $y \neq x$. Since U is open, y is an isolated point of A, that is, $y \in A \setminus d_\tau(A)$. Hence, $x \in d_\tau(A \setminus d_\tau(A))$. The inclusion $d_\tau(A \setminus d_\tau(A)) \subseteq d_\tau(A)$ follows from the monotonicity of d_τ. Therefore Statement (ii) holds.

Suppose that (ii) holds and let $A \subseteq X$ be nonempty. We show that A has an isolated point. If $d_\tau A$ is empty, we are done. Otherwise, take any $x \in d_\tau A$. Since x is a limit of isolated points of A, there must be at least one such point. □

We notice that the transitivity principle $d_\tau d_\tau A \subseteq d_\tau A$ topologically means that the set $d_\tau A$, for any $A \subseteq X$, is closed. We recall the following standard equivalent characterization an easy proof of which we shall omit.

Proposition 2. *For any topological space (X, τ), the following statements are equivalent:*

(i) Every $x \in X$ is an intersection of an open and a closed set;
(ii) For each $A \subseteq X$, the set $d_\tau A$ is closed.

Topological spaces satisfying either of these conditions are called *T_d-spaces*. Condition (i) shows that T_d is a weak separation property located between T_0 and T_1. Thus, Proposition 1 yields, as a corollary, the modal proof of the following well-known fact.

Corollary 1. *All scattered spaces are T_d.*

We have seen in Theorem 3 that each scattered space equipped with a topological derivative operator is a Magari algebra. The following result by Esakia [28] shows that any Magari algebra on $\mathscr{P}(X)$ can be described in this way.

Theorem 4. (Esakia) *If $(\mathscr{P}(X), \delta)$ is a Magari algebra, then X bears a unique topology τ for which $\delta = d_\tau$. Moreover, τ is scattered.*

Proof We first remark that if $(\mathscr{P}(X), \delta)$ is a Magari algebra, then the operator $c(A) := A \cup \delta A$ satisfies the Kuratowski axioms of the topological closure: $c\varnothing = \varnothing$, $c(A \cup B) = cA \cup cB$, $A \subseteq cA$, $ccA = cA$. This defines a topology τ on X in which a set A is τ-closed iff $A = c(A)$ iff $\delta A \subseteq A$. If ν is any topology such that $\delta = d_\nu$, then ν has the same closed sets, that is, $\nu = \tau$. So if the required topology exists, it is unique. To show that $\delta = d_\tau$ we need an auxiliary lemma. □

Lemma 1. *Suppose $(\mathscr{P}(X), \delta)$ is Magari. Then, for all $x \in X$,*

(i) $x \notin \delta(\{x\})$;
(ii) $x \in \delta A \iff x \in \delta(A \setminus \{x\})$.

Proof (i) By Axiom M2 we have $\delta\{x\} \subseteq \delta(\{x\} \setminus \delta\{x\})$. If $x \in \delta\{x\}$ then $\delta(\{x\} \setminus \delta\{x\}) = \delta\varnothing = \varnothing$. Hence, $\delta\{x\} = \varnothing$, a contradiction.

(ii) $x \in \delta A$ implies $x \in \delta((A \setminus \{x\}) \cup \{x\}) = \delta(A \setminus \{x\}) \cup \delta\{x\}$. By (i), $x \notin \delta\{x\}$, hence $x \in \delta(A \setminus \{x\})$. The other implication follows from the monotonicity of δ. □

Lemma 2. *Suppose* $(\mathscr{P}(X), \delta)$ *is Magari and* τ *is the associated topology. Then* $\delta = d_\tau$.

Proof Let $d = d_\tau$; we show that for any set $A \subseteq X$ $dA = \delta A$. Notice that for any B, $cB = dB \cup B = \delta B \cup B$. Assume $x \in \delta A$. Then $x \in \delta(A \setminus \{x\}) \subseteq c(A \setminus \{x\}) \subseteq d(A \setminus \{x\}) \cup (A \setminus \{x\})$. Since $x \notin A \setminus \{x\}$, we obtain $x \in d(A \setminus \{x\})$. By the monotonicity of d, $x \in dA$. Similarly, if $x \in dA$ then $x \in d(A \setminus \{x\})$. Hence, $x \in c(A \setminus \{x\}) = \delta(A \setminus \{x\}) \cup (A \setminus \{x\})$. Since $x \notin A \setminus \{x\}$ we obtain $x \in \delta A$. □

From this lemma and Theorem 3 we also infer that τ is a scattered topology.

Theorem 4 shows that to study a natural set-theoretic interpretation of provability logic means to study the semantics of $\Diamond$ as a derivative operation on a scattered topological space. Derivative semantics of modality was first suggested in the fundamental paper by McKinsey and Tarski [48]. See [43] for a detailed survey of such semantics for arbitrary topological spaces. The emphasis in this chapter is on the logics related to formal provability and scattered topological spaces.

10.3 Topological Completeness Theorems

Natural examples of scattered topological spaces come from orderings. Two examples will play an important role below.

Let $(X, \prec)$ be a strict partial ordering. The *left topology* or the *downset topology* $\tau_\leftarrow$ on $(X, \prec)$ is given by all sets $A \subseteq X$ such that $\forall x, y\ (y \prec x \in A \Rightarrow y \in A)$. We obviously have that $(X, \prec)$ is well-founded iff $(X, \tau_\leftarrow)$ is scattered. The *right topology* or the *upset topology* is defined similarly.

The left topology is, in general, non-Hausdorff. More natural is the *interval topology* on a linear ordering $(X, <)$, which is generated by all open intervals $(\alpha, \beta) = \{x \in X \mid \alpha < x < \beta\}$ such that $\alpha, \beta \in X \cup \{\pm\infty\}$ and $\alpha < \beta$. The interval topology refines both the left topology and the right topology and is scattered on any ordinal [52].

Given a topological space (X, τ), we denote the logic of its derivative algebra $(\mathscr{P}(X), d_\tau)$ by $\mathrm{Log}(X, \tau)$, and we let $\mathrm{Log}(\mathscr{C})$ denote the logic of (the class of derivative algebras associated with) a class $\mathscr{C}$ of topological spaces. Thus, if $\mathscr{C}$ is a class of scattered spaces, $\mathrm{Log}(\mathscr{C})$ is a normal modal logic extending **GL**.

Esakia [28] has noted that the completeness theorem for **GL** w.r.t. its Kripke semantics (see [22, 51]) implies that **GL** is the modal logic of scattered spaces. In fact, if $(X, \prec)$ is a strict partial ordering, then the modal algebra associated with the Kripke frame $(X, \prec)$ is the same as the derivative algebra of (X, τ) where τ is its upset topology. This implies that any modal logic of a class of strict partial orders, including **GL**, is complete w.r.t. topological derivative semantics.

We can also note that **GL** is the logic of a single countable scattered space. Abashidze [1] and Blass [18] independently proved a stronger completeness result.

Theorem 5. (Abashidze, Blass) *Let* $\alpha \geq \omega^\omega$ *be any ordinal equipped with the interval topology. Then* $\mathrm{Log}(\alpha) = \mathbf{GL}$.

Thus, **GL** is complete w.r.t. a natural scattered topological space. The rest of this section is devoted to a new proof of this result. We need some technical prerequisites that will be also useful later in this chapter.

Ranks and d-maps. An equivalent characterization of scattered spaces is often given in terms of the following transfinite *Cantor–Bendixson sequence* of subsets of a topological space (X, τ):

- $d_\tau^0 X = X$; $d_\tau^{\alpha+1} X = d_\tau(d_\tau^\alpha X)$ and
- $d_\tau^\alpha X = \bigcap_{\beta<\alpha} d_\tau^\beta X$ if α is a limit ordinal.

It is easy to show by transfinite induction that for any (X, τ), all sets $d_\tau^\alpha X$ are closed and that $d_\tau^\alpha X \supseteq d_\tau^\beta X$ whenever $\alpha \leq \beta$.

Theorem 6. (Cantor) (X, τ) *is scattered iff* $d_\tau^\alpha X = \varnothing$ *for some ordinal* α.

Proof Let $d = d_\tau$. If (X, τ) is scattered then we have $d^\alpha X \supset d^{\alpha+1} X$ for each α such that $d^\alpha X \neq \varnothing$. By cardinality arguments this yields an α such that $d^\alpha X = \varnothing$.

Conversely, suppose $A \subseteq X$ is nonempty. Let α be the least ordinal such that $A \nsubseteq d^\alpha X$. Obviously, α cannot be a limit ordinal, hence $\alpha = \beta + 1$ for some β and there is an $x \in A \setminus d^{\beta+1} X$. Since $A \subseteq d^\beta X$, we also have $x \in d^\beta X$. Since $x \notin d^{\beta+1} X = d(d^\beta X)$, x is isolated in the relative topology of $d^\beta X$, and hence in the relative topology of $A \subseteq d^\beta X$. □

Call the least α such that $d_\tau^\alpha X = \varnothing$ the *Cantor–Bendixson rank* of X and denote it by $\rho_\tau(X)$. Let On denote the class of all ordinals. Then the *rank function* $\rho_\tau : X \to$ On is defined by

$$\rho_\tau(x) := \min\{\alpha : x \notin d_\tau^{\alpha+1}(X)\}.$$

Notice that ρ_τ maps X onto $\rho_\tau(X) = \{\alpha : \alpha < \rho_\tau(X)\}$. Also, $\rho_\tau(x) \geq \alpha$ iff $x \in d_\tau^\alpha X$. We omit the subscript τ whenever there is no danger of confusion.

Example 1. For an ordinal equipped with its *left topology*, $\rho(\alpha) = \alpha$ for all α. When the same ordinal is equipped with its *interval topology*, ρ is the function ℓ defined by $\ell(0) = 0$; $\ell(\alpha) = \beta$ if $\alpha = \gamma + \omega^\beta$ for some γ, β. By the Cantor normal form theorem for any $\alpha > 0$, such a β is uniquely determined, thus ℓ is well-defined. Notice that $\ell(\alpha) = 0$ iff α is a non-limit ordinal.

Let (X, τ_X) and (Y, τ_Y) be topological spaces, and let d_X, d_Y denote the corresponding derivative operators. A map $f : X \to Y$ is called a *d-map* if f is continuous, open and *pointwise discrete*, that is, $f^{-1}(y)$ is a discrete subspace of X for each $y \in Y$. d-maps are well known to satisfy the properties expressed in the following lemma (see [16]).

Lemma 3.

(i) $f^{-1}(d_Y(A)) = d_X(f^{-1}(A))$ *for any* $A \subseteq Y$;
(ii) $f^{-1} : (\mathscr{P}(Y), d_Y) \to (\mathscr{P}(X), d_X)$ *is a homomorphism of derivative algebras;*

(iii) If f is onto, then $\mathrm{Log}(X, \tau_X) \subseteq \mathrm{Log}(Y, \tau_Y)$.

Property (i) is easy to check directly; (ii) follows from (i), and (iii) follows from (ii). Each of the conditions (i) and (ii) is equivalent to f being a d-map.

A proof of the following lemma can be found in [5].

Lemma 4. *Let Ω be the ordinal $\rho_\tau(X)$ taken with its left topology. Then*

(i) $\rho_\tau : X \twoheadrightarrow \Omega$ is an onto d-map;
(ii) If $f : X \to \lambda$ is a d-map, where λ is an ordinal with its left topology, then $f(X) = \Omega$ and $f = \rho_\tau$.

An immediate corollary is that the rank function is preserved under d-maps.

The d-sum construction. The constructions of summing up structures, in particular, topological spaces or orderings 'along' another structure play an important role in various branches of logic and mathematics (see, e.g., [34]). Here we present another construction of this type, called *d-sum*, which can be used to recursively build both finite trees and ordinals. Given a tree T, one can construct a new tree by 'plugging in' other trees in place of the leaves of T. Similarly, given an ordinal α, one can 'plug in' new ordinals α_i for each isolated point $i \in \alpha$ to obtain another ordinal. The d-sum construction turned out to be rather useful for proving topological completeness theorems. Its particular case called *d-product* serves as a tool in the proof of topological completeness of **GLP** in [5].

Definition 1 Let X be a topological space and let $\{Y_j \mid j \in iso(X)\}$ be a collection of spaces indexed by the set $iso(X)$ of isolated points of X. We uniquely extend it to the collection $\{Y_j \mid j \in X\}$ by letting $Y_j = \{j\}$ for all $j \in dX$.

We define the *d-sum* (Z, τ_Z) of $\{Y_j\}$ over X (denoted $\sum^d_{j\in X} Y_j$) as follows. The base set is the disjoint union $Z := \bigsqcup_{j\in X} Y_j$. Define the map $\pi : Z \to X$ by putting $\pi(y) = j$ whenever $y \in Y_j$. Now let the topology τ_Z consist of the sets $V \cup \pi^{-1}(U)$ where V is open in the topological sum $\bigsqcup_{j\in iso(X)} Y_j$ and U is open in X. It is not difficult to check that τ_Z qualifies for a topology.

Example 2. (trees) Consider finite irreflexive trees equipped with the upset topology. Note that the leaves of a tree are the isolated points in the topology. Therefore, taking the d-sum of trees T_i over a tree T simply means plugging in T_i's in place of the leaves of T.

Let us call an *n-fork* a tree $\mathfrak{F}_n = (W_n, R_n)$, where $W_n = \{r, w_0, w_1, \ldots, w_{n-1}\}$ and $R_n = \{(r, w_i) \mid 0 \leq i < n\}$. Observe that any finite tree is either an irreflexive point, or an n-fork, or can be obtained (possibly in several ways) as a d-sum of trees of smaller depth.

Example 3. (ordinals) Consider ordinals equipped with the interval topology. If $(\alpha_i)_{i\in\beta}$ is a family of ordinals such that $\alpha_i = 1$ for limit i, then the d-sum $\sum^d_{i\in\beta} \alpha_i$ is homeomorphic to the ordinal sum $\sum_{i\in\beta} \alpha_i$. This can be checked directly by examining the descriptions of neighborhoods in respective spaces. Thus, a d-sum of ordinals along another ordinal is homeomorphic to an ordinal.

The following lemma shows that d-sums, in a way, commute with d-maps.

Lemma 5. *Let X and X' be two spaces and let $\{Y_j \mid j \in iso(X)\}$ and $\{Y'_k \mid k \in iso(X')\}$ be collections of spaces indexed by $iso(X)$ and $iso(X')$, respectively. Suppose further that $f : X \to X'$ is an onto d-map, and for each $j \in iso(X)$ there is an onto d-map $f_j : Y_j \to Y'_{f(j)}$. Then there exists an onto d-map $g : \sum^d_{j\in X} Y_j \to \sum^d_{k\in X'} Y'_k$.*

Proof First note that since f is a d-map, $f(j)$ is isolated in X' iff j is isolated in X. Indeed, by openness of f, if $\{j\} \in \tau$, then $\{f(j)\} \in \tau'$. Conversely, if $f(j)$ is isolated, then $f^{-1}f(j)$ is both open and discrete by continuity and pointwise discreteness of f. Hence, any point in $f^{-1}f(j)$, and j in particular, is isolated in X. For convenience, let us denote $f_* \equiv f\restriction_{d_\tau X}$ and $f^* \equiv f\restriction_{iso(X)}$. It follows that $f^* : iso(X) \to iso(X')$ and $f_* : d_\tau X \to d_{\tau'} X'$ are well-defined onto maps and $f = f^* \cup f_*$. Thus, in particular, the space $Y'_{f(j)}$ in the formulation of the theorem is well-defined.

Take g to be the set-theoretic union $g = f_* \cup \bigcup_{j\in iso(X)} f_j$. We show that g is a d-map. Let π and π' be the 'projection' maps associated with $\sum^d_{j\in X} Y_j$ and $\sum^d_{k\in X'} Y'_k$, respectively. To show that g is open, take $W = V \cup \pi^{-1}(U) \in \tau_Z$. Then $g(W) = g(V) \cup g(\pi^{-1}(U))$. That $g(V)$ is open in the topological sum of Y'_k is clear from the openness of the maps f_j. Moreover, from the definition of g and the fact that all f_j are onto it can be easily deduced that $g(\pi^{-1}(U)) = \pi'^{-1}(f(U))$. Since f is an open map, it follows that $g(W)$ is open in τ'_Z. To see that g is continuous, take $W' = V' \cup \pi'^{-1}(U') \in \tau'_Z$. Then $g^{-1}(W') = g^{-1}(U') \cup g^{-1}(\pi'^{-1}(U'))$. Again, the openness of $g^{-1}(U')$ is trivial. It is also easily seen that $g^{-1}(\pi'^{-1}(U')) = \pi^{-1}(f^{-1}(U'))$. It follows that $g^{-1}(W')$ is open in τ_Z. To see that g is pointwise discrete is straightforward, given that f and all the f_j are pointwise discrete. □

The following lemma is crucial for a proof of Theorem 5.

Lemma 6. *For each finite irreflexive tree T there exists a countable ordinal $\alpha < \omega^\omega$ and an onto d-map $f : \alpha \twoheadrightarrow T$.*

Proof The proof proceeds by induction on the depth of T. It is clear that the claim is true for a one-point tree. If T is an n-fork $\mathfrak{F}_n$ we define a d-map $f : \omega + 1 \twoheadrightarrow \mathfrak{F}_n$ by letting $f(x) := w_{x \bmod n}$ for $x < \omega$ and $f(\omega) := r$.

Now consider a tree T of depth $n > 1$ and suppose the claim is true for all trees of depth less than n. Clearly T can be presented as a d-sum of trees of strictly smaller depth in various ways. Using the induction hypothesis, each of the smaller trees is an image of a countable ordinal under a d-map. Applying Lemma 5 and observing that a countable d-sum of countable ordinals is a countable ordinal produces a countable ordinal α and an onto d-map $f : \alpha \twoheadrightarrow T$. Since the rank function is preserved under d-maps, the rank of α is equal to the rank of T, that is, to n. It follows that $\alpha < \omega^\omega$, which completes the proof. □

Now we prove Theorem 5.

Proof Take a non-theorem φ of **GL**. Then φ can be refuted on a finite irreflexive tree T by theorem 2. By Lemma 6, there exists an ordinal $\beta < \omega^\omega$ that maps onto T via a d-map. By Lemma 3 (iii), φ can be refuted on β. But β is an open subspace of α. It follows that φ can be refuted on α. □

Another, perhaps the simplest, proof of Theorem 5 appeared recently in [17, Theorem 3.5]. It relied on a direct proof of Lemma 6 rather than on Lemma 5. However, we believe that our approach illuminates the underlying recursive mechanism and may lead to additional insights in more complicated situations (see [5]).

10.4 Topological Semantics of Linearity Axioms

For a gödelian theory T consider the 0-generated subalgebra $\mathcal{L}_T^0$ of $(\mathcal{L}_T, \Diamond_T)$, that is, the subalgebra generated by $\top$. If $\mathrm{ch}(\mathcal{L}_T, \Diamond_T) = \infty$, then also $\mathrm{ch}(\mathcal{L}_T^0, \Diamond_T) = \infty$. In fact, the modal logic of the Magari algebra $(\mathcal{L}_T^0, \Diamond_T)$ is known (see [37]) to be **GL.3** which is obtained from **GL** by adding the following axiom:

$$(.3) \qquad \Diamond p \wedge \Diamond q \to \Diamond(p \wedge q) \vee \Diamond(p \wedge \Diamond q) \vee \Diamond(\Diamond p \wedge q).$$

This is the so called 'linearity axiom' and, as the name suggests, its finite rooted Kripke frames are precisely the finite strict linear orders. Since **GL.3** is Kripke complete (see, e.g., [24]), its topological completeness is immediate. However, it is not immediately clear what kind of scattered spaces does the linearity axiom isolate. To characterize GL.3-spaces, let us first simplify the axiom (.3). Consider the following formula:

$$(lin) \qquad \Box(\Box^+ p \vee \Box^+ q) \to \Box p \vee \Box q,$$

where $\Box^+\varphi$ is a shorthand for $\varphi \wedge \Box\varphi$.

Lemma 7. *In **GL** the schema* (.3) *is equivalent to* (lin).

Proof To show that $(lin) \vdash_{\mathbf{GL}}$ (.3), witness the following syntactic argument. Observe that the dual form of (lin) looks as follows:

$$\Diamond p \wedge \Diamond q \to \Diamond(\Diamond^+ p \wedge \Diamond^+ q) \qquad (*)$$

where $\Diamond^+\varphi := \varphi \vee \Diamond\varphi$. Furthermore, an instance of the **GL** axiom looks as follows:

$$\Diamond(\Diamond^+ p \wedge \Diamond^+ q) \to \Diamond(\Diamond^+ p \wedge \Diamond^+ q \wedge \Box(\Box^+ \neg p \vee \Box^+ \neg q)).$$

By the axiom (lin) we also have: $\Box(\Box^+\neg p \vee \Box^+\neg q) \to (\Box\neg p \vee \Box\neg q)$. So, using the monotonicity of $\Diamond$ we obtain:

$$\Diamond p \wedge \Diamond q \to \Diamond(\Diamond^+ p \wedge \Diamond^+ q \wedge (\Box\neg p \vee \Box\neg q)).$$

By boolean logic

$$\Diamond^+ p \wedge \Diamond^+ q \leftrightarrow (p \wedge q) \vee (p \wedge \Diamond q) \vee (\Diamond p \wedge q) \vee (\Diamond p \wedge \Diamond q) \qquad (**)$$

and

$$(\Box\neg p \vee \Box\neg q) \leftrightarrow \neg(\Diamond p \wedge \Diamond q).$$

Using these, together with the monotonicity of $\Diamond$ we finally arrive at:

$$\Diamond p \wedge \Diamond q \to \Diamond((p \wedge q) \vee (p \wedge \Diamond q) \vee (\Diamond p \wedge q)),$$

which is equivalent to (.3) since $\Diamond$ distributes over $\vee$.

To show the converse, we observe that (.3) implies (lin) even in the system **K**. Indeed, the formula $(*)$, which is the dual form of (lin), can be rewritten, using $(**)$ and the distribution of $\Diamond$ over $\vee$ as follows:

$$\Diamond p \wedge \Diamond q \to \Diamond(p \wedge q) \vee \Diamond(p \wedge \Diamond q) \vee \Diamond(\Diamond p \wedge q) \vee \Diamond(\Diamond p \wedge \Diamond q),$$

which is clearly a weakening of (.3). Therefore (.3) $\vdash_{\mathbf{GL}} (lin)$. □

It follows that a scattered space is a GL.3-space iff it validates (lin). To characterize such spaces, consider the following definition.

Definition 2 Call a scattered space *primal* if for each $x \in X$ and $U, V \in \tau$, $\{x\} \cup U \cup V \in \tau$ implies $\{x\} \cup U \in \tau$ or $\{x\} \cup V \in \tau$.

It can be shown that X is primal iff the collection of punctured open neighborhoods of each non-isolated point is prime filter in the Heyting algebra τ.

Theorem 7. *Let X be a scattered space. Then $X \vDash (lin)$ iff X is primal.*

Proof Let X be a scattered space together with a valuation ν. Let $P := \nu(p)$ and $Q := \nu(q)$ denote the truth-sets of p and q, respectively. Then the truth sets of $\Box^+ p$ and $\Box^+ q$ are $I_\tau P$ and $I_\tau Q$, where I_τ is the interior operator of X. We write $x \vDash \varphi$ for $X, x \vDash_\nu \varphi$.

Suppose X is primal and for some valuation $x \vDash \Box(\Box^+ p \vee \Box^+ q)$. Then there exists an open neighborhood W of x such that $W \setminus \{x\} \vDash \Box^+ p \vee \Box^+ q$. In other words, $W \setminus \{x\} \subseteq I_\tau P \cup I_\tau Q$. Let $U = W \cap I_\tau P \in \tau$ and $V = W \cap I_\tau Q \in \tau$. Then $\{x\} \cup U \cup V = W \in \tau$. It follows that either $\{x\} \cup U \in \tau$ or $\{x\} \cup V \in \tau$. Hence $x \vDash \Box p$ or $x \vDash \Box q$. This proves that $X \vDash (lin)$.

Suppose now X is not primal. Then there exist $x \in X$ and $U, V \in \tau$ such that $\{x\} \cup U \cup V \in \tau$, but $\{x\} \cup U \notin \tau$ and $\{x\} \cup V \notin \tau$. Take a valuation such that $P = U$ and $Q = V$. Then clearly $x \vDash \Box(\Box^+ p \vee \Box^+ q)$. However, neither $x \vDash \Box p$ nor $x \vDash \Box q$ is true. Indeed, if, for example, $x \vDash \Box p$, then there exists an open neighborhood W of x such that $W \setminus \{x\} \subseteq P = U$. But then $\{x\} \cup U = W \cup U \in \tau$, which is a contradiction. This shows that $X \nvDash (lin)$. □

Example 4. (primal spaces) The left topology of any well-founded linear order is clearly primal. To give an example of a primal space not coming from order, consider any countable set A, a point $b \notin A$ and a free ultrafilter $\mathbf{u}$ over A. Then the set $A \cup \{b\}$ with the topology $\wp(A) \cup \{U \cup \{b\} \mid U \in \mathbf{u}\}$ is easily seen to be primal. This space is homeomorphic to a subspace of the Stone-Čech compactification of a countable discrete space A defined by $A \cup \{\mathbf{u}\}$.

The primal scattered spaces are closely related to *maximal scattered* spaces of [5]. A scattered space is called *maximal* if it does not have any proper refinements with the same rank function. It is easy to see that each maximal scattered space is primal, but there are primal spaces which are not maximal. The two notions do coincide for the scattered spaces of finite rank. It follows that the logic of maximal scattered spaces is **GL.3**.

10.5 GLP-Algebras and Polymodal Provability Logic

A natural generalization of provability logic **GL** to a language with infinitely many modal diamonds $\langle 0 \rangle$, $\langle 1 \rangle$, … has been introduced in 1986 by Japaridze [40]. He interpreted $\langle 1 \rangle \varphi$ as an arithmetical statement expressing the *ω-consistency of* φ over a given gödelian theory T.[2] Similarly, $\langle n \rangle \varphi$ was interpreted as the consistency of the extension of $T + \varphi$ by n nested applications of the ω-rule.

While the logic of each of the individual modalities $\langle n \rangle$ over Peano Arithmetic was known to coincide with **GL** by a relatively straightforward extension of the Solovay theorem [20], Japaridze found a complete axiomatization of the *joint* logic of the modalities $\langle n \rangle$ for all $n \in \omega$. This result involved considerable technical difficulties and lead to one of the first genuine extensions of Solovay's arithmetical fixed-point construction. Later, Japaridze's work has been simplified and extended by Ignatiev [39] and Boolos [21]. In particular, Ignatiev showed that **GLP** is complete for more general sequences of 'strong' provability predicates in arithmetic and analyzed the variable-free fragment of **GLP**. Boolos included a treatment of **GLB** (the fragment of **GLP** with just two modalities) in his popular book on provability logic [22].

More recently, **GLP** has found interesting applications in proof-theoretic analysis of arithmetic [2, 6, 7, 9] which stimulated some further interest in the study of modal-logical properties of **GLP** [11, 15, 23, 38]. For such applications, the algebraic language appears to be more natural and a different choice of the interpretation of the provability predicates is needed. The relevant structures have been introduced in [6] under the name of *graded provability algebras*.

[2] A gödelian theory U is ω-consistent if its extension by unnested applications of the ω-rule $U' := U + \{\forall x\, \varphi(x) : \forall n\, U \vdash \varphi(\underline{n})\}$ is consistent.

Recall that an arithmetical formula is called Π_n if it can be obtained from a formula containing only bounded quantifiers $\forall x \leq t$ and $\exists x \leq t$ by a prefix of n alternating blocks of quantifiers starting from $\forall$. Arithmetical Σ_n-formulas are defined dually.

Let T be a gödelian theory. T is called *n-consistent* if T together with all true arithmetical Π_n-sentences is consistent. (Alternatively, T is n-consistent iff every Σ_n-sentence provable in T is true.) Let $n\text{-}\mathsf{Con}(T)$ denote an arithmetical formula expressing the n-consistency of T (it can be defined using the standard Π_n-definition of truth for Π_n-sentences in arithmetic). Since we assume T to be recursively enumerable, it is easy to check that the formula $n\text{-}\mathsf{Con}(T)$ itself belongs to the class Π_{n+1}.

The n-consistency formula induces an operator $\langle n\rangle_T$ acting on the Lindenbaum–Tarski algebra $\mathscr{L}_T$:

$$\langle n\rangle_T : [\varphi] \longmapsto [n\text{-}\mathsf{Con}(T+\varphi)].$$

The dual *n-provability* operators are defined by $[n]_T x = \neg\langle n\rangle_T \neg x$ for all $x \in \mathscr{L}_T$. Since every true Π_n-sentence is assumed to be an axiom for n-provability, we notice that every true Σ_{n+1}-sentence must be n-provable. Moreover, this latter fact is formalizable in T, so we obtain the following lemma (see [54]). (By the abuse of notation we denote by $[n]_T\varphi$ the arithmetical formula expressing the n-provability of φ in T.)

Lemma 8. *For each true Σ_{n+1}-formula $\sigma(x)$, $T \vdash \forall x\,(\sigma(x) \to [n]_T\sigma(\underline{x}))$.*

As a corollary we obtain a basic observation probably due to Smorynski [54].

Proposition 3. *For each $n \in \omega$, the structure $(\mathscr{L}_T, \langle n\rangle_T)$ is a Magari algebra.*

A proof of this fact consists of verifying the Hilbert–Bernays–Löb derivability conditions for $[n]_T$ in T and of deducing from them, in the usual way, an analog of Löb's theorem for $[n]_T$.

The structure $(\mathscr{L}_T, \{\langle n\rangle_T : n \in \omega\})$ is called the *graded provability algebra of T* or the *GLP-algebra of T*. Apart from the identities inherited from the structure of Magari algebras for each $\langle n\rangle$, it satisfies the following principles for all $m < n$:

P1. $\langle m\rangle x \leq [n]\langle m\rangle x$;
P2. $\langle n\rangle x \leq \langle m\rangle x$.

The validity of P1 follows from Lemma 8 because the formula $\langle m\rangle_T\varphi$, for any φ, belongs to the class Π_{m+1}. P2 holds since $\langle n\rangle_T\varphi$ asserts the consistency of a stronger theory than $\langle m\rangle_T\varphi$ for $m < n$.

In general, we call a *GLP-algebra* a structure $(M, \{\langle n\rangle : n \in \omega\})$ such that each $(M, \langle n\rangle)$ is a Magari algebra and conditions P1, P2 (that are equivalent to identities) are satisfied for all $x \in M$.

At this point it is worth noticing that condition P1 has an equivalent form that has proved to be quite useful in the study of GLP-algebras.

Lemma 9. *Modulo the other identities of GLP-algebras, P1 is equivalent to*

P1'. $\langle n\rangle y \wedge \langle m\rangle x = \langle n\rangle(y \wedge \langle m\rangle x)$ *for all* $m < n$.

Proof First, we prove P1′. We have $y \wedge \langle m\rangle x \leq y$, hence $\langle n\rangle(y \wedge \langle m\rangle x) \leq \langle n\rangle y$. Similarly, by P2 and transitivity, $\langle n\rangle(y \wedge \langle m\rangle x) \leq \langle n\rangle\langle m\rangle x \leq \langle m\rangle\langle m\rangle x \leq \langle m\rangle x$. Hence, $\langle n\rangle(y \wedge \langle m\rangle x) \leq \langle n\rangle y \wedge \langle m\rangle x$. In the other direction, by P1, $\langle n\rangle y \wedge \langle m\rangle x \leq \langle n\rangle y \wedge [n]\langle m\rangle x$. However, as in any modal algebra, we also have $\langle n\rangle y \wedge [n]z \leq \langle n\rangle(y \wedge z)$. It follows that $\langle n\rangle y \wedge [n]\langle m\rangle x \leq \langle n\rangle(y \wedge \langle m\rangle x)$. Thus, P1′ is proved.

To infer P1 from P1′ it is sufficient to prove that $\langle m\rangle x \wedge \neg[n]\langle m\rangle x = \bot$. We have that $\neg[n]\langle m\rangle x = \langle n\rangle\neg\langle m\rangle x$. Therefore, by P1′, $\langle m\rangle x \wedge \langle n\rangle\neg\langle m\rangle x = \langle n\rangle(\neg\langle m\rangle x \wedge \langle m\rangle x) = \langle n\rangle\bot = \bot$, as required. □

An equivalent formulation of Japaridze's arithmetical completeness theorem is that any identity of $(\mathscr{L}_T, \{\langle n\rangle_T : n \in \omega\})$ follows from the identities of GLP-algebras [40]. It is somewhat strengthened to the current formulation in [13, 39].

Theorem 8. (Japaridze) *Suppose* T *is gödelian,* T *contains Peano Arithmetic, and* $\mathrm{ch}(\mathscr{L}_T, \langle n\rangle_T) = \infty$ *for each* $n < \omega$*. Then, an identity holds in* $(\mathscr{L}_T, \{\langle n\rangle_T : n \in \omega\})$ *iff it holds in all GLP-algebras.*

We note that the condition $\mathrm{ch}(\mathscr{L}_T, \langle n\rangle_T) = \infty$, for each $n \in \omega$, is equivalent to $T + n\text{-}\mathsf{Con}(T)$ being consistent for each $n \in \omega$, and is clearly necessary for the validity of Japaridze's theorem.

The logic of all GLP-algebras can also be axiomatized as a Hilbert-style calculus (see the footnote in Sect. 10.1). The corresponding system **GLP** was originally introduced by Japaridze. **GLP** is formulated in the language of propositional logic enriched by modalities $[n]$ for all $n \in \omega$. The axioms of **GLP** are those of **GL**, formulated for each $[n]$, as well as the two analogs of P1 and P2 for all $m < n$:

P1. $\langle m\rangle\varphi \to [n]\langle m\rangle\varphi$;
P2. $[m]\varphi \to [n]\varphi$.

The inference rules of **GLP** are modus ponens and $\varphi/[n]\varphi$ for each $n \in \omega$.

We let $\mathbf{GLP}_n$ denote the fragment of **GLP** in the language with the first n modalities; thus **GLB** is $\mathbf{GLP}_2$.

For any modal formula φ, $\mathbf{GLP} \vdash \varphi$ iff the identity $t_\varphi = \top$ holds in all **GLP**-algebras. Hence, **GLP** coincides with the logic of all GLP-algebras as well as with the logic of the GLP-algebra of T for any theory T such that $T + n\text{-}\mathsf{Con}(T)$ is consistent for each $n < \omega$.

10.6 GLP-Spaces

Topological semantics for **GLP** has been first considered in [14]. The main difficulty in the modal-logical study of **GLP** comes from the fact that it is incomplete with respect to its relational semantics; that is, **GLP** is the logic of no class of *frames* [22]. Even though a suitable class of relational *models* for which **GLP** is sound and complete was developed in [11], these models are not so easy to handle. So, it is natural to consider a generalization of the topological semantics we have for **GL**. As it turns out, topological semantics provides another natural class of GLP-algebras which is interesting in its own right, and also due to its analogy with the proof-theoretic GLP-algebras.

As before, we are interested in GLP-algebras of the form $(\mathscr{P}(X), \{\langle n\rangle : n \in \omega\})$, where $\mathscr{P}(X)$ is the boolean algebra of subsets of a given set X. Since each $(\mathscr{P}(X), \langle n\rangle)$ is a Magari algebra, the operator $\langle n\rangle$ is the derivative operator with respect to some uniquely defined scattered topology on X. Thus, we come to the following definition [14].

A polytopological space $(X, \{\tau_n : n \in \omega\})$ is called a *GLP-space* if the following conditions hold for each $n \in \omega$:

D0. (X, τ_n) is a scattered space;
D1. For each $A \subseteq X$, $d_{\tau_n}(A)$ is τ_{n+1}-open;
D2. $\tau_n \subseteq \tau_{n+1}$.

We notice that the last two conditions directly correspond to conditions P1 and P2 of GLP-algebras. By a *GLP$_m$-space* we mean a space $(X, \{\tau_n : n < m\})$ satisfying conditions D0–D2 for the first m topologies.

Proposition 4. *(i) If $(X, \{\tau_n : n \in \omega\})$ is a GLP-space, then the structure $(\mathscr{P}(X), \{d_{\tau_n} : n \in \omega\})$ is a GLP-algebra.*
(ii) If $(\mathscr{P}(X), \{\langle n\rangle : n \in \omega\})$ is a GLP-algebra, then there are uniquely defined topologies $\{\tau_n : n \in \omega\}$ on X such that $(X, \{\tau_n : n \in \omega\})$ is a GLP-space and $\langle n\rangle = d_{\tau_n}$ for each $n < \omega$.

Proof (i) Suppose $(X, \{\tau_n : n \in \omega\})$ is a GLP-space. Let $d_n := d_{\tau_n}$ denote the corresponding derivative operators and let $\tilde{d}_n$ denote its dual $\tilde{d}_n(A) := X \setminus d_n(X \setminus A)$.[3] By Theorem 3 $(\mathscr{P}(X), d_n)$ is a Magari algebra for each $n \in \omega$. Notice that $A \in \tau_n$ iff $A \subseteq \tilde{d}_n A$. If $m < n$, then $d_m A \in \tau_n$, so $d_m A \subseteq \tilde{d}_n d_m A$, hence P1 holds. Since $\tau_n \subseteq \tau_{n+1}$, we have $d_{n+1} A \subseteq d_n A$, thus P2 holds.

(ii) Let $(\mathscr{P}(X), \{\langle n\rangle : n \in \omega\})$ be a GLP-algebra. Since each of the algebras $(\mathscr{P}(X), \langle n\rangle)$ is Magari, by Theorem 4 a scattered topology τ_n on X is defined for which $\langle n\rangle = d_{\tau_n}$. In fact, we have $U \in \tau_n$ iff $U \subseteq [n]U$. We check that conditions D1 and D2 are met.

Suppose A is τ_n-closed, that is, $\langle n\rangle A \subseteq A$. Then $\langle n+1\rangle A \subseteq \langle n\rangle A \subseteq A$ by P2. Hence, A is τ_{n+1}-closed. Thus, $\tau_n \subseteq \tau_{n+1}$.

By P1 for any set A we have $\langle n\rangle A \subseteq [n+1]\langle n\rangle A$. Hence, $d_{\tau_n}(A) = \langle n\rangle A \in \tau_{n+1}$. Thus, $(X, \{\tau_n : n \in \omega\})$ is a GLP-space. □

To obtain examples of GLP-spaces let us first consider the case of two modalities. The following basic example is due to Esakia (private communication, see [14]).

[3] There is no conventional name for the dual of the derivative operator. Sometimes it is denoted by t. Here we choose the notation $\tilde{d}$ to emphasize its connection with d.

Example 5. Consider a bitopological space $(\Omega; \tau_0, \tau_1)$, where Ω is an ordinal, τ_0 is its left topology, and τ_1 is its interval topology. Esakia noticed that this space is a model of **GLB**, that is, in our terminology, a GLP_2-space. In fact, for any $A \subseteq \Omega$ the set $d_0(A) = (\min A, \Omega)$ is an open interval, whenever A is not empty. Hence, D1 holds (the other two conditions are immediate). Esakia also noticed that such spaces can never be complete for **GLP** as the linearity axiom (.3) holds for $\langle 0 \rangle$.

In general, to define GLP_n-spaces for $n > 1$, we introduce an operation $\tau \longmapsto \tau^+$ on topologies on a given set X. This operation plays a central role in the study of GLP-spaces.

Given a topological space (X, τ), let τ^+ be the coarsest topology containing τ such that each set of the form $d_\tau(A)$, with $A \subseteq X$, is open in τ^+. Thus, τ^+ is generated by τ and $\{d_\tau(A) : A \subseteq X\}$. Clearly, τ^+ is the coarsest topology on X such that $(X; \tau, \tau^+)$ is a GLP_2-space. Sometimes we call τ^+ the *derivative topology* of (X, τ).

Getting back to Esakia's example, it is easy to verify that, on any ordinal Ω, the derivative topology of the left topology coincides with the interval topology. (In fact, any open interval is an intersection of a downset and an open upset.)

Example 6. Even though we are mainly interested in scattered spaces, the derivative topology makes sense for arbitrary spaces. The reader can check that if τ is the *coarsest* topology on a set X (whose open sets are just X and $\varnothing$), then τ^+ is the *cofinite* topology on X (whose open sets are exactly the cofinite subsets of X together with $\varnothing$). On the other hand, if τ is the cofinite topology, then $\tau^+ = \tau$. We note that the logic of the cofinite topology on an infinite set is **KD45** (see [57]).

For scattered spaces, τ^+ is always strictly finer than τ, unless τ is discrete. We present a proof using the language of Magari algebras.

Proposition 5. *If (X, τ) is scattered, then $d_\tau(X)$ is not open, unless $d_\tau(X) = \varnothing$.*

Proof The set $d_\tau(X)$ corresponds to the element $\Diamond\top$ in the associated Magari algebra; $d_\tau(X)$ being open means $\Diamond\top \leq \Box\Diamond\top$. By M2 we have $\Box\Diamond\top \leq \Box\bot = \neg\Diamond\top$. Hence, $\Diamond\top \leq \neg\Diamond\top$, that is, $\Diamond\top = \bot$. This means $d_\tau(X) = \varnothing$. □

We will see later that τ^+ can be much finer than τ. Notice that if τ is T_d, then each set of the form $d_\tau(A)$ is τ-closed. Hence, it will be clopen in τ^+. Thus, τ^+ is obtained by adding to τ new clopen sets. In particular, τ^+ will be zero-dimensional if so is τ.[4]

Iterating the plus operation yields a GLP-space. Let (X, τ) be a scattered space. Define: $\tau_0 := \tau$ and $\tau_{n+1} := \tau_n^+$. Then $(X, \{\tau_n : n \in \omega\})$ is a GLP-space that will be called the GLP-space *generated from* (X, τ) or simply the *generated GLP-space*.

Thus, from any scattered space we can always produce a GLP-space in a natural way. The question is whether this space will be nontrivial, that is, whether we can guarantee that the topologies τ_n are non-discrete.

[4] Recall that a topological space is zero-dimensional if it has a base of clopen sets.

In fact, the next observation from [14] shows that for many natural τ already the topology τ^+ will be discrete. Recall that a topological space X is *first-countable* if every point $x \in X$ has a countable basis of open neighborhoods.

Proposition 6. *If (X, τ) is Hausdorff and first-countable, then τ^+ is discrete.*

Proof It is easy to see that if (X, τ) is first-countable and Hausdorff, then every point $a \in d_\tau(X)$ is a (unique) limit point of a countable sequence of points $A = \{a_n\}_{n\in\omega}$. Hence, there is a set $A \subseteq X$ such that $d_\tau(A) = \{a\}$. By D1 this means that $\{a\}$ is τ^+-open. □

Thus, if τ is the interval topology on a countable ordinal, then τ^+ is discrete. The same holds, for example, if τ is the (non-scattered) topology of the real line.

We remark that the left topology τ on any countable ordinal $> \omega$ yields an example of a non-Hausdorff first-countable space such that τ^+ is non-discrete. In the following section we will also see that if τ is the interval topology on any ordinal $> \omega_1$, then τ^+ is non-discrete (ω_1 is its least non-isolated point). However, we do not have any topological characterization of spaces (X, τ) such that τ^+ is discrete. (See, however, Proposition 8, which provides a characterization in terms of d-reflection.)

Given an arbitrary scattered topology τ, it is natural to ask about the separation properties of τ^+. In fact, for τ^+ we can infer a bit more separation than for an arbitrary scattered topology. Recall that a topological space X is T_1 if for any two different points $a, b \in X$ there is an open set U such that $a \in U$ and $b \notin U$.

Proposition 7. *Let (X, τ) be any topological space. Then (X, τ^+) is T_1.*

Proof Let $a, b \in X$, $a \neq b$. Consider the set $B := d_\tau(\{b\})$, which is open in τ^+. We either have $a \in B$ (and $b \notin B$ by definition) or a belongs to the complement of the closure of $\{b\}$. □

The following example shows that, in general, τ^+ need not always be Hausdorff.

Example 7. Let $(X, \prec)$ be a strict partial ordering on $X := \omega \cup \{a, b\}$, where ω is taken with its natural order, a and b are $\prec$-incomparable, and $n \prec a, b$ for all $n \in \omega$. Let τ be the left topology on $(X, \prec)$. Since $\prec$ is well-founded, τ is scattered.

Notice that for any $A \subseteq X$ we have $d_\tau(A) = \{x \in X : \exists y \in A\ y \prec x\}$. Hence, if A intersects ω, then $d_\tau(A)$ contains an end-segment of ω. Otherwise, $d_\tau(A) = \varnothing$. It follows that a base of open neighborhoods of a in τ^+ consists of sets of the form $I \cup \{a\}$, where I is an end-segment of ω. Similarly, sets of the form $I \cup \{b\}$ are a base of open neighborhoods of b. But any two such sets have a non-empty intersection.

10.7 d-Reflection

In the next section we are going to describe in some detail the GLP-space generated from the left topology on the ordinals. Strikingly, we will see that it naturally leads to some of the central notions of combinatorial set theory, such as Mahlo operation and stationary reflection. In fact, part of our analysis can be easily stated using the language of modal logic for arbitrary generated GLP-spaces. In this section we provide a necessary setup and characterize the topologies of a generated GLP-space in terms of what we call *d-reflection*.[5]

Throughout this section we fix a topological space (X, τ) and let $d = d_\tau$.

Definition 3 A point $a \in X$ is called *d-reflexive* if $a \in dX$ and, for each $A \subseteq X$,

$$a \in dA \Rightarrow a \in d(dA).$$

In modal logic terms this means that the formula $\Diamond\top \wedge (\Diamond p \to \Diamond\Diamond p)$ is valid at $a \in X$ for any evaluation of the variable p in (X, τ).

Similarly, a point $a \in X$ is called *m-fold d-reflexive* if $a \in dX$ and for each $A_1, \dots, A_m \subseteq X$,

$$a \in dA_1 \cap \dots \cap dA_m \Rightarrow a \in d(dA_1 \cap \dots \cap dA_m).$$

2-fold d-reflexive points will also be called *doubly d-reflexive* points. Expressed with the help of the modal language, $a \in X$ is doubly d-reflexive iff the formula $\Diamond\top \wedge (\Diamond p \wedge \Diamond q \to \Diamond(\Diamond p \wedge \Diamond q))$ is valid at a for any evaluation of p, q.

Lemma 10. *Let (X, τ) be a T_d-space. Each doubly d-reflexive point $x \in X$ is m-fold d-reflexive for any finite m.*

Proof The argument goes by induction on $m \geq 2$. Suppose $x \in dA_1 \cap \dots \cap dA_{m+1}$, then $x \in dA_1 \cap \dots \cap dA_m$ and $x \in dA_{m+1}$. By induction hypothesis, $x \in d(dA_1 \cap \dots \cap dA_m)$ and by 2-fold reflection $x \in d(d(dA_1 \cap \dots \cap dA_m) \cap dA_{m+1})$. However, by T_d property $d(dA_1 \cap \dots \cap dA_m) \subseteq dA_1 \cap \dots \cap dA_m$, hence $x \in d(dA_1 \cap \dots \cap dA_m \cap dA_{m+1})$, as required. □

Proposition 8. *Let (X, τ) be a T_d-space. A point $x \in X$ is doubly d-reflexive iff x is a limit point of (X, τ^+).*

Proof For the (if) direction, we give an argument in the algebraic format. In fact, it is sufficient to show the following inequality in the algebra of (X, τ) for any elements $p, q \subseteq X$:

$$\langle 1\rangle\top \wedge \langle 0\rangle p \wedge \langle 0\rangle q \leq \langle 0\rangle(\langle 0\rangle p \wedge \langle 0\rangle q).$$

Notice that by Lemma 9, $\langle 1\rangle\top \wedge \langle 0\rangle p = \langle 1\rangle(\top \wedge \langle 0\rangle p) = \langle 1\rangle\langle 0\rangle p$. Hence, using P1′ once again, we obtain: $\langle 1\rangle\top \wedge \langle 0\rangle p \wedge \langle 0\rangle q = \langle 1\rangle\langle 0\rangle p \wedge \langle 0\rangle q = \langle 1\rangle(\langle 0\rangle p \wedge \langle 0\rangle q)$. The latter formula can be weakened to $\langle 0\rangle(\langle 0\rangle p \wedge \langle 0\rangle q)$ by P2, as required.

[5] Curiously, the reader may notice that the notion of *reflection principle* as used in provability logic and formal arithmetic matches very nicely the notions such as *stationary reflection* in set theory. (As far as we know, the two terms have evolved completely independently from one another.)

For the (only if) direction, it is sufficient to show that each doubly d-reflexive point of (X, τ) is a limit point of τ^+. Suppose x is doubly d-reflexive. By Lemma 10, x is m-fold d-reflexive. Any basic open subset of τ^+ has the form $U := A_0 \cap dA_1 \cap \cdots \cap dA_m$, where $A_0 \in \tau$. Assume $x \in U$, we have to find a point $y \neq x$ such that $y \in U$.

Since $x \in dA_1 \cap \cdots \cap dA_m$, by m-fold d-reflexivity we obtain $x \in d(dA_1 \cap \cdots \cap dA_m)$. Since A_0 is an open neighborhood of x, there is a $y \in A_0$ such that $y \neq x$ and $y \in dA_1 \cap \cdots \cap dA_m$. Hence, $y \in U$ and $y \neq x$, as required. □

Let d^+ denote the derivative operator associated with τ^+. We obtain the following characterization of derived topology in terms of neighborhoods.

Proposition 9. *Let (X, τ) be a T_d-space. A subset $U \subseteq X$ contains a τ^+-neighborhood of $x \in X$ iff one of the following two cases holds:*

(i) x is not doubly d-reflexive and $x \in U$;
(ii) x is doubly d-reflexive and there is an $A \in \tau$ and a B such that $x \in A \cap dB \subseteq U$.

Proof Since (i) ensures that x is τ^+-isolated by Proposition 8, each condition is clearly sufficient for U to contain a τ^+-neighborhood of x. To prove the converse, assume that U contains a τ^+-neighborhood of x. This means $x \in A \cap dA_1 \cap \cdots \cap dA_m \subseteq U$ for some $A, A_1, \ldots, A_m$ with $A \in \tau$. If x is τ^+-isolated, condition (i) holds. Otherwise, $x \in d^+X$. Let $B := dA_1 \cap \cdots \cap dA_m$. Since B is closed in τ we have $dB \subseteq B$, hence $A \cap dB \subseteq U$. It remains to show that $x \in A \cap dB$. By Lemma 9, $B \cap d^+X = d^+B \subseteq dB$. Hence, $x \in A \cap B \cap d^+X \subseteq A \cap dB$. □

Remark 1. Since in clause (ii) of Proposition 9 the set A is open, we have $A \cap dB = A \cap d(A \cap B)$ for any B. Hence, we may assume $B \subseteq A$.

Corollary 2. *Let (X, τ) be a T_d-space. Then, for all $x \in X$ and $A \subseteq X$, $x \in d^+A$ iff the following two conditions hold:*

(i) x is doubly d-reflexive;
(ii) For all $B \subseteq X$, $x \in dB \Rightarrow x \in d(A \cap dB)$.

Proof The fact that (i) and (ii) are necessary is proved using Proposition 8 and the inequality $d^+A \cap dB = d^+(A \cap dB) \subseteq d(A \cap dB)$. We prove that (i) and (ii) are sufficient. Assume $x \in U \in \tau^+$. By Proposition 9 we may assume that U has the form $V \cap dB$, where $V \in \tau$. By (ii), from $x \in dB$ we obtain $x \in d(A \cap dB)$. Hence, there is a $y \neq x$ such that $y \in V$ and $y \in A \cap dB$. It follows that $y \in A$ and $y \in V \cap dB = U$. ⊣ □

10.8 The Ordinal GLP-Space

Here we discuss the GLP-space generated from the left topology on the ordinals, that is, the GLP-space $(\Omega; \{\tau_n : n \in \omega\})$, where Ω is a fixed ordinal, τ_0 is the left topology on Ω and $\tau_{n+1} = \tau_n^+$ for each $n \in \omega$. The material in this section comes from a so far unpublished manuscript of the first author [10]. Our basic findings are summarized in the following table, to which we provide extended comments below.

The rows of the table correspond to topologies τ_n. The first column contains the name of the topology (the first two are standard, the third one is introduced in [14], the fourth one is introduced here). The second column indicates the first limit point of τ_n, which is denoted θ_n. The last column describes the derivative operator associated with τ_n. We note that θ_3 is a large cardinal which is sometimes referred to as *the first cardinal reflecting for pairs of stationary sets* (see below), but we know no special notation for this cardinal.

	Name	θ_n	$d_n(A)$
τ_0	Left	1	$\{\alpha : A \cap \alpha \neq \varnothing\}$
τ_1	Interval	ω	$\{\alpha \in Lim : A \cap \alpha$ is unbounded in $\alpha\}$
τ_2	Club	ω_1	$\{\alpha : \mathrm{cf}(\alpha) > \omega$ and $A \cap \alpha$ is stationary in $\alpha\}$
τ_3	Mahlo	θ_3	$\ldots \ldots$

We have already seen that the derivative topology of the left topology is exactly the interval topology. Therefore, basic facts related to the first two rows of the table are rather clear. We turn to the next topology τ_2.

Club topology. Recall that the *cofinality* $\mathrm{cf}(\alpha)$ of a limit ordinal α is the least order type of a cofinal subset of α; $\mathrm{cf}(\alpha) := 0$ if $\alpha \notin \mathrm{Lim}$. (We use the words *cofinal in* α and *unbounded in* α as synonyms.) An ordinal α is *regular* if $\mathrm{cf}(\alpha) = \alpha$.

To characterize τ_2 we apply Proposition 9, hence it is useful to see what corresponds to the notion of doubly d-reflexive point of the interval topology.

Lemma 11. *For any ordinal α, α is d_1-reflexive iff α is doubly d_1-reflexive iff* $\mathrm{cf}(\alpha) > \omega$.

Proof d_1-reflexivity of α means that $\alpha \in \mathrm{Lim}$ and, for all $A \subseteq \alpha$, if A is cofinal in α, then $d_1(A)$ is cofinal in α. If $\mathrm{cf}(\alpha) = \omega$, then there is an increasing sequence $(\alpha_n)_{n\in\omega}$ such that $\sup\{\alpha_n : n \in \omega\} = \alpha$. Then, for $A := \{\alpha_n : n \in \omega\}$ we obviously have $d_1(A) = \{\alpha\}$, hence A violates the reflexivity property. Therefore, d_1-reflexivity of α implies $\mathrm{cf}(\alpha) > \omega$.

Now we show that $\mathrm{cf}(\alpha) > \omega$ implies α is doubly d_1-reflexive. Suppose $\mathrm{cf}(\alpha) > \omega$ and $A, B \subseteq \alpha$ are both cofinal in α. We show that $d_1 A \cap d_1 B$ is cofinal in α. Assume $\beta < \alpha$. Using the cofinality of A, B we can construct an increasing sequence $(\gamma_n)_{n\in\omega}$ above β such that $\gamma_n \in A$ for even n, and $\gamma_n \in B$ for odd n. Let $\gamma := \sup\{\gamma_n : n < \omega\}$. Obviously, both A and B are cofinal in γ whence $\gamma \in d_1 A \cap d_1 B$. Since $\mathrm{cf}(\alpha) > \omega$ and $\mathrm{cf}(\gamma) = \omega$, we have $\gamma < \alpha$. □

Corollary 3. *Limit points of τ_2 are exactly the ordinals of uncountable cofinality.*

It turns out that topology τ_2 is strongly related to the well-known concept of a *club filter*, i.e., the filter generated by all clubs on a limit ordinal. Recall that a subset $C \subseteq \alpha$ is called a *club* in α if C is closed in the interval topology of α and unbounded in α.

Proposition 10. *Assume* $\mathrm{cf}(\alpha) > \omega$. *The following statements are equivalent:*

(i) *U contains a τ_2-neighborhood of α;*
(ii) *There is a $B \subseteq \alpha$ such that $\alpha \in d_1 B \subseteq U$;*
(iii) *$\alpha \in U$ and U contains a club in α;*
(vi) *$\alpha \in U$ and $U \cap \alpha$ belongs to the club filter on α.*

Proof Statement (ii) implies (iii) since $\alpha \cap d_1 B$ is a club in α whenever $\alpha \in d_1 B$. Statement (iii) implies (iv) for obvious reasons.

Statement (iv) implies (i). If C is a club in α, then $C \cup \{\alpha\}$ contains a τ_2-neighborhood $d_1 C$ of α. Indeed, $d_1 C$ is τ_2-open, contains α, and $d_1 C \subseteq C \cup \{\alpha\}$ since C is τ_1-closed in α.

Statement (i) implies (ii). Assume U contains a τ_2-neighborhood of α. Since $\mathrm{cf}(\alpha) > \omega$, by Lemma 11 and Proposition 9 there is an $A \in \tau_1$ and a B_1 such that $\alpha \in A \cap d_1 B_1 \subseteq U$. Since A is a τ_1-neighborhood of α, by Proposition 9 again there are $A_0 \in \tau_0$ and B_0 such that $\alpha \in A_0 \cap d_0 B_0$. Since τ_0 is the left topology, we may assume that A_0 is the minimal τ_0-neighborhood $[0, \alpha]$ of α. Besides, we have $\alpha \in d_0 B_0 \cap d_1 B_1 = d_1(B_1 \cap d_0 B_0) \subseteq U$. Since $[0, \alpha]$ is τ_1-clopen, $d_1(C \cap \alpha) = [0, \alpha] \cap d_1 C$ for any C, so we can take $B_1 \cap d_0 B_0 \cap \alpha$ for B. □

Corollary 4. *τ_2 is the unique topology on Ω such that*

- *If $\mathrm{cf}(\alpha) \leq \omega$, then α is an isolated point;*
- *If $\mathrm{cf}(\alpha) > \omega$, then, for any $U \subseteq \Omega$, U contains a neighborhood of α iff $\alpha \in U$ and U contains a club in α.*

Hence, we may call τ_2 the *club topology*.

The derivative operation for the club topology is also well known in set theory. Recall the following definition for $\mathrm{cf}(\alpha) > \omega$.

A subset $A \subseteq \alpha$ is called *stationary in* α if A intersects every club in α. Observe that this happens exactly when α is a limit point of A in τ_2, so

$$d_2(A) = \{\alpha : \mathrm{cf}(\alpha) > \omega \text{ and } A \cap \alpha \text{ is stationary in } \alpha\}.$$

The map d_2 is usually called the *Mahlo operation* (see [41], where d_2 is denoted Tr). Its main significance is associated with the notion of Mahlo cardinal, one of the basic examples of large cardinals in set theory. Let Reg denote the class of regular cardinals; the ordinals in $d_2(\mathrm{Reg})$ are called *weakly Mahlo cardinals*. Their existence implies the consistency of ZFC, as well as the consistency of ZFC together with the assertion 'inaccessible cardinals exist.'

Now we turn to topology τ_3.

Stationary reflection and Mahlo topology. Since the open sets of τ_3 are generated by the Mahlo operation, we call τ_3 *Mahlo topology*. It turns out to be intrinsically connected with *stationary reflection*, an extensively studied phenomenon in set theory (see [32, Chaps. 1, 15]).

We adopt the following terminology. An ordinal λ is called *reflecting* if $\mathrm{cf}(\lambda) > \omega$ and, whenever A is stationary in λ, there is an $\alpha < \lambda$ such that $A \cap \alpha$ is stationary in

α. Similarly, λ is *doubly reflecting* if $\mathrm{cf}(\lambda) > \omega$ and whenever A, B are stationary in λ there is an $\alpha < \lambda$ such that both $A \cap \alpha$ and $B \cap \alpha$ are stationary in α.

Mekler and Shelah's notion of *reflection cardinal* [49] is somewhat more general than the one given here, however it has the same consistency strength. Reflection for pairs of stationary sets has been introduced by Magidor [47]. Since d_2 coincides with the Mahlo operation, we immediately obtain the following statement.

Proposition 11. *(i) λ is reflecting iff λ is d_2-reflexive;*
(ii) λ is doubly reflecting iff λ is doubly d_2-reflexive;
(iii) λ is a non-isolated point in τ_3 iff λ is doubly reflecting.

Together with the next proposition this yields a characterization of Mahlo topology in terms of neighborhoods.

Proposition 12. *Suppose λ is doubly reflecting. For any subset $U \subseteq \Omega$, the following conditions are equivalent:*

(i) U contains a τ_3-neighborhood of λ;
(ii) $\lambda \in U$ and there is a $B \subseteq \lambda$ such that $\lambda \in d_2 B \subseteq U$;
(iii) $\lambda \in U$ and there is a τ_2-closed (in the relative topology of λ) stationary $C \subseteq \lambda$ such that $C \subseteq U$.

Notice that the notion of τ_2-closed stationary C in (iii) is the analog of the notion of club for the τ_2-topology.

Proof Condition (ii) implies (iii). Since λ is reflecting, if $\lambda \in d_2 B$, then $\lambda \in d_2 d_2 B$, that is, $\lambda \cap d_2 B$ is stationary in λ. So we may take $C := \lambda \cap d_2 B$.

Condition (iii) implies (ii). If C is τ_2-closed and stationary in λ, then $d_2 C \subseteq C \cup \{\lambda\} \subseteq U$ and $\lambda \in d_2 C$. Thus, $\lambda \cap d_2 C$ can be taken for B.

Condition (ii) implies (i). If (ii) holds, U contains a subset of the form $d_2 B$. The latter is τ_3-open and contains λ, thus it is a neighborhood of λ.

For the converse direction, we note that by Proposition 9 U contains a subset of the form $A \cap d_2 B$, where $A \in \tau_2$, $B \subseteq A$ and $\lambda \in A \cap d_2 B$. Since A is a τ_2-neighborhood of λ, by Proposition 10 there is a set B_1 such that $\lambda \in [0, \lambda] \cap d_1 B_1 \subseteq A$. Then

$$\lambda \in [0, \lambda] \cap d_1 B_1 \cap d_2 B = [0, \lambda] \cap d_2(B \cap d_1 B_1).$$

Since $[0, \lambda]$ is clopen, we obtain $\lambda \in d_2 C$ with $C := B \cap d_1 B_1 \cap \lambda$. □

Reflecting and doubly reflecting cardinals are large cardinals in the sense that their existence implies consistency of ZFC. They have been studied by Mekler and Shelah [49] and Magidor [47] who investigated their consistency strength and related them to some other well-known large cardinals. By a result of Magidor, the existence of a doubly reflecting cardinal is equiconsistent with the existence of a *weakly compact cardinal*.[6] More precisely, the following proposition holds.

[6] Weakly compact cardinals are the same as Π^1_1-indescribable cardinals, see below.

Proposition 13. *(i) If λ is weakly compact, then λ is doubly reflecting.*
(ii) (Magidor) *If λ is doubly reflecting, then λ is weakly compact in L.*

Here, the first item is well known and easy. Magidor originally proved the analog of the second item for $\lambda = \aleph_2$ and stationary sets of ordinals of countable cofinality in $\aleph_2$. However, it has been remarked by Mekler and Shelah [49] that essentially the same proof yields the stated claim.[7]

Corollary 5. *Assertion "τ_3 is non-discrete" is equiconsistent with the existence of a weakly compact cardinal.*

Corollary 6. *If* ZFC *is consistent, then it is consistent with* ZFC *that τ_3 is discrete and hence that* **GLP**$_3$ *is incomplete w.r.t. any ordinal space.*

Recall that θ_n denotes the first non-isolated point of τ_n (in the space of all ordinals). We have: $\theta_0 = 1$, $\theta_1 = \omega$, $\theta_2 = \omega_1$, θ_3 is the first doubly reflecting cardinal.

ZFC does not know much about the location of θ_3, however the following facts are interesting.

- θ_3 is regular, but not a successor of a regular cardinal;
- While weakly compact cardinals are non-isolated, θ_3 need not be weakly compact: If infinitely many supercompact cardinals exist, then there is a model, where $\aleph_{\omega+1}$ is doubly reflecting [47];
- If θ_3 is a successor of a singular strong limit cardinal, then it is consistent that infinitely many Woodin cardinals exist, see [56].[8]

Further topologies. Further topologies of the ordinal GLP-space do not seem to have prominently occurred in set-theoretic work. They yield some large cardinal notions, for the statement that τ_n is non-discrete (equivalently, θ_n exists) implies the existence of a doubly reflecting cardinal for any $n > 2$. We do not know whether cardinals θ_n coincide with any of the standard large cardinal notions.

Here we give a sufficient condition for the topology τ_{n+2} to be non-discrete. We show that if there exists a Π^1_n*-indescribable cardinal*, then τ_{n+2} is non-discrete.

Let Q be a class of second order formulas over the standard first order set-theoretic language enriched by a unary predicate R. We assume Q to contain at least the class of all first order formulas (denoted Π^1_0). We shall consider standard models of that language of the form $(V_\alpha, \in, R)$, where α is an ordinal, V_α is the α-th class in the cumulative hierarchy, and R is a subset of V_α.

We would like to give a definition of Q-indescribable cardinals in topological terms. They can then be defined as follows.

Definition 4 For any sentence $\varphi \in Q$ and any $R \subseteq V_\kappa$, let $U_\kappa(\varphi, R)$ denote the set $\{\alpha \leq \kappa : (V_\alpha, \in, R \cap V_\alpha) \vDash \varphi\}$. The *$Q$-describable topology τ_Q* on Ω is generated by a subbase consisting of sets $U_\kappa(\varphi, R)$ for all $\kappa \in \Omega$, $\varphi \in Q$, and $R \subseteq V_\kappa$.

[7] The first author thanks J. Cummings for clarifying this.

[8] Stronger results have been announced, see [50].

As an exercise, the reader can check that the intervals $(\alpha, \kappa]$ are open in any τ_Q (consider $R = \{\alpha\}$ and $\varphi = \exists x\ (x \in R)$). The main strength of the Q-describable topology, however, comes from the fact that a second order variable R is allowed to occur in φ. So, all subsets of Ω that can be 'described' in this way are open in τ_Q.

Let d_Q denote the derivative operator for τ_Q. An ordinal $\kappa < \Omega$ is called *Q-indescribable* if it is a limit point of τ_Q. In other words, κ is Q-indescribable iff $\kappa \in d_Q(\Omega)$ iff $\kappa \in d_Q(\kappa)$.

It is not difficult to show that, whenever Q is any of the classes Π^1_n, the sets $U_\kappa(\varphi, R)$ actually form a base for τ_Q. Hence, our definition of Π^1_n-indescribable cardinals is equivalent to the standard one given in [42]: κ is Q-indescribable iff, for all $R \subseteq V_\kappa$ and all sentences $\varphi \in Q$,

$$(V_\kappa, \in, R) \vDash \varphi \ \Rightarrow\ \exists \alpha < \kappa\ (V_\alpha, \in, R \cap V_\alpha) \vDash \varphi.$$

It is well known that *weakly compact cardinals* coincide with the Π^1_1-indescribable ones (see [41]). From this it is easy to conclude that the Mahlo topology τ_3 is contained in $\tau_{\Pi^1_1}$. The following more general proposition was suggested to the first author by Philipp Schlicht (see [10]).

Proposition 14. *For any $n \geq 0$, τ_{n+2} is contained in $\tau_{\Pi^1_n}$.*

Proof We shall show that for each n, there is a Π^1_n-formula $\varphi_{n+1}(R)$ such that

$$\kappa \in d_{n+1}(A) \iff (V_\kappa, \in, A \cap \kappa) \vDash \varphi_{n+1}(R). \qquad (**)$$

This implies that for each $\kappa \in d_{n+1}(A)$, the set $U_\kappa(\varphi_{n+1}, A \cap \kappa)$ is a $\tau_{\Pi^1_n}$-open subset of $d_{n+1}(A)$ containing κ. Hence, each $d_{n+1}(A)$ is $\tau_{\Pi^1_n}$-open. Since τ_{n+2} is generated over τ_{n+1} by the open sets of the form $d_{n+1}(A)$ for various A, we have $\tau_{n+2} \subseteq \tau_{\Pi^1_n}$.

We prove $(**)$ by induction on n. For $n = 0$, notice that $\kappa \in d_1(A)$ iff ($\kappa \in \mathrm{Lim}$ and $A \cap \kappa$ is unbounded in κ) iff

$$(V_\kappa, \in, A \cap \kappa) \vDash \forall \alpha\ \exists \beta\ (R(\beta) \wedge \alpha < \beta).$$

For the induction step recall that by Corollary 2, $\kappa \in d_{n+1}(A)$ iff

(i) κ is doubly d_n-reflexive;
(ii) $\forall Y \subseteq \kappa\ (\kappa \in d_n(Y) \to \exists \alpha < \kappa\ (\alpha \in A \wedge \alpha \in d_n(Y)))$.

By the induction hypothesis, for some $\varphi_n(R) \in \Pi^1_{n-1}$, we have

$$\alpha \in d_n(A) \iff (V_\alpha, \in, A \cap \alpha) \vDash \varphi_n(R).$$

Hence, part (ii) is equivalent to

$$(V_\kappa, \in, A \cap \kappa) \vDash \forall Y \subseteq \mathrm{On}\ (\varphi_n(Y) \to \exists \alpha\ (R(\alpha) \wedge \varphi_n^{V_\alpha}(Y \cap \alpha))).$$

Here, φ^{V_α} means the relativization of all quantifiers in φ to V_α. We notice that V_α is first order definable, hence the complexity of $\varphi_n^{V_\alpha}$ remains in the class Π^1_{n-1}. So, the resulting formula is Π^1_n.

To treat part (i) we recall that $\kappa < \Omega$ is doubly d_n-reflexive iff $\kappa \in d_n(\Omega)$ and

$$\forall Y_1, Y_2 \subseteq \kappa\ (\kappa \in d_n(Y_1) \cap d_n(Y_2) \to \exists \alpha < \kappa\ \alpha \in d_n(Y_1) \cap d_n(Y_2)).$$

Similarly to the above, using the induction hypothesis this can be rewritten as a Π^1_n-formula. □

Corollary 7. *If there is a Π^1_n-indescribable cardinal $\kappa < \Omega$, then τ_{n+2} has a non-isolated point.*

Corollary 8. *If for each n there is a Π^1_n-indescribable cardinal $\kappa < \Omega$, then all τ_n are non-discrete.*

By the result of Magidor [47] we know that θ_3 need not be weakly compact in some models of ZFC (e.g. in a model, where $\theta_3 = \aleph_{\omega+1}$). Hence, in general, the condition of the existence of Π^1_n-indescribable cardinals is not a necessary one for the nontriviality of the topologies τ_{n+2}. However, Bagaria et al. [4] prove that in L the Π^1_n-indescribable cardinals coincide with the limit points of τ_{n+2}.

10.9 Topological Completeness Results for GLP

As in the case of the unimodal language (cf. Sect. 10.3), one can ask two basic questions: Is **GLP** complete w.r.t. the class of all GLP-spaces? Is **GLP** complete w.r.t. some fixed natural GLP-space?

In the unimodal case, both questions received positive answers due to Esakia and Abashidze–Blass, respectively. Now the situation is more complicated.

The first question was initially studied by Beklemishev et al. in [14], where only some partial results were obtained. It was proved that the bimodal system **GLB** is complete w.r.t. GLP$_2$-spaces of the form (X, τ, τ^+), where X is a well-founded partial ordering and τ is its left topology. A proof of this result was based on the Kripke model techniques coming from [11].

Already at that time it was clear that these techniques cannot be immediately generalized to GLP$_3$-spaces since the third topology τ^{++} on such orderings is sufficiently similar to the club topology. From the results of Blass [18] (see Theorem 10 below) it was known that some stronger set-theoretic assumptions would be needed to prove completeness w.r.t. such topologies. Moreover, without any large cardinal assumptions it was not even known whether a GLP-space with a non-discrete third topology could exist at all.

First examples of GLP-spaces in which all topologies are non-discrete are constructed in [5], where also the stronger fact of topological completeness of **GLP** w.r.t. the class of all (countable, Hausdorff) GLP-spaces is established.

Theorem 9. *(i)* $\mathrm{Log}(\mathscr{C}) = \mathbf{GLP}$*, where* $\mathscr{C}$ *is the class of all GLP-spaces.*
(ii) There is a countable Hausdorff GLP-space X *such that* $\mathrm{Log}(X) = \mathbf{GLP}$.

In fact, X is the ordinal $\varepsilon_0 = \sup\{\omega, \omega^\omega, \omega^{\omega^\omega}, \ldots\}$ equipped with a sequence of topologies refining the interval topology. However, these topologies cannot be first-countable and are, in fact, defined using non-constructive methods such as Zorn's lemma.[9] In this sense, it is not an example of a *natural* GLP-space. The proof of this theorem introduces the techniques of maximal and limit-maximal extensions of scattered spaces. It falls outside the present survey (see [5]).

The question whether **GLP** is complete w.r.t. some natural GLP-space is still open. Some partial results concerning the GLP-space generated from the interval topology on the ordinals (in the sense of the plus operation) are described below. Here, we call this space the *ordinal GLP-space*. (The space described in Sect. 10.8 is not an exact model of **GLP** as the left topology validates the linearity axiom.)

As we know from Corollary 6, it is consistent with ZFC that the Mahlo topology is discrete. Hence, it is consistent that **GLP** is incomplete w.r.t. the ordinal GLP-space. However, is it consistent with ZFC that **GLP** is complete w.r.t. the ordinal GLP-space? To this question we do not know a full answer. A pioneering work has been done by Blass [18] who studied the question of completeness of the Gödel–Löb logic **GL** w.r.t. a semantics equivalent to the topological interpretation w.r.t. the *club topology* τ_2. He used the language of filters rather than that of topological spaces as is more common in set theory.

Theorem 10. (Blass)

(i) If $V = L$ *and* $\Omega \geq \aleph_\omega$*, then* **GL** *is complete w.r.t.* (Ω, τ_2).
(ii) If there is a weakly Mahlo cardinal, there is a model of ZFC *in which* **GL** *is incomplete w.r.t.* (Ω, τ_2) *for any* Ω.

A corollary of (i) is that the statement "**GL** is complete w.r.t. τ_2" is consistent with ZFC (provided ZFC is consistent). In fact, instead of $V = L$ Blass used the so-called *square principle* for all $\aleph_n$, $n < \omega$, which holds in L by the results of Ronald Jensen. A proof of (i) is based on an interesting combinatorial construction using the techniques of splitting stationary sets.

A proof of (ii) is much easier. It uses a model of Harrington and Shelah in which $\aleph_2$ is reflecting for stationary sets of ordinals of countable cofinality [35]. Assuming Mahlo cardinals exist, they have shown that the following statement holds in some model of ZFC:

> If S is a stationary subset of $\aleph_2$ such that $\forall \alpha \in S\, \mathrm{cf}(\alpha) = \omega$, then there is a $\beta < \alpha$ (of cofinality ω_1) such that $S \cap \beta$ is stationary in β.

In fact, this statement can be expressed in the language of modal logic. First, we remark that this principle implies its generalization to all ordinals λ of cofinality $\aleph_2$ (consider an increasing continuous function mapping $\aleph_2$ to a club in λ). Second, we

[9] It seems to be interesting to study the question of topological completeness of **GLP** in the absence of the full axiom of choice, possibly with the axiom of determinacy.

remark that for the club topology the formula $\Diamond^n\top$ represents the class of ordinals of cofinality at least $\aleph_n$. This is a straightforward generalization of Lemma 11. Thus, the formula $\Box^3\bot\wedge\Diamond^2\top$ represents the subclass of Ω consisting of ordinals of cofinality ω_2.

Hence, the above reflection principle amounts to the validity of the following modal formula:

$$\Box^3\bot\wedge\Diamond^2\top\wedge\Diamond(p\wedge\Box\bot)\to\Diamond^2(p\wedge\Box\bot). \qquad (*)$$

In fact, if the antecedent is valid in λ, then $\mathrm{cf}(\lambda)=\omega_2$ and the interpretation of $p\wedge\Box\bot$ is a set S consisting of ordinals of countable cofinality such that $S\cap\lambda$ is stationary in λ. The consequent just states that this set reflects. Thus, formula $(*)$ is valid in (Ω,τ_2) for any Ω. Since this formula is clearly not provable in **GL**, the topological completeness fails for (Ω,τ_2).

Thus, Blass managed to give an exact consistency strength of the statement "**GL** is incomplete w.r.t. τ_2".

Corollary 9. *"**GL** is incomplete w.r.t. τ_2" is consistent iff it is consistent that Mahlo cardinals exist.*

It is possible to generalize these results to the case of bimodal logic **GLB** [12]. The situation remains essentially unchanged, although a proof of Statement (i) of Theorem 10 needs considerable adaptation.

Theorem 11. *If $V=L$ and $\Omega\geq\aleph_\omega$, then **GLB** is complete w.r.t. $(\Omega;\tau_1,\tau_2)$.*

10.10 Topologies for the Variable-Free Fragment of GLP

A natural topological model for the variable-free fragment of **GLP** has been introduced by Icard [38]. It is not a GLP-space and thus it is not a model of the full **GLP** (nor even of **GLB**). However, it is sound and complete for the variable-free fragment of **GLP**. It gives a convenient tool for the study of this fragment, which plays an important role in proof-theoretic applications of the polymodal provability logic. Here we give a simplified presentation of Icard's polytopological space.

Let Ω be an ordinal and let $\ell:\Omega\to\Omega$ denote the rank function for the interval topology on Ω (see Example 1). We define $\ell^0(\alpha)=\alpha$ and $\ell^{k+1}(\alpha)=\ell\ell^k(\alpha)$.

Icard's topologies υ_n, for each $n\in\omega$, are defined as follows. Let υ_0 be the left topology, and let υ_n be generated by υ_0 and all sets of the form

$$U^m_\beta:=\{\alpha\in\Omega:\ell^m(\alpha)>\beta\}$$

for $m<n$ and $\beta<\Omega$.

Clearly, υ_n is an increasing sequence of topologies. In fact, υ_1 is the interval topology. We let d_n and ρ_n denote the derivative operator and the rank function for υ_n, respectively. We have the following characterizations.

Lemma 12. *(i)* $\ell : (\Omega, \upsilon_{n+1}) \to (\Omega, \upsilon_n)$ *is a d-map;*
(ii) υ_{n+1} *is the coarsest topology* ν *on* Ω *such that* ν *contains the interval topology and* $\ell : (\Omega, \nu) \to (\Omega, \upsilon_n)$ *is continuous;*
(iii) ℓ^n *is the rank function of* υ_n*, that is,* $\rho_n = \ell^n$*;*
(vi) υ_{n+1} *is generated by* υ_n *and* $\{d_n^{\alpha+1}(\Omega) : \alpha < \rho_n(\Omega)\}$.

Proof (i) The map $\ell : (\Omega, \upsilon_{n+1}) \to (\Omega, \upsilon_n)$ is continuous. In fact, $\ell^{-1}[0, \beta)$ is open in the interval topology υ_1 since $\ell : (\Omega, \upsilon_1) \to (\Omega, \upsilon_0)$ is its rank function, hence a d-map. Also, if $m < n$, then $\ell^{-1}(U_\beta^m) = U_\beta^{m+1}$, hence it is open in υ_{n+1}.

The map ℓ is open. Notice that υ_{n+1} is generated by υ_1 and some sets of the form $\ell^{-1}(U)$, where $U \in \upsilon_n$. A base of υ_{n+1} consists of sets of the form $V \cap \ell^{-1}(U)$ for some $V \in \upsilon_1$ and $U \in \upsilon_n$. We have $\ell(V \cap \ell^{-1}(U)) = \ell(V) \cap U$. $\ell(V)$ is υ_0-open since $\ell : (\Omega, \upsilon_1) \to (\Omega, \upsilon_0)$ is a d-map and $V \in \upsilon_1$. Hence, the image of any basic open in υ_{n+1} is open in υ_n.

The map ℓ is pointwise discrete since $\ell^{-1}\{\alpha\}$ is discrete in the interval topology υ_1, hence in υ_{n+1}.

(ii) By (i), $\ell : (\Omega, \upsilon_{n+1}) \to (\Omega, \upsilon_n)$ is continuous, hence $\nu \subseteq \upsilon_{n+1}$. On the other hand, if $\ell : (\Omega, \nu) \to (\Omega, \upsilon_n)$ is continuous, then $\ell^{-1}(U_\beta^m) \in \nu$ for each $m < n$. Therefore, $U_\beta^m \in \nu$ for all m such that $1 \le m \le n$. Since ν also contains the interval topology, we have $\upsilon_{n+1} \subseteq \nu$.

(iii) By (i), we have that $\rho_n \circ \ell$ is a d-map from (Ω, υ_{n+1}) to (Ω, υ_0). Hence, it coincides with the rank function for υ_{n+1}, $\rho_{n+1} = \rho_n \circ \ell$. The claim follows by an easy induction on n.

(iv) By (iii),

$$d_n^{\beta+1}(\Omega) = \{\alpha \in \Omega : \rho_n(\alpha) > \beta\} = \{\alpha \in \Omega : \ell^n(\alpha) > \beta\} = U_\beta^n.$$

Obviously, υ_{n+1} is generated by υ_n and U_β^n for all β. Hence, the claim. □

We call an *Icard space* a polytopological space of the form $(\Omega; \upsilon_0, \upsilon_1, \ldots)$. Icard originally considered just $\Omega = \varepsilon_0$. We are going to give an alternative proof of the following theorem [38].

Theorem 12. (Icard) *Let* φ *be a variable-free* **GLP**-*formula.*

(i) If **GLP** $\vdash \varphi$*, then* $(\Omega; \upsilon_0, \upsilon_1, \ldots) \vDash \varphi$.
(ii) If $\Omega \ge \varepsilon_0$ *and* **GLP** $\nvdash \varphi$*, then* $(\Omega; \upsilon_0, \upsilon_1, \ldots) \nvDash \varphi$.

Proof Within this proof we abbreviate $(\Omega; \upsilon_0, \upsilon_1, \ldots)$ by Ω. To prove part (i) we first remark that all topologies υ_n are scattered, hence all axioms of **GLP** except for P1 are valid in Ω. Moreover, $\mathrm{Log}(\Omega)$ is closed under the inference rules of **GLP**. Thus, we only have to show that the variable-free instances of axiom P1 are valid in Ω. This is sufficient because any derivation of a variable-free formula in **GLP** can be replaced by a derivation in which only the variable-free formulas occur (replace all the variables by the constant $\top$).

Let φ be a variable-free formula. We denote by φ^* its uniquely defined interpretation in Ω. The validity of an instance of P1 for φ amounts to the fact that $d_m(\varphi^*)$ is open in υ_n, whenever $m < n$. Thus, we have to prove the following proposition.□

Proposition 15. *For any variable-free formula φ, $d_n(\varphi^*)$ is open in υ_{n+1}.*

Let φ^+ denote the result of replacing in φ each modality $\langle n\rangle$ by $\langle n+1\rangle$. We need the following auxiliary claim.

Lemma 13. *If φ is variable-free, then $\ell^{-1}(\varphi^*) = (\varphi^+)^*$.*

Proof This goes by induction on the build-up of φ. The cases of constants and boolean connectives are easy. Suppose $\varphi = \langle n\rangle\psi$. We notice that since $\ell : (\Omega, \upsilon_{n+1}) \to (\Omega, \upsilon_n)$ is a d-map, we have $\ell^{-1}(d_n(A)) = d_{n+1}(\ell^{-1}(A))$ for any $A \subseteq \Omega$. Therefore, $\ell^{-1}(\varphi^*) = \ell^{-1}(d_n(\psi^*)) = d_{n+1}(\ell^{-1}(\psi^*)) = d_{n+1}((\psi^+)^*) = (\varphi^+)^*$, as required. □

We prove Proposition 15 in two steps. First, we show that it holds for a subclass of variable-free formulas called *ordered formulas*. Then we show that any variable-free formula is equivalent in Ω to an ordered one.

A formula φ is called *ordered* if no modality $\langle m\rangle$ occurs within the scope of $\langle n\rangle$ in φ for any $m < n$. The *height of* φ is the index of its maximal modality.

Lemma 14. *If $\langle n\rangle\varphi$ is ordered, then $d_n(\varphi^*)$ is open in υ_{n+1}.*

Proof This goes by induction on the height of $\langle n\rangle\varphi$. If it is 0, then $n = 0$. If $n = 0$, the claim is obvious since $d_0(A)$ is open in υ_1 for any $A \subseteq \Omega$. If $n > 0$, since $\langle n\rangle\varphi$ is ordered, we observe that $\langle n\rangle\varphi$ has the form $(\langle n-1\rangle\psi)^+$ for some ψ. The height of $\langle n-1\rangle\psi$ is less than that of $\langle n\rangle\varphi$. Hence, by the induction hypothesis, $(\langle n-1\rangle\psi)^* \in \upsilon_n$. Since $\ell : (\Omega, \upsilon_{n+1}) \to (\Omega, \upsilon_n)$ is continuous, we conclude that $\ell^{-1}(\langle n-1\rangle\psi)^*$ is open in υ_{n+1}. By Lemma 13, this set coincides with $(\langle n\rangle\varphi)^* = d_n(\varphi^*)$. □

Lemma 15. *Any variable-free formula φ is equivalent in Ω to an ordered one.*

Proof We argue by induction on the complexity of φ. The cases of boolean connectives and constants are easy. Suppose φ has the form $\langle n\rangle\psi$, where we may assume ψ to be in disjunctive normal form $\psi = \bigvee_i \bigwedge_j \pm\langle n_{ij}\rangle\psi_{ij}$. By the induction hypothesis, we may assume all the subformulas $\langle n_{ij}\rangle\psi_{ij}$ (and ψ itself) are ordered. Since $\langle n\rangle$ commutes with disjunction, it will be sufficient to show that for each i the formula $\theta_i := \langle n\rangle \bigwedge_j \pm\langle n_{ij}\rangle\psi_{ij}$ can be ordered.

By Lemma 14 each set $(\langle n_{ij}\rangle\psi_{ij})^*$ is open in υ_n whenever $n_{ij} < n$. Being a derived set, it is also closed in $\upsilon_{n_{ij}}$ and hence in υ_n. Thus, all such sets are clopen.

If U is open, then $d(A \cap U) = d(A) \cap U$ for any topological space. In particular, for any $A \subseteq \Omega$ and $n_{ij} < n$, $d_n(A \cap (\pm\langle n_{ij}\rangle\psi_{ij})^*) = d_n(A) \cap (\pm\langle n_{ij}\rangle\psi_{ij})^*$. This allows us to bring all the conjuncts $\pm\langle n_{ij}\rangle\psi_{ij}$ from under the $\langle n\rangle$ modality in θ_i. The resulting conjunction is ordered. □

This concludes the proof of Proposition 15 and thereby of Part (i).

A variable-free formula A is called a *word* if it is built-up from $\top$ only using connectives of the form $\langle n\rangle$ for any $n \in \omega$. We write $A \vdash B$ for **GLP** $\vdash A \to B$.

To prove Part (ii), we shall rely on the following fundamental lemma about the variable-free fragment of **GLP**. For a proof of this lemma we refer to [6, 8].

Lemma 16. *(i) Every variable-free formula is equivalent in* **GLP** *to a boolean combination of words;*
(ii) For any words A and B, either $A \vdash \langle 0\rangle B$, or $B \vdash \langle 0\rangle A$, or A and B are equivalent;
(iii) Conjunction of words is equivalent to a word.

We prove Part (ii) of Theorem 12 in a series of lemmas. First, we show that any word is true at some point in Ω provided $\Omega \geq \varepsilon_0$.

Lemma 17. *For any word A, $\varepsilon_0 \in A^*$.*

Proof We know that $\rho_n(\varepsilon_0) = \ell^n(\varepsilon_0) = \varepsilon_0$. Hence, $\varepsilon_0 \in d_n(\Omega)$ for each n. Assume n exceeds all the indices of modalities in A and $A = \langle m\rangle B$. By Proposition 15 the set B^* is open in υ_n. By the induction hypothesis $\varepsilon_0 \in B^*$. Hence, $\varepsilon_0 \in d_n(B^*) \subseteq d_m(B^*) = A^*$. This proves the claim. □

Applying this lemma to the word $\langle 0\rangle A$ we obtain the following corollary.

Corollary 10. *For every word A, there is an $\alpha < \varepsilon_0$ such that $\alpha \in A^*$.*

Let $\min(A^*)$ denote the least ordinal $\alpha \in \Omega$ such that $\alpha \in A^*$.

Lemma 18. *For any words A, B, if $A \nvdash B$, then* $\min(A^*) \notin B^*$.

Proof If $A \nvdash B$, then, by Lemma 16 (ii), $B \vdash \langle 0\rangle A$. Therefore, by the soundness of **GLP** in Ω, $B^* \subseteq d_0(A^*)$. It follows that for each $\beta \in B^*$ there is an $\alpha \in A^*$ such that $\alpha < \beta$. Thus, $\min(A^*) \notin B^*$. □

Now we are ready to prove Part (ii). Assume φ is variable-free and **GLP** $\nvdash \varphi$. By Lemma 16 (i) we may assume that φ is a boolean combination of words. Writing φ in conjunctive normal form we observe that it is sufficient to prove the claim only for formulas φ of the form $\bigwedge_i A_i \to \bigvee_j B_j$, where A_i and B_j are words. Moreover, $\bigwedge_i A_i$ is equivalent to a single word A.

Since **GLP** $\nvdash \varphi$ we have $A \nvdash B_j$ for each j. Let $\alpha = \min(A^*)$. By Lemma 18 we have $\alpha \notin B_j^*$ for each j. Hence, $\alpha \notin (\bigvee_j B_j)^*$ and $\alpha \notin \varphi^*$. This means that $\Omega \nvDash \varphi^*$.

10.11 Further Results

Topological semantics of polymodal provability logic has been extended to the language with transfinitely many modalities. A logic $\mathbf{GLP}_\Lambda$ having modalities $[\alpha]$ for all ordinals $\alpha < \Lambda$ is introduced in [8]. It was intended for the proof-theoretic analysis of predicative theories and is currently being actively investigated for that purpose.

David Fernandez and Joost Joosten undertook a thorough study of the variable-free fragment of that logic mostly in connection with the arising ordinal notation systems (see [25, 27] for a sample). In particular, they found a suitable generalization of Icard's polytopological space and showed that it is complete for that fragment [26]. Fernandez [30] also proved topological completeness of the full $\mathbf{GLP}_\Lambda$ by generalizing the results of [5].

The ordinal GLP-space is easily generalized to transfinitely many topologies $(\tau_\alpha)_{\alpha<\Lambda}$ by letting τ_0 be the left topology, $\tau_{\alpha+1} := \tau_\alpha^+$ and, for limit ordinals λ, τ_λ be the topology generated by all τ_α such that $\alpha < \lambda$. This space is a natural model of $\mathbf{GLP}_\Lambda$ and has been studied quite recently by Bagaria [3] and further by Bagaria et al. [4]. In particular, the three authors proved that in L the limit points of τ_{n+2} are Π^1_n-indescribable cardinals. The question posed in [14] whether the non-discreteness of τ_{n+2} is equiconsistent with the existence of Π^1_n-indescribable cardinals still appears to be open.

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