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Modal languages for topology: Expressivity and definability☆

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ABSTRACT

In this paper we study the expressive power and definability for (extended) modal languages interpreted on topological spaces. We provide topological analogues of the van Benthem characterization theorem and the Goldblatt–Thomason definability theorem in terms of the well-established first-order topological language $\mathcal{L}_t$.

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1. Introduction

Modal logic, as a language for talking about topological spaces, has been studied for at least 60 years. Originally, the motivations for this study were purely mathematical. More recently, computer science applications have lead to a revival of interest, giving rise to new logics of space, many of which are based on (extensions of) modal languages ([24,2,22,36], to name a few).

The design of such logics is usually guided by considerations involving expressive power and computational complexity. Within the landscape of possible spatial languages, the basic modal language interpreted on topological spaces can be considered a minimal extreme. It has a low computational complexity, but also a limited expressive power. There are various well-known ways of increasing the expressive power of the basic modal language (over topological structures), while retaining its nice computational properties. Some are more syntactically flavored, such as adding the global modality, the

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difference modality, or nominals; some rely on a change of semantics, such as interpreting the diamond operator by taking the derived set rather than the closure of a set [3]. Scattered results have appeared in the literature comparing the expressive power of these languages. For example, Shehtman [32] shows that adding the global modality increases the expressive power of the language, and Gabelaia [15] demonstrates that adding the nominals (as well as adding the difference modality) further increase the expressive power; However, these efforts have not delineated the precise boundaries of expressivity of the respective languages over topological structures. In fact, no machinery has been known so far that is suited for answering such questions.

Questions like these *have* been successfully studied for Kripke semantics, using classical first-order model theory. This is enabled through the standard translation, which maps many extensions of the modal language into first-order logic. Both expressivity and frame definability questions have been answered in this way, for the basic modal language and for extensions (cf. for instance [6,33]). However, a similar first-order 'common ground' for topological structures seems to be missing.

We show in this paper that, although the very definition of a topological space, and of many interesting topological properties, is higher-order, for the purposes of studying modal expressivity and definability, a suitable first-order 'common ground' language exists, just as in the case of Kripke semantics. In fact, the language in question, $\mathcal{L}_t$, was proposed already in the literature on topological model theory [14]. We show that the basic modal language, as well as many of its extensions, can be translated into $\mathcal{L}_t$ by means of a standard translation, and that expressivity and definability questions can be studied systematically in the context of $\mathcal{L}_t$. Among other things, we obtain the following results, which can be seen as topological generalizations of the Van Benthem theorem and the Goldblatt–Thomason theorem, respectively:

Theorem 31. *Let $\phi(x)$ be any $\mathcal{L}_t$ formula with one free variable. Then $\phi(x)$ is equivalent to (the standard translation of) a modal formula iff $\phi(x)$ is invariant under topo-bisimulations.*

Theorem 43. *Let K be a class of topological spaces definable in $\mathcal{L}_t$. Then K is definable in the basic modal language iff K is closed under topological sums, open subspaces and images of interior maps, while the complement of K is closed under Alexandroff extensions.*

We give similar characterizations for some extensions of the modal language, containing nominals, the global modality, the difference modality, and the $\downarrow$-binder (for a summary of our main results, see Section 6). Characterizations such as these help explain why certain languages (in this case the basic modal language) are natural to consider. They can also guide us in finding languages that provide the appropriate level of expressivity for an application.

We consider the main contribution of this work to be the way it connects two independent, and well-developed lines of investigation: the study of modal logics for topology, and first-order topological model theory. We expect that the connection we identify will be useful in answering other questions in modal logic for topology as well. Several directions for further research can be found in Section 6.

Outline of the paper

The structure of the paper is as follows: Section 2 contains basic notions from topology, topological model theory, and the topological semantics for modal logic. Section 3 is the core of the paper: in Section 3.1 we characterize the expressivity of the basic modal language; Theorem 42 of Section 3.3 is the main technical result that is used extensively in subsequent sections, while in Section 3.4 we compare definability in the basic modal language with first-order definability. Section 4 provides the proper algebraic perspective on these results. In Section 5, we consider a number of extensions of the basic modal language and characterize definability in these richer languages. Finally, we conclude in Section 6.

2. Preliminaries

In this section we recall some basic notions from topology, topological model theory, and the topological semantics for modal logic.

2.1. Topological spaces

Definition 1 (*Topological Spaces*). A *topological space* (X, τ) is a non-empty set X together with a collection $\tau \subseteq \wp(X)$ of subsets that contains $\emptyset$ and X and is closed under finite intersections and arbitrary unions. The members of τ are called *open sets* or simply *opens*. We often use the same letter to denote both the set and the topological space based on this set: $X = (X, \tau)$.

If $A \subseteq X$ is a subset of the space X, by $\mathbb{I}A$ (read: 'interior A') one denotes the greatest open contained in A (i.e. the union of all the opens contained in A). Thus $\mathbb{I}$ is an operator over the subsets of the space X. It is called the *interior operator*.

Complements of open sets are called *closed*. The *closure* operator, which is a dual of the interior operator, is defined as $\mathbb{C}A = -\mathbb{I}-A$ where '$-$' stands for the set-theoretic complementation. Observe that $\mathbb{C}A$ is the least closed set containing A.

A standard example of a topological space is the real line $\mathbb{R}$, where a set is considered to be open if it is a union of open intervals (a, b).

For technical reasons, at times it will be useful to consider topological bases—collections of sets that generate a topology.

Definition 2 (*Topological Bases*). A *topological base* σ is a collection $\sigma \subseteq \wp(X)$ of subsets of a set X such that closing σ under arbitrary unions gives a topology on X (i.e., such that $(X, \{\bigcup \sigma' \mid \sigma' \subseteq \sigma\})$ is a topological space). The latter requirement is in fact equivalent to the conjunction of the following conditions:

(1) $\emptyset \in \sigma$,
(2) $\bigcup \sigma = X$,
(3) For all $A, B \in \sigma$ and $x \in A \cap B$, there is a $C \in \sigma$ such that $x \in C$ and $C \subseteq A \cap B$.

For (X, σ) a topological base, we denote by $\widehat{X} = (X, \widehat{\sigma})$ the topological space it generates, i.e., the topological space obtained by closing σ under arbitrary unions. Furthermore, we say that σ *is a base for* $\widehat{\sigma}$.

For example, a base for the standard topology on the reals is the set of open intervals $\{(a, b) \mid a \leq b\}$.

2.2. The basic modal language

We recall syntax and the topological semantics for the basic modal language.

Definition 3 (*The Basic Modal Language*). The *basic modal language* $\mathcal{ML}$ consists of a set of propositional letters $\textsc{Prop} = \{p_1, p_2, \ldots\}$, the boolean connectives $\wedge$, $\neg$, the constant truth $\top$ and a modal box $\Box$. Modal formulas are built according to the following recursive scheme:

$$\phi ::= \top \mid p_i \mid \phi \wedge \phi \mid \neg\phi \mid \Box\phi.$$

We use $\Diamond\phi$ as an abbreviation for $\neg\Box\neg\phi$. Unless specifically indicated otherwise, we will always assume that the set of propositional letters is countably infinite.

Nowadays, the best-known semantics for $\mathcal{ML}$ is the Kripke semantics. In this paper, however, we study the topological semantics, according to which modal formulas denote regions in a topological space. The regions denoted by the propositional letters are specified in advance by means of a *valuation*, and $\wedge$, $\neg$ and $\Box$ are interpreted as intersection, complementation and the interior operator. Formally:

Definition 4 (*Topological Models*). A *topological model* $\mathfrak{M}$ is a tuple (X, ν) where $X = (X, \tau)$ is a topological space and the valuation $\nu : \textsc{Prop} \to \wp(X)$ sends propositional letters to subsets of X.

Definition 5 (*Topological Semantics of the Basic Modal Language*). Truth of a formula ϕ at a point w in a topological model $\mathfrak{M}$ (denoted by $\mathfrak{M}, w \models \phi$) is defined inductively:

$\mathfrak{M}, w \models \top$		always
$\mathfrak{M}, w \models p$	iff	$w \in \nu(p)$
$\mathfrak{M}, w \models \phi \wedge \psi$	iff	$\mathfrak{M}, w \models \phi$ and $\mathfrak{M}, w \models \psi$
$\mathfrak{M}, w \models \neg\phi$	iff	$\mathfrak{M}, w \not\models \phi$
$\mathfrak{M}, w \models \Box\phi$	iff	$\exists O \in \tau$ such that $w \in O$ and $\forall v \in O.(\mathfrak{M}, v \models \phi)$

If $\mathfrak{M}, w \models \phi$ for all $w \in A$ for some $A \subset X$, we write $A \models \phi$. Further, $\mathfrak{M} \models \phi$ (ϕ is *valid* in $\mathfrak{M}$) means that $\mathfrak{M}, w \models \phi$ for all $w \in X$. We write $X \models \phi$ (ϕ is valid in X) when $(X, \nu) \models \phi$ for any valuation ν. If K is a class of topological spaces we write $\mathsf{K} \models \phi$ when $X \models \phi$ for each $X \in \mathsf{K}$.

Each modal formula ϕ defines a set of points in a topological model (namely the set of points at which it is true). With a slight overloading of notation, we will sometimes denote this set by $\nu(\phi)$. It is not hard to see that $\nu(\Box\phi) = \mathbb{I}\nu(\phi)$.

We extend the notions of truth and validity to the sets of modal formulas in the usual way (e.g., $X \models \Gamma$ means that $X \models \phi$ for each $\phi \in \Gamma$).

Definition 6 (*Modal Definability*). A set of modal formulas Γ *defines* a class K of spaces if, for any space X,

$$X \in \mathsf{K} \text{ iff } X \models \Gamma.$$

A class of topological spaces is said to be *modally definable* if there exists a set of modal formulas that defines it. A topological property is said to be modally definable (or, defined by a set of formulas Γ) if the class of all spaces that have the property is modally definable (is defined by Γ).

Given a class K of spaces, the set of modal formulas $\{\phi \in \mathcal{ML} \mid \mathsf{K} \models \phi\}$ ("*the modal logic of* K") is denoted by $\mathrm{Log}(\mathsf{K})$. Conversely, given a set of modal formulas Γ, the class of spaces $\{X \mid X \models \Gamma\}$ is denoted by $\mathrm{Sp}(\Gamma)$. Thus, in this notation, a class K is modally definable iff $\mathrm{Sp}(\mathrm{Log}(\mathsf{K})) = \mathsf{K}$.

The following example illustrates the concept of modal definability.

Definition 7 (*Hereditary Irresolvability*). A subset $A \subseteq X$ of a space X is said to be *dense* in X if $\mathbb{C}A = X$ (or, equivalently, if A intersects each non-empty open in X). A topological space X is called *irresolvable* if it cannot be decomposed into two disjoint dense subsets. It is *hereditarily irresolvable* (HI) if all its subspaces[1] are irresolvable.

[1] Recall that a *subspace* of a space X is a non-empty subset $A \subseteq X$ endowed with the relative topology $\tau_A = \{O \cap A \mid O \in \tau\}$.

Theorem 8. *The modal formula* $\Box(\Box(p \to \Box p) \to p) \to \Box p$ *(Grz) defines the class of hereditarily irresolvable spaces.*

Proof. Follows from the results in [12] and [4]. For purposes of illustration, we will give a direct proof, inspired by [4].

We are to show that X is HI iff $X \models (Grz)$.

First note that $X \models (Grz)$ iff $X \models \Diamond\neg p \to \Diamond(\neg p \wedge \Box(p \to \Box p))$ iff $X \models \Diamond q \to \Diamond(q \wedge \neg\Diamond(\neg q \wedge \Diamond q))$ iff $\forall A \subseteq X.[\mathbb{C}A \subseteq \mathbb{C}(A - \mathbb{C}(\mathbb{C}A - A))]$.

Suppose X is not HI. Then there exists a non-empty subset $A \subseteq X$ and two disjoint sets $B, B' \subset A$ such that $A \subseteq \mathbb{C}B \cap \mathbb{C}B'$. We show that $\mathbb{C}B \not\subseteq \mathbb{C}(B - \mathbb{C}(\mathbb{C}B - B))$ so X does not make (Grz) valid. Indeed, since $A \subseteq \mathbb{C}B$ it is clear that $B' \subseteq \mathbb{C}B - B$, hence $B \subseteq A \subseteq \mathbb{C}B' \subseteq \mathbb{C}(\mathbb{C}B - B)$ and $\mathbb{C}B \not\subseteq \mathbb{C}(B - \mathbb{C}(\mathbb{C}B - B)) = \emptyset$.

Suppose $X \not\models (Grz)$. Then there exists a non-empty subset $A \subseteq X$ such that $\mathbb{C}A \not\subseteq \mathbb{C}(A - \mathbb{C}(\mathbb{C}A - A))$. Denote $Y = \mathbb{C}A$. We will show that Y is not HI thus proving that X is not HI (it is easily seen that a closed subspace of an HI space must itself be HI). Since Y is a closed subspace of X the operator $\mathbb{C}_Y$ coincides with $\mathbb{C}$ on subsets of Y. Thus $Y \not\subseteq \mathbb{C}_Y(A - \mathbb{C}_Y(Y - A)) = \mathbb{C}_Y\mathbb{I}_Y A$. It follows that A is dense in Y while $\mathbb{I}_Y A$ is not dense in Y. Then there exists a subset $U \subseteq Y$ that is open in the relative topology of Y such that $\emptyset = \mathbb{I}_Y A \cap U = \mathbb{I}_Y(U \cap A) = \mathbb{I}_Y((U \cap (Y - U)) \cup (U \cap A)) = \mathbb{I}_Y(U \cap (A \cup (Y - U))) = U \cap \mathbb{I}_Y(A \cup (Y - U)) = U - \mathbb{C}_Y(U - A)$. This implies that $U \subseteq \mathbb{C}_Y(U - A)$. But at the same time $U \subseteq \mathbb{C}_Y(U \cap A)$ since U is open in Y and A is dense in Y. As $U = (U - A) \cup (U \cap A)$ it follows that U is decomposed into two disjoint dense in U subsets $U - A$ and $U \cap A$, so U is resolvable. Thus Y is not HI and hence X is not HI either. $\dashv$

One of the central questions in this paper is which properties of topological spaces are definable in the basic modal language and its various extensions.

2.3. The topological correspondence language $\mathcal{L}_t$

In the relational semantics, the Van Benthem theorem and the Goldblatt–Thomason theorem characterize the expressive power of the basic modal language by comparing it to the 'golden standard' of first-order logic. In the topological setting, it is less clear what the golden standard should be. Let us imagine for a moment a perfect candidate for a 'first-order correspondence language for topological semantics of modal logic'. Such a language should have the usual kit of nice properties of first-order languages like Compactness and the Löwenheim–Skolem theorem; it should be able to express topological properties in a natural way; moreover, it should be close enough to the usual mathematical language used for speaking about topologies so that we could determine easily whether a given topological property is expressible in it or not; and it should be suitable for translating modal formulas into it nicely.

The language $\mathcal{L}_t$ which we describe in this section satisfies all these requirements. Moreover, its model theory has been quite well studied and the corresponding machinery will serve us well in the following sections. With the exception of Theorems 11 and 19, all results on $\mathcal{L}_t$ discussed in this section, and much more, can be found in the classical monograph on topological model theory by Flum and Ziegler [14].

Before defining $\mathcal{L}_t$, we will first introduce the two-sorted first-order language $\mathcal{L}^2$. In its usual definition, this language can contain predicate symbols of arbitrary arity. Here, however, since the models we intend to describe are the *topological models* introduced in the previous section, we will restrict attention to a specific signature, containing a unary predicate for each propositional letter $p \in \textsc{Prop}$.

Definition 9 (*The Quantified Topological Language $\mathcal{L}^2$*). $\mathcal{L}^2$ is a two-sorted first-order language: it has terms that are intended to range over elements, and terms that are intended to range over open sets. Formally, the alphabet is constituted by a countably infinite set of "point variables" $x, y, z, \ldots$ a countably infinite set of "open variables" $U, V, W, \ldots$, unary predicate symbols P_p corresponding to propositional letters $p \in \textsc{Prop}$ and a binary predicate symbol ε that relates point variables with open variables. The formulas of $\mathcal{L}^2$ are given by the following recursive definition:

$$\phi ::= \top \mid x = y \mid U = V \mid P_p(x) \mid x\varepsilon U \mid \neg\phi \mid \phi \wedge \phi \mid \exists x.\phi \mid \exists U.\phi$$

where x, y are point variables and U, V are open variables. The usual shorthand notations (e.g., $\forall$ for $\neg\exists\neg$) apply.

Due to the chosen signature, formulas of $\mathcal{L}^2$ can be naturally interpreted in topological models (relative to assignments that send point variables to elements of the domain and open variables to open sets). However, as we show in Appendix A, under this semantics, $\mathcal{L}^2$ is rather ill-behaved: it lacks the usual model-theoretic features such as Compactness, the Löwenheim–Skolem theorem and the Łoś theorem. For this reason, we will first consider a more general semantics in terms of *basoid models*.

Definition 10 (*Basoid Models*). A *basoid model* is a tuple (X, σ, ν) where X is a non-empty set, $\sigma \subseteq \wp(X)$ is a topological base, and the valuation $\nu : \textsc{Prop} \to \wp(X)$ sends propositional letters to subsets of X.

Interpret $\mathcal{L}^2$ on a basoid model as follows: point variables range over X, open variables range over σ, the valuation ν determines the meaning of the unary predicates P_p, while ε is interpreted as the set-theoretic membership relation.

Under this interpretation, $\mathcal{L}^2$ displays all the usual features of a first-order language, including Compactness, the Löwenheim–Skolem property and the Łoś theorem [14]. [2] As we mentioned already, these properties are lost if we further restrict attention to topological models.

Theorem 11. *$\mathcal{L}^2$ interpreted on topological models lacks Compactness, Löwenheim–Skolem and Interpolation, and is Π^1_1-hard for validity.*

The proof can be found in Appendix A.

Thus, in order to work with topological models *and* keep the nice first-order properties we need to somehow 'tame' $\mathcal{L}^2$. This is where $\mathcal{L}_t$, a well-behaved fragment of $\mathcal{L}^2$, enters the picture. Let us call an $\mathcal{L}^2$ formula α *positive (negative) in an open variable U* if all free occurrences of U are under an even (odd) number of negation signs.

Definition 12 (*The Language $\mathcal{L}_t$*). $\mathcal{L}_t$ contains all atomic $\mathcal{L}^2$-formulas and is closed under conjunction, negation, quantification over the point variables and the following restricted form of quantification over open variables:

- if α is positive in the open variable U, and x is a point variable, then $\forall U.(x\varepsilon U \to \alpha)$ is a formula of $\mathcal{L}_t$,
- if α is negative in the open variable U, and x is a point variable, then $\exists U.(x\varepsilon U \wedge \alpha)$ is a formula of $\mathcal{L}_t$.

(recall that $\phi \to \psi$ is simply an abbreviation for $\neg(\phi \wedge \neg\psi)$).

The reason $\mathcal{L}_t$ is particularly well suited for describing topological models lies in the following observation: $\mathcal{L}_t$-formulas cannot distinguish between a basoid model and the topological model it generates. More precisely, for any basoid model $\mathfrak{M} = (X, \sigma, \nu)$, let $\widehat{\mathfrak{M}} = (X, \widehat{\sigma}, \nu)$, where $\widehat{\sigma}$ is the topology generated by the topological base σ.

Theorem 13 (*$\mathcal{L}_t$ is the Base-Invariant Fragment of $\mathcal{L}^2$*). *For any $\mathcal{L}_t$-formula $\alpha(x_1, \ldots, x_n, U_1, \ldots, U_m)$, basoid model $\mathfrak{M} = (X, \sigma, \nu)$, and for all $d_1, \ldots, d_n \in X$ and $O_1, \ldots, O_m \in \sigma$,*

$$\mathfrak{M} \models \alpha\,[d_1, \ldots, d_n, O_1, \ldots, O_m] \quad \textit{iff } \widehat{\mathfrak{M}} \models \alpha\,[d_1, \ldots, d_n, O_1, \ldots, O_m]\,.$$

Moreover, every $\mathcal{L}^2$-formula $\phi(x_1, \ldots, x_n, U_1, \ldots, U_m)$ satisfying this invariance property is equivalent on topological models to an $\mathcal{L}_t$-formula with the same free variables.

It follows that $\mathcal{L}_t$ satisfies appropriate analogues of Compactness, the Löwenheim–Skolem property, and the Łoś theorem *relative to the class of topological models*. Let us start with the Löwenheim–Skolem property. Call a topological model $\mathfrak{M} = (X, \tau, \nu)$ *countable* if X is countable and τ has a countable base.

Theorem 14 (*Löwenheim–Skolem for $\mathcal{L}_t$*). *Let Γ be any set of $\mathcal{L}_t$-formulas (in a countable signature). If Γ has an infinite topological model, then it has a countable topological model.*

Next, we will discuss an analogue of Łoś's theorem for $\mathcal{L}_t$. First we need to define ultraproducts of topological models.

Definition 15 (*Ultraproducts of Basoid Models*). Let $(\mathfrak{M}_i)_{i\in I}$ be an indexed family of basoid models, where $\mathfrak{M}_i = (X_i, \sigma_i, \nu_i)$, and let $\mathfrak{D}$ be an ultrafilter over the index set I. Define an equivalence relation $\sim_{\mathfrak{D}}$ on $\prod_{i\in I} X_i$ as follows:

$$x \sim_{\mathfrak{D}} y \quad \text{iff } \{i \mid x_i = y_i\} \in \mathfrak{D}.$$

We define the *ultraproduct* $\prod_{\mathfrak{D}} \mathfrak{M}_i$ to be (X, σ, ν), where $X = (\prod_{i\in I} X_i)/_{\sim_{\mathfrak{D}}}$, $\sigma = \{(\prod_{i\in I} O_i)/_{\sim_{\mathfrak{D}}} \mid \text{each } O_i \in \sigma_i\}$, and $\nu(p) = (\prod_{i\in I} \nu_i(p))/_{\sim_{\mathfrak{D}}}$.

If $\mathfrak{M}_i = \mathfrak{M}_j$ for all $i, j \in I$, then $\prod_{\mathfrak{D}} \mathfrak{M}_i$ is called an *ultrapower*.

It is not hard to see that, under this definition, every ultraproduct of basoid models is again a basoid model. The same does *not* hold for topological models. Hence, rather than the basoid ultrapower $\prod_{\mathfrak{D}} \mathfrak{M}_i$, we will use the topological model it generates, i.e., $\widehat{\prod_{\mathfrak{D}} \mathfrak{M}_i}$. We will call the latter the *topological ultraproduct* (or, *topological ultrapower*, if all factor models coincide). Note that, by Theorem 13, the topological ultraproduct $\widehat{\prod_{\mathfrak{D}} \mathfrak{M}_i}$ cannot be distinguished from the basoid ultraproduct $\prod_{\mathfrak{D}} \mathfrak{M}_i$ in $\mathcal{L}_t$.

Theorem 16 (*Łoś Theorem for $\mathcal{L}_t$*). *Let α be any $\mathcal{L}_t$-sentence, $(\mathfrak{M}_i)_{i\in I}$ an indexed set of topological models, and $\mathfrak{D}$ an ultrafilter over I. Then*

$$\widehat{\prod_{\mathfrak{D}} \mathfrak{M}_i} \models \alpha \quad \textit{iff } \{i \in I \mid \mathfrak{M}_i \models \alpha\} \in \mathfrak{D}.$$

In particular, if $\mathfrak{N}$ is a topological ultrapower of $\mathfrak{M}$, then for all $\mathcal{L}_t$-formulas ϕ and assignments g, $\mathfrak{M} \models \phi\,[g]$ iff $\mathfrak{N} \models \phi\,[f \cdot g]$, where $f : \mathfrak{M} \to \mathfrak{N}$ is the natural diagonal embedding.

A typical use of ultraproducts is for proving compactness.

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[2] Essentially, this is due to the fact that, within the class of all two-sorted first-order structures, the basoid models can be defined up to isomorphism by conjunction of the following sentences of $\mathcal{L}^2$ (cf. Definition 2, see also [16, p. 14]):

$$\begin{aligned}
\text{Ext} &\equiv \forall U, V.(U = V \leftrightarrow \forall x.(x\varepsilon U \leftrightarrow x\varepsilon V))\\
\text{Union} &\equiv \forall x.\exists U.(x\varepsilon U)\\
\text{Empty} &\equiv \exists U.\forall x.(\neg x\varepsilon U)\\
\text{Bas} &\equiv \forall x.\forall U, V.(x\varepsilon U \wedge x\varepsilon V \to \exists W.(x\varepsilon W \wedge \forall z.(z\varepsilon W \to z\varepsilon U \wedge z\varepsilon V))).
\end{aligned}$$

Theorem 17 (*Compactness for $\mathcal{L}_t$*). *Let Γ be any set of $\mathcal{L}_t$-formulas. If every finite subset of Γ is satisfiable in a topological model, then Γ itself is satisfiable in a topological ultraproduct of these models.*

Another common use of ultraproducts is for obtaining *saturated models*. One can generalize this construction to topological models, provided that the notion of saturation is defined carefully enough. The following definition of saturatedness is probably not the most general, but will suffice for the purposes of this paper.

Definition 18 (*$\mathcal{L}_t$-Saturatedness*). By an *$\mathcal{L}_t$-type* we will mean a set of $\mathcal{L}_t$-formulas $\Gamma(x)$ having exactly one free point variable x and no free open variables. An open set O in a topological model is called *point-saturated* if, whenever all finite subtypes of an $\mathcal{L}_t$-type $\Gamma(x)$ are realized somewhere in O, then $\Gamma(x)$ itself is realized somewhere in O. A topological model $\mathfrak{M} = (X, \tau, \nu)$ is said to be *$\mathcal{L}_t$-saturated* if the following conditions hold:

(1) The entire space X is point-saturated.
(2) The collection of all point-saturated open sets forms a base for the topology. Equivalently, for each point d with open neighborhood O, there is a point-saturated open subneighborhood $O' \subseteq O$ of d.
(3) Every point d has an open neighborhood O_d such that, for all $\mathcal{L}_t$-formulas $\phi(x)$, if $\phi(x)$ holds throughout *some* open neighborhood of d then $\phi(x)$ holds throughout O_d.

Theorem 19. *Every topological model $\mathfrak{M}$ has an $\mathcal{L}_t$-saturated topological ultrapower. This holds regardless of the cardinality of the language.*

Proof. Let $\mathfrak{M}$ be any topological model. It follows from classical model-theoretic results that $\mathfrak{M}$ has a basoid ultrapower $\prod_{\mathfrak{D}} \mathfrak{M} = (X, \sigma, \nu)$ that is countably saturated (in the classical sense, for the language $\mathcal{L}^2$) [10, Theorem 6.1.4 and 6.1.8]. We claim that $\widehat{\prod_{\mathfrak{D}} \mathfrak{M}}$ is $\mathcal{L}_t$-saturated.

In what follows, with *basic open sets* we will mean open sets from the basoid model $\prod_{\mathfrak{D}} \mathfrak{M}$.

(1) Suppose every finite subset of an $\mathcal{L}_t$-type $\Gamma(x)$ is satisfied by some point in $\widehat{\prod_{\mathfrak{D}} \mathfrak{M}}$. In other words, for every finite $\Gamma'(x) \subseteq \Gamma$, $\widehat{\prod_{\mathfrak{D}} \mathfrak{M}} \models \exists x. \bigwedge \Gamma'(x)$. Note that the latter formula belongs to $\mathcal{L}_t$. It follows by base invariance (Theorem 13) that every finite subset of $\Gamma(x)$ is satisfied by some point in $\prod_{\mathfrak{D}} \mathfrak{M}$. Hence, by the countable saturatedness of this basoid model, there is a point d satisfying all formulas of $\Gamma(x)$. Applying the base invariance again, we conclude that d still satisfies all formulas of $\Gamma(x)$ in $\widehat{\prod_{\mathfrak{D}} \mathfrak{M}}$.
(2) Let d be any point and O any open neighborhood of d. By definition, O is a union of basic open sets from $\prod_{\mathfrak{D}} \mathfrak{M}$. It follows that d must have a basic open subneighborhood O'. Of course, O' is still an open neighborhood of d in $\widehat{\prod_{\mathfrak{D}} \mathfrak{M}}$. By the same argument as before, we know that O' is point-saturated—just consider the type $\Sigma(x) = \{x \varepsilon O'\} \cup \Gamma(x)$.
(3) Let d be any point and let Σ be the collection of all $\mathcal{L}_t$-formulas $\phi(x)$ that hold throughout some open neighborhood of d. Recall that each open neighborhood of d contains a basic open subneighborhood of d. It follows that each $\phi(x) \in \Sigma$ holds throughout some basic open neighborhood of d.

    Next, we will proceed using the language $\mathcal{L}^2$, and the fact that $\prod_{\mathfrak{D}} \mathfrak{M}$ is countably saturated as a model for this language. Consider the following set of $\mathcal{L}^2$-formulas (where $\mathbf{d}$ is used as a parameter referring to d, and U is a free open variable):

    $$\Gamma(U) = \{\mathbf{d} \varepsilon U\} \cup \{\forall y.(y \varepsilon U \to \phi(y)) \mid \phi(y) \in \Sigma\}.$$

    Every finite subset of $\Gamma(U)$ holds throughout some basic open neighborhood of d (in $\prod_{\mathfrak{D}} \mathfrak{M}$). This follows from the definition of Σ, the base invariance of $\mathcal{L}_t$, and the fact that every open neighborhood of d contains a basic open neighborhood.

    It follows by the countable saturatedness of $\prod_{\mathfrak{D}} \mathfrak{M}$ with respect to $\mathcal{L}^2$ that there is a basic open set O_d satisfying all formulas in $\Gamma(U)$. In particular, O_d is an open neighborhood of d and (applying base invariance once more) all formulas in Σ hold throughout O_d in $\widehat{\prod_{\mathfrak{D}} \mathfrak{M}}$. $\dashv$

We can conclude that $\mathcal{L}_t$ is model theoretically quite well behaved. Computationally, $\mathcal{L}_t$ is unfortunately less well behaved.

Theorem 20. *The $\mathcal{L}_t$-theory of all topological spaces is undecidable, even in the absence of unary predicates. The same holds for T_0-spaces, for T_1-spaces, and for T_2-spaces. The $\mathcal{L}_t$-theory of topological models based on T_3-spaces, on the other hand, is decidable.*

The next natural question is which topologically interesting properties we can express in this language. Table 1 lists some examples of properties that can be expressed in $\mathcal{L}_t$ (where $x \not\varepsilon U$ is used as shorthand for $\neg(x \varepsilon U)$, $\forall U_x.\alpha$ as shorthand for $\forall U.(x \varepsilon U \to \alpha)$, and $\exists U_x.\alpha$ as shorthand for $\exists U.(x \varepsilon U \wedge \alpha)$). Recall that the *separation axioms* $T_0 - T_5$ are properties of spaces that allow separating distinct points and/or disjoint closed sets [11].

A typical example of a property not expressible in $\mathcal{L}_t$ is *connectedness*.

Definition 21 (*Connectedness*). A topological space (X, τ) is said to be *connected* if $\emptyset$ and X are the only sets that are both open and closed.

Table 1
Some examples of properties that can be expressed in $\mathcal{L}_t$.

T_0	$\forall xy.(x \neq y \to \exists U_x.(y \notin U) \vee \exists V_y.(x \notin V))$
T_1	$\forall xy.(x \neq y \to \exists U_x.(y \notin U))$
T_2	$\forall xy.(x \neq y \to \exists U_x.\exists V_y.\forall z.(z \notin U \vee z \notin V))$
Regular	$\forall x.\forall U_x.\exists V_x.\forall y.(y \varepsilon U \vee \exists V'_y.\forall z.(z \varepsilon V' \to z \notin V))$
T_3	$T_2 \wedge$ Regular
Discrete	$\forall x.\exists U_x.\forall y.(y \varepsilon U \to y = x)$
Alexandroff	$\forall x.\exists U_x.\forall V_x.\forall y.(y \varepsilon V \to y \varepsilon U)$

Theorem 22 ([14, page 8]). *Connectedness is not expressible in $\mathcal{L}_t$.*

Note that connectedness *is* expressible in $\mathcal{L}^2$, namely by the sentence $\forall U, U'.(\forall x.(x \varepsilon U \leftrightarrow x \notin U') \to (\forall x.(x \varepsilon U) \vee \forall x.(x \notin U)))$.

We have the following translation from the basic modal language to $\mathcal{L}_t$.[3]

Definition 23 (*Standard Translation*). The *standard translation ST* from the basic modal language $\mathcal{ML}$ into $\mathcal{L}_t$ is defined inductively:

$$\begin{array}{lcl} ST_x(\top) & = & \top \\ ST_x(p) & = & P_p(x) \\ ST_x(\neg\phi) & = & \neg ST_x(\phi) \\ ST_x(\phi \wedge \psi) & = & ST_x(\phi) \wedge ST_x(\psi) \\ ST_x(\Box\phi) & = & \exists U.(x \varepsilon U \wedge \forall y.(y \varepsilon U \to ST_y(\phi))) \end{array}$$

where x, y are distinct point variables and U is an open variable.

Theorem 24. *For $\mathfrak{M}$ a topological model and $\varphi \in \mathcal{ML}$ a modal formula, $\mathfrak{M}, a \models \varphi$ iff $\mathfrak{M} \models ST_x(\varphi)[a]$*

Proof. By induction on the complexity of φ. $\dashv$

In other words, modal formulas can be seen as $\mathcal{L}_t$-formulas in one free variable, and sets of modal formulas can be seen as $\mathcal{L}_t$-*types* in the sense of Definition 18. This shows that all the above results on $\mathcal{L}_t$ also apply to modal formulas. For example,

Theorem 25 (*Löwenheim–Skolem Theorem for $\mathcal{ML}$*). *Let $\Sigma \subseteq \mathcal{ML}$ be a set of modal formulas (in a countable signature). If Σ is satisfied in a topological model, then it is satisfied in a countable topological model.*

3. The basic modal language

The expressive power of the basic modal language *on relational structures* is relatively well understood. The Van Benthem theorem characterizes the modally definable *properties of points in Kripke models*, in terms of bisimulations, while the Goldblatt–Thomason theorem characterizes modal definability of *classes of Kripke frames*, in terms of closure under operations such as disjoint union.

In this section, we will prove topological analogues of these results. First, we present a topological version of Van Benthem's theorem, using the notion of *topo-bisimulations* [1]. Next, we identify four operations on topological spaces that preserve validity of modal formulas. Finally, we apply these closure conditions in order to determine which $\mathcal{L}_t$-definable classes are modally definable, and vice versa.

3.1. Topological bisimulations

In this section we characterize the modal fragment of $\mathcal{L}_t$ in terms of topo-bisimulations.

Definition 26. Consider topological models $\mathfrak{M} = (X, \nu)$ and $\mathfrak{M}' = (X', \nu')$. A non-empty relation $Z \subseteq X \times X'$ is a *topo-bisimulation between $\mathfrak{M}$ and $\mathfrak{M}'$* if the following conditions are met for all $x \in X$ and $x' \in X'$:

Zig If xZx' and $x \in O \in \tau$ then there exists $O' \in \tau'$ such that $x' \in O'$ and for all $y' \in O'$ there exists a $y \in O$ such that yZy'.
Zag If xZx' and $x' \in O' \in \tau'$ then there exists $O \in \tau$ such that $x \in O$ and for all $y \in O$ there is a $y' \in O'$ such that yZy'.
Atom If xZx' then $x \in \nu(p)$ iff $x' \in \nu'(p)$ for all $p \in$ Prop.

Elements $x \in X$ and $x' \in X'$ are said to be *bisimilar*, denoted by $(\mathfrak{M}, x) \underline{\leftrightarrow} (\mathfrak{M}', x')$, if there exists a bisimulation Z between $\mathfrak{M}$ and $\mathfrak{M}'$ such that xZx'.

[3] In fact, a slight variation of this translation shows that modal formulas can be mapped to $\mathcal{L}_t$-formulas containing at most two point variables and one open variable.

This definition can be formulated more naturally if we use some standard mathematical notation. For a binary relation $Z \subseteq X \times X'$ and a set $A \subseteq X$, let us denote by $Z[A]$ the *image* $\{x' \in X' \mid \exists x \in A.(xZx')\}$, and let us define the *preimage* $Z^{-1}[A']$ of a set $A' \subseteq X'$ analogously.

Proposition 27. *The **Zig** and **Zag** conditions in Definition 26 are equivalent to the following:*

Zig$'$ *For all* $O \in \tau, Z[O] \in \tau'$.
Zag$'$ *For all* $O' \in \tau, Z^{-1}[O'] \in \tau$.

Proof. We will only show the equivalence for **Zig**$'$, the proof for **Zag**$'$ is analogous. In one direction, suppose that Z satisfies **Zig**, and take an open $O \in \tau$. The **Zig** condition ensures that, for each $x' \in Z[O]$, we can find an open neighborhood $O' \in \tau'$ with $x' \in O'$, such that $O' \subseteq Z[O]$. It follows that $Z[O]$, being the union of these neighborhoods, is open in τ'. For the other direction, suppose $Z[O] \in \tau'$ holds for all $O \in \tau$. Consider an arbitrary $x \in O \in \tau$ and $x' \in X'$ such that xZx'. Then $Z[O]$ qualifies for an open neighborhood O' of x' satisfying the condition **Zig** since $x' \in Z[O] \in \tau'$. $\dashv$

In what follows, we will freely use this equivalent formulation whenever it is convenient. Topo-bisimulations are closely linked with the notion of modal equivalence.

Definition 28. We say that two pointed topological models $(\mathfrak{M}, x)$ and $(\mathfrak{M}', x')$ are *modally equivalent* and write $(\mathfrak{M}, x) \leftrightsquigarrow (\mathfrak{M}', x')$ if for all formulas $\phi \in \mathcal{ML}$, $(\mathfrak{M}, x) \models \phi$ iff $(\mathfrak{M}', x') \models \phi$.

Theorem 29 ([1]). *For arbitrary topological pointed models $(\mathfrak{M}, x)$ and $(\mathfrak{M}', x')$, if $(\mathfrak{M}, x) \underline{\leftrightarrow} (\mathfrak{M}', x')$ then $(\mathfrak{M}, x) \leftrightsquigarrow (\mathfrak{M}', x')$.*

Proof. The proof proceeds via straightforward induction on the complexity of modal formulas. We only treat the case $\phi = \Box\psi$.

Suppose $(\mathfrak{M}, x) \models \Box\psi$. Then there exists an open neighborhood O of x such that $O \models \psi$. By **Zig** we obtain that $Z[O]$ is an open neighborhood of x' and, by induction hypothesis, $Z[O] \models \psi$. Therefore $(\mathfrak{M}', x') \models \Box\psi$. The other direction is proved similarly. $\dashv$

The converse does not hold in general, but it holds on a restricted class of $\mathcal{L}_t$-saturated topological models.

Theorem 30. *Let $\mathfrak{M}$ and $\mathfrak{M}'$ be $\mathcal{L}_t$-saturated topological models, and suppose that $(\mathfrak{M}, x) \leftrightsquigarrow (\mathfrak{M}', x')$. Then $(\mathfrak{M}, x) \underline{\leftrightarrow} (\mathfrak{M}', x')$.*

Proof. Let $\mathfrak{M} = (X, \tau, \nu)$ and $\mathfrak{M}' = (X', \tau', \nu')$, and let $Z \subseteq X \times X'$ be the modal indistinguishability relation (i.e., xZx' iff $(\mathfrak{M}, x) \leftrightsquigarrow (\mathfrak{M}', x')$). We will show that Z is a topo-bisimulation, and hence, $(\mathfrak{M}, x) \underline{\leftrightarrow} (\mathfrak{M}', x')$. That the **Atom** condition holds follows immediately from the construction of Z. In the remainder of this proof, we will show that **Zag** holds. The case for **Zig** is analogous.

Consider any a, a' such that aZa', and let $O' \in \tau'$ be an open neighborhood of a'. Since $\mathfrak{M}'$ is $\mathcal{L}_t$-saturated, we may assume that O' is point-saturated (if not, just take a point-saturated subneighborhood of a'). We need to find an open neighborhood O of a such that for each $b \in O$ there exists a $b' \in O'$ with bZb'.

By $\mathcal{L}_t$-saturatedness of $\mathfrak{M}$, we know that a has an open neighborhood O_a such that, for every modal formula ϕ, if $a \models \Box\phi$ then ϕ holds throughout O_a. Dually, this means that

(*) For any $b \in O_a$ and modal formula ϕ, if $b \models \phi$ then $a \models \Diamond\phi$.

Proof. To show that O_a meets the requirements of the **Zag** condition, consider any $b \in O_a$. We will find a $b' \in O'$ such that bZb'. Let Σ_b be the set of modal formulas true at b. Every finite subset of Σ_b is satisfied somewhere in O'. For, consider any finite $\Sigma' \subseteq \Sigma_b$. Then by (*), $\mathfrak{M}, a \models \Diamond \bigwedge \Sigma'$, and hence $\mathfrak{M}', a' \models \Diamond \bigwedge \Sigma'$. Therefore $\bigwedge \Sigma'$ must be satisfied somewhere in O'. Recall that O' is point-saturated. We conclude that there is a point $b' \in O'$ satisfying Σ_b. It follows that $(\mathfrak{M}, b) \leftrightsquigarrow (\mathfrak{M}', b')$, and hence bZb'. $\dashv$

Combining this with Theorem 19, we obtain

Theorem 31. *An $\mathcal{L}_t$-formula $\alpha(x)$ is invariant under topo-bisimulations iff it is equivalent to the standard translation of a modal formula.*

Proof. Easily adapted from the proof of the Van Benthem Characterization Theorem for relational semantics (see e.g. [6, Theorem 2.68] for details). $\dashv$

3.2. Validity preserving operations

In this section, we use topo-bisimulations for showing that three natural operations on topological spaces (topological sums, open subspaces and interior maps) preserve validity of modal formulas.

3.2.1. Topological sums

The *topological sum* (also called *disjoint union*, *direct sum*, or *coproduct*) of a family of disjoint topological spaces $(X_i, \tau_i)_{i\in I}$, denoted by $\biguplus_{i\in I}(X_i, \tau_i)$, is the space (X, τ) with $X = \bigcup_{i\in I} X_i$ and $\tau = \{O \subseteq X \mid \forall i \in I.(O \cap X_i \in \tau_i)\}$. For non-disjoint spaces, the topological sum is obtained by taking appropriate isomorphic copies. In what follows, when working with topological sums, we will tacitly assume that the spaces involved are disjoint (cf. [11, pp. 123–126]).

Theorem 32. *Let $(X_i)_{i\in I}$ be a family of topological spaces and let ϕ be a modal formula. Then $\biguplus_{i\in I} X_i \models \phi$ iff $\forall i \in I.(X_i \models \phi)$*

Proof. There is a natural topo-bisimulation Z_i between X_i and $X = \biguplus_{i\in I} X_i$:

$$Z_i = \{(x, x) \mid x \in X_i\}.$$

Suppose that $X_i \models \phi$ for each $i \in I$. In order to show that $X \models \phi$, consider any valuation ν and point x. Clearly x must belong to X_j for some $j \in I$. Let ν_j be the restriction of ν to X_j, i.e., $\nu_j(p) = \nu(p) \cap X_j$ for each $p \in$ Prop. It is easily seen that Z_j is a topo-bisimulation between $((X, \nu), x)$ and $((X_j, \nu_j), x)$. Since $X_j \models \phi$, we obtain by Theorem 29 that $(X, \nu), x \models \phi$. This argument was independent of ν and x, and therefore we may conclude that $X \models \phi$.

The other direction is established similarly, and follows also from Theorem 34 below. ⊣

The above lemma can immediately be put to use to show that compactness and connectedness are not modally definable. Recall that a space is said to be *compact* if any open cover of the space contains a finite subcover, and a space is said to be *connected* if it does not contain a proper non-empty subset that is both closed and open.

Corollary 33. *The class of connected spaces and the class of compact spaces are not modally definable.*

Proof. In view of Theorem 32 it suffices to note that while each space $X_i = (\{i\}, \{X_i, \emptyset\})$ (a singleton set equipped with the only possible topology) is both connected and compact, the topological sum $X = \biguplus_{i\in\omega} X_i$ is neither connected nor compact. ⊣

Incidentally, the class of connected spaces is definable in a modal language with the *global modality* [32]. We discuss the global modality and the connectedness axiom in Section 5.1 below.

Typical examples of properties that *are* preserved under taking disjoint union are *disconnectedness*, as well as T_0, T_1, T_2, and discreteness.

3.2.2. Open subspaces

Given a topological space (X, τ) and an open subset $O \in \tau$, there is a natural topology on O induced by τ, namely $\tau_O = \{A \subseteq O \mid A \in \tau\}$, or, equivalently, $\tau_O = \{A \cap O \mid A \in \tau\}$ (cf. [11, pp. 111–112]). An *open subspace* of X is any space (O, τ_O) for $O \in \tau$, $O \neq \emptyset$.

Theorem 34. *Let (X, τ) be a space and (O, τ_O) an open subspace, and let ϕ a modal formula. If $X \models \phi$ then $O \models \phi$.*

Proof. Suppose $(O, \nu), x \not\models \phi$, for some valuation ν and point $x \in O$. We can view ν also as a valuation for X. The inclusion map is then a topo-bisimulation between $((O, \nu), x)$ and $((X, \nu), x)$. It follows by Theorem 29 that $(X, \nu), x \not\models \phi$. ⊣

Theorem 34 provides us with another way to prove that connectedness is not modally definable: the real line $\mathbb{R}$ with the usual topology is connected, but its open subspace $\mathbb{R}-\{0\}$ is not. Using Theorem 34 we can also show that *disconnectedness* is not modally definable. We call a space disconnected if it is not connected. Since the two-point discrete space is disconnected, while its one-point open subspaces are connected, we obtain the following corollary:

Corollary 35. *The class of disconnected spaces is not modally definable.*

Typical examples of properties that *are* preserved under taking open subspaces are T_0, T_1, T_2, *density-in-itself* and *being a Baire space*.

3.2.3. Images of interior maps

The third operation that we will consider is taking images of *interior maps* (also known as *continuous open maps*). A map $f : X_1 \to X_2$ between topological spaces (X_1, τ_1) and (X_2, τ_2) is said to be *open* if $f(O) \in \tau_2$ for each $O \in \tau_1$ (i.e. images of opens are open), and *continuous* if $f^{-1}(O) \in \tau_1$ for each $O \in \tau_2$ (i.e. preimages of opens are open). If f is both open and continuous, it is called an *interior map*. Note that homeomorphisms are simply bijective interior maps (cf. [11, pp. 57-67]).

Theorem 36. *Let X_1 and X_2 be topological spaces and $f : X_1 \to X_2$ a surjective interior map. For all modal formulas ϕ, if $X_1 \models \phi$ then $X_2 \models \phi$.*

Proof. By contraposition: suppose $(X_2, \nu_2), x_2 \not\models \phi$ for some ν_2, x_2. Let x_1 be any element of X_1 such that $f(x_1) = x_2$ (recall that f is surjective), and let ν_1 be the valuation on X_1 defined by $\nu_1(p) = f^{-1}[\nu_1(p)]$. By construction, the graph of f is a topo-bisimulation between $((X_1, \nu_1), x_1)$ and $((X_2, \nu_2), x_2)$ (cf. Proposition 27). It follows by Theorem 29 that $(X_1, \nu_1), x_1 \not\models \phi$. ⊣

Not many properties of spaces are preserved under taking images of interior maps.

It is known that the real line $\mathbb{R}$ with its usual topology obeys all separation axioms T_i for $i \in \{0, D, 1, 2, 3, 3\frac{1}{2}, 4, 5\}$. As a corollary of Theorem 36, we obtain that none of these are definable in the basic modal language.

Corollary 37. *The separation axioms T_i with $i \in \{0, D, 1, 2, 3, 3\frac{1}{2}, 4, 5\}$ are not definable in the basic modal language.*

Proof. Consider the interior map from the real line $\mathbb{R}$ with the standard topology onto $X = \{1, 2\}$ equipped with the trivial topology $\tau = \{\emptyset, X\}$, sending the rationals to 1 and the irrationals to 2. It is easy to verify that the reals obey all separation axioms, while X obeys none. As surjective interior maps preserve modal validity, none of the separation axioms can be defined by a formula in the basic modal language. $\dashv$

We will show in Section 5 that extending the modal language can help us in defining some of the lower separation axioms.

Examples of properties that *are* preserved under interior maps are being HI, extremally disconnected, compact, connected or separable (in fact, the latter three are even preserved by continuous maps).

For further application of the preservation results presented in this section, as well as related techniques for establishing (un)definability of topological properties such as submaximality, being nodec, door, maximal, perfectly disconnected, etc., see the recent paper [3].

3.3. Alexandroff extensions

In this section, we introduce a fourth operation on topological spaces: the formation of *Alexandroff extensions*. It allows one to turn arbitrary spaces into Alexandroff spaces. We will show that this construction reflects the validity of modal formulas, and we will identify a connection between Alexandroff extensions and topological ultraproducts.

Definition 38 (*Alexandroff Extensions*). A filter $\mathcal{F} \subseteq \wp(X)$ over a topological space (X, τ) is called *open* if for all $A \in \mathcal{F}$, also $\mathbb{I}A \in \mathcal{F}$. The *Alexandroff extension* of a space (X, τ) is the space $X^* = (\mathtt{Uf}X, \tau^*)$, where $\mathtt{Uf}X$ is the set of ultrafilters over X, and τ^* is the topology over $\mathtt{Uf}X$ generated by the sets of the form $\{\mathfrak{u} \in \mathtt{Uf}X \mid \mathcal{F} \subseteq \mathfrak{u}\}$ for $\mathcal{F}$ an open filter over X.

Theorem 39. *For any space X, X^* is Alexandroff.*

Proof. For any point $\mathfrak{u} \in X^*$ consider a filter $\mathcal{F}$ generated by all open sets that belong to $\mathfrak{u}$. Then the set $\{\mathfrak{v} \in X^* \mid \mathcal{F} \subseteq \mathfrak{v}\}$ is a least open neighborhood of $\mathfrak{u}$. It follows that $\mathfrak{v}$ is in the least open neighborhood of $\mathfrak{u}$ iff for each $\mathbb{I}A \in \mathfrak{u}$ we have $\mathbb{I}A \in \mathfrak{v}$ iff $\mathbb{C}A \in \mathfrak{u}$ for each $A \in \mathfrak{v}$. $\dashv$

Note that the map $\pi : X \to X^*$ that sends $a \in X$ to the corresponding principal ultrafilter π_a need not be open, or even continuous [5, Example 5.13]. Indeed the image $\pi(X)$, as a subspace of X^*, might not be homeomorphic to X—as soon as X is T_1 the subspace $\pi(X)$ is discrete. Nevertheless, it is worth mentioning that the topology τ^* preserves the information about the original topology τ in a curious way. It is an easy exercise for the reader familiar with ultrafilter convergence (see, e.g., [11, pp. 91–93]) that $\mathfrak{u} \in X^*$ belongs to the least open neighborhood of the principal ultrafilter π_a in X^* iff $\mathfrak{u} \to a$ (i.e. $\mathfrak{u}$ converges to $a \in X$ according to τ).

Basic open sets of the Alexandroff extension X^* have a nice characterization that follows immediately from their definition. For any topological space X and subset $A \subseteq X$, let $A^* = \{\mathfrak{u} \in X^* \mid A \in \mathfrak{u}\}$. It easily seen that:

- $\{a\}^* = \{\pi_a\}$;
- $(A \cap B)^* = A^* \cap B^*$, $(A \cup B)^* = A^* \cup B^*$;
- A^* is open iff A is open.

Now, the basic open sets of X^* are precisely the sets of the form $\bigcap_{A \in \mathcal{F}} A^*$ for $\mathcal{F}$ an open filter on X.

We saw in the earlier sections that some topological constructions preserve modal validity. Now we show that formation of the Alexandroff extension *anti-preserves* modal validity.

Theorem 40. *Let X be a topological space and X^* its Alexandroff extension. For all modal formulas ϕ, if $X^* \models \phi$ then $X \models \phi$.*

Proof. By contraposition: suppose $X \not\models \phi$. Then there exists a valuation ν such that $\nu(\neg\phi) \neq \emptyset$. Let ν^* be the valuation on X^* defined by $\nu^*(p) = \{\mathfrak{u} \in X^* \mid \nu(p) \in \mathfrak{u}\}$. We will show that

(*) For any $\psi \in \mathcal{ML}$ and $\mathfrak{u} \in X^*$, $\mathfrak{u} \models \psi$ iff $\nu(\psi) \in \mathfrak{u}$.

This gives us the intended result: since $\nu(\neg\phi) \neq \emptyset$, we can extend $\nu(\neg\phi)$ to an ultrafilter. It follows that $\nu^*(\neg\phi) \neq \emptyset$, so $X^* \not\models \phi$, as required.

We will prove (*) by induction on the complexity of the formula ψ. The propositional case is taken care of by the definition of ν^*, the cases for the boolean connectives are rather obvious, so we only address the modality case. Let ψ therefore be of the form $\Box\xi$.

[$\Rightarrow$] Suppose $\mathfrak{u} \models \Box\xi$. Then $\mathfrak{u}$ has an open neighborhood (restrict to the element of the base without loss of generality) $O = \{\mathfrak{v} \in X^* \mid \mathcal{F} \subseteq \mathfrak{v}\}$ such that $\mathcal{F}$ is an open filter over X and $\mathfrak{v} \models \xi$ holds for all $\mathfrak{v} \in O$. In other words,

$$\mathcal{F} \subseteq \mathfrak{v} \Rightarrow \mathfrak{v} \models \xi.$$

By the induction hypothesis, this can be rephrased as

$$\mathcal{F} \subseteq \mathfrak{v} \Rightarrow \nu(\xi) \in \mathfrak{v}$$

for all $\mathfrak{v} \in X^*$. This indicates that $\nu(\xi) \in \mathcal{F}$. As $\mathcal{F}$ is an open filter, we obtain $\mathbb{I}\nu(\xi) \in \mathcal{F}$. Since $\mathfrak{u}$ extends $\mathcal{F}$, it follows, that $\mathbb{I}\nu(\xi) = \nu(\Box\xi) \in \mathfrak{u}$.

[$\Leftarrow$] Suppose $\nu(\Box\xi) \in \mathfrak{u}$. Then $\mathbb{I}\nu(\xi) \in \mathfrak{u}$. Consider any ultrafilter $\mathfrak{v}$ from the least open neighborhood of $\mathfrak{u}$. Clearly $\mathbb{I}\nu(\xi) \in \mathfrak{v}$. By $\mathbb{I}\nu(\xi) \subseteq \nu(\xi)$ we get $\nu(\xi) \in \mathfrak{v}$. By the induction hypothesis $\mathfrak{v} \in \nu^*(\xi)$. As $\mathfrak{v}$ was arbitrarily chosen from the least open neighborhood of $\mathfrak{u}$, we arrive at $\mathfrak{u} \models \Box\xi$. $\dashv$

We can immediately conclude that

Corollary 41. *The class of Alexandroff spaces is not modally definable.*

Proof. Indeed, suppose a formula α defines the class of Alexandroff spaces. Take an arbitrary non-Alexandroff space X. Then X^* is Alexandroff, so $X^* \models \alpha$ and by the above theorem $X \models \alpha$. It follows that X is Alexandroff, contrary to our assumption. $\dashv$

The following key theorem (which can be seen as a topological analogue of [6, Theorem 3.17]) connects Alexandroff extensions to topological ultrapowers.

Theorem 42. *For every topological space $X = (X, \tau)$ there exists a topological ultrapower $\widehat{\prod_{\mathfrak{D}} X}$ and a surjective interior map $f : \widehat{\prod_{\mathfrak{D}} X} \to X^*$. In a picture:*

$$\begin{tikzcd} \widehat{\prod_{\mathfrak{D}} X} \arrow[d] \arrow[dr, "f"] & \\ X & X^* \end{tikzcd}$$

Proof. Let us consider an $\mathcal{L}^2$-based language containing a unary predicate P_A for each $A \subseteq X$, interpreted naturally on X, i.e., $(P_A)^X = A$. In what follows we will treat X as a topological model for this (possibly uncountable) language. By Theorem 19, X has an $\mathcal{L}_t$-saturated topological ultrapower $\widehat{\prod_{\mathfrak{D}} X}$. Denote $Y \equiv \widehat{\prod_{\mathfrak{D}} X}$. The following $\mathcal{L}_t$-sentences are clearly true in X, and hence, by Theorem 16, also in Y:

(1) $Y \models \exists x.P_A(x)$ for each non-empty $A \subseteq X$,
(2) $Y \models \forall x.(P_A(x) \wedge P_B(x) \leftrightarrow P_{A\cap B}(x))$ for each $A, B \subseteq X$,
(3) $Y \models \forall x.(\neg P_A(x) \leftrightarrow P_{-A}(x))$ for each $A \subseteq X$,
(4) $Y \models \forall x.(P_{\mathbb{I}A}(x) \leftrightarrow \exists U.(x\varepsilon U \wedge \forall y.(y\varepsilon U \to P_A(y))))$ for each $A \subseteq X$,
(5) $Y \models \forall x.(P_{\mathbb{C}A}(x) \leftrightarrow \forall U.(x\varepsilon U \to \exists y.(y\varepsilon U \wedge P_A(y))))$ for each $A \subseteq X$.

We define the desired interior map $f : Y \to X^*$ in the following way:

$$f(a) = \{A \subseteq X \mid a \in (P_A)^Y\}.$$

In the remainder of this proof, we will demonstrate that f is indeed a surjective interior map from Y to X^*. First we show that f is a well-defined onto map.

- For any $a \in Y$, $f(a)$ is an *ultrafilter* over X.
  Recall that an ultrafilter over X is any set $\mathfrak{u}$ of subsets of X satisfying (i) $A \cap B \in \mathfrak{u}$ iff both $A \in \mathfrak{u}$ and $B \in \mathfrak{u}$, and (ii) $A \in \mathfrak{u}$ iff $(X-A) \notin \mathfrak{u}$. By (2) and (3) above, $f(a)$ indeed satisfies these properties.
- f is *surjective* (i.e., every ultrafilter over X is $f(a)$ for some $a \in Y$).
  Take $\mathfrak{u} \in X^*$, and let $\Gamma_{\mathfrak{u}}(x) = \{P_A(x) \mid A \in \mathfrak{u}\}$. It follows from (1) and (2) that every finite subset of $\Gamma_{\mathfrak{u}}(x)$ is satisfied by some point in Y. Since Y is point-saturated, there exists $a \in Y$ satisfying $\Gamma_{\mathfrak{u}}(x)$, hence $f(a) = \mathfrak{u}$.

Next we show that f is open and continuous. Note that by Proposition 27 it suffices to prove that the graph of f is a topo-bisimulation.

Take arbitrary $a \in Y$ and let O_a be as described in Definition 18. Let $O_{\mathfrak{u}}$ be a least open neighborhood of $\mathfrak{u} = f(a)$. We proceed by verifying the conditions **Zig** and **Zag** for the pair $(a, \mathfrak{u})$.

- **Zig**. Take arbitrary O' such that $a \in O'$. By $\mathcal{L}_t$-saturatedness of Y, there exists a *point-saturated* $O \subseteq O'$ such that $a \in O$.
  Take arbitrary $\mathfrak{v} \in O_{\mathfrak{u}}$. We will find a $b \in O$ such that $\mathfrak{v} = f(b)$. Let

  $$\Gamma_{\mathfrak{v}}(x) = \{P_A(x) \mid A \in \mathfrak{v}\}.$$

  Every finite subset of $\Gamma_{\mathfrak{v}}(x)$ is satisfied somewhere in O. Indeed, if $P_{A_1}, \ldots, P_{A_n} \in \Gamma_{\mathfrak{v}}$, denote $B \equiv \bigcap_i A_i$. Then $B \in \mathfrak{v}$ and hence $\mathbb{C}B \in \mathfrak{u}$. Therefore $Y \models P_{\mathbb{C}B}(a)$. It follows by (5) that P_B holds somewhere in O. By the point-saturatedness of O we may conclude that some $b \in O$ satisfies all of $\Gamma_{\mathfrak{v}}(x)$, and hence $f(b) = \mathfrak{v}$.
- **Zag**. It suffices to show that for any $b \in O_a$ we have $f(b) \in O_{\mathfrak{u}}$. Suppose the contrary. Then we have $b \in O_a$ and $f(b) \notin O_{\mathfrak{u}}$. The latter means that there exists a set $A \subseteq X$ such that $A \in f(b)$ but $\mathbb{C}A \notin \mathfrak{u}$. From $A \in f(b)$ we obtain $Y \models P_A(b)$. While $\mathbb{C}A \notin \mathfrak{u}$ iff $-\mathbb{C}A \in \mathfrak{u}$ iff $\mathbb{I}-A \in \mathfrak{u}$ iff $Y \models P_{\mathbb{I}-A}(a)$ iff $Y \models \exists U.[a\varepsilon U \wedge \forall y.(y\varepsilon U \to P_{-A}(y))]$ iff $P_{-A}(x)$ is true throughout some open neighborhood of a iff $P_{-A}(x)$ is true throughout O_a, which contradicts $Y \models P_A(b)$ since $b \in O_a$. $\dashv$

3.4. Modal definability vs $\mathcal{L}_t$-definability

In this section we are seeking *necessary and sufficient* conditions, in the spirit of the Goldblatt–Thomason theorem, for a class of topological spaces to be modally definable. We have already found some necessary conditions: we have seen that every modally definable class of topological spaces is closed under the formation of *topological sums*, *open subspaces* and *interior images* and reflects *Alexandroff extensions*. Our aim is to prove a converse, in other words, to characterize modal definability in terms of these closure properties.

Theorem 43. *Let* K *be any* $\mathcal{L}_t$*-definable class of topological spaces. Then* K *is modally definable iff it is closed under taking open subspaces, interior images, topological sums and it reflects Alexandroff extensions.*

Proof. We will only prove the difficult right-to-left direction. The left-to-right direction already follows from Theorems 32, 34, 36 and 40.

Let K be any class satisfying the given closure conditions. Take the set $\mathrm{Log}(\mathsf{K})$ of modal formulas valid on K. We will show that, whenever $X \models \mathrm{Log}(\mathsf{K})$, then $X \in \mathsf{K}$. In other words, $\mathrm{Log}(\mathsf{K})$ *defines* K.

Suppose $X \models \mathrm{Log}(\mathsf{K})$ for some topological space X. Introduce a propositional letter p_A for each subset $A \subseteq X$, and let ν be the natural valuation on X for this (possibly uncountable) language, i.e. $\nu(p_A) = A$ for all $A \subseteq X$. Let Δ be the set of all modal formulas of the following forms (where A, B range over subsets of X):

$$\begin{array}{lcl} p_{A\cap B} & \leftrightarrow & p_A \wedge p_B \\ p_{-A} & \leftrightarrow & \neg p_A \\ p_{\mathbb{I}A} & \leftrightarrow & \Box p_A \\ p_{\mathbb{C}A} & \leftrightarrow & \Diamond p_A. \end{array}$$

By definition, Δ is valid on $\mathfrak{M} = (X, \nu)$. Note that the standard translations of the formulas in Δ correspond exactly to the formulas listed in conditions (2)–(5) of Theorem 42 (in the corresponding $\mathcal{L}_t$-language, which has a one-place predicate $P_A(x)$ for each $A \subseteq X$). What is missing is the condition (1). The following claim addresses this.

Claim: For each $a \in X$ there is a model $\mathfrak{N}_a = (Y_a, \mu_a)$ with $Y_a \in \mathsf{K}$, such that $\mathfrak{N}_a \models \Delta$ and some point in $\mathfrak{N}_a$ satisfies p_a.

Proof. Take any $a \in X$, and let $\Delta_a = \{\Box\varphi \mid \varphi \in \Delta\} \cup \{p_{\{a\}}\}$. As a first step, we will show that there is a topological model $\mathfrak{K}$ based on a space in K, such that some point a' of $\mathfrak{K}$ satisfies Δ_a. By the compactness of $\mathcal{L}_t$ (Theorem 17), it suffices to show that every finite conjunction δ of formulas in Δ_a is satisfiable on K. Since δ is satisfied at a in $\mathfrak{M}$ and $\mathfrak{M} \models \mathrm{Log}(\mathsf{K})$, $\neg\delta$ cannot belong to $\mathrm{Log}(\mathsf{K})$. Hence δ is satisfiable on K.

By Theorem 19 we may assume $\mathfrak{K}$ is $\mathcal{L}_t$-saturated. Let $O_{a'}$ be an open neighborhood of a' as described in Definition 18, and let $\mathfrak{N}_a$ be the submodel of $\mathfrak{K}$ based on $O_{a'}$. Then $\mathfrak{N}_a$ satisfies all requirements of the claim. $\dashv$

Note how, in the above argument, we used the fact that K is $\mathcal{L}_t$-definable (for the compactness argument, and for the saturation), and that it is closed under taking open subspaces. Next, we will use the fact that K is closed under taking *topological sums*.

Let $Y = \biguplus_{a\in X} Y_a$, and let $\mathfrak{N} = (Y, \mu)$, where μ is obtained from the μ_a's in the obvious way. By closure under taking topological sums, $Y \in \mathsf{K}$. Moreover, by Theorem 32, $\mathfrak{N} \models \Delta$. Finally, each p_A, for non-empty $A \subseteq X$, holds at some point in $\mathfrak{N}$ (more precisely, at some point in $\mathfrak{N}_a$ for any $a \in A$). It follows (using the standard translation) that the conditions (1)–(5) from the proof of Theorem 42 hold for $\mathfrak{N}$.

We can now proceed as in the proof of Theorem 42, and construct an interior map from an ultrapower of $\mathfrak{N}$ onto the Alexandroff extension X^* of X. Since K is closed under topological ultrapowers (Theorem 16) and images of interior maps, and reflects Alexandroff extensions, we conclude that $X \in \mathsf{K}$. $\dashv$

Inspection of the proof shows that Theorem 43 applies not only to $\mathcal{L}_t$-definable classes but to any class of spaces closed under ultraproducts. In fact, by Lemma 44 below, closure under ultra*powers* suffices. Some further improvements are still possible. Most importantly, using algebraic techniques, we will show in the next section that *closure under Alexandroff extensions* already suffices. For the complete picture, see Corollary 53.

3.4.0.1. The opposite question. Theorem 43 characterizes, among all $\mathcal{L}_t$-definable classes of topological spaces, those that are modally definable. It makes sense to ask the opposite question: *which modally definable classes of spaces are* $\mathcal{L}_t$*-definable?* In classical modal logic the answer was provided by Van Benthem in [35] (see also [19]). We follow the route paved in these papers. First we prove a topological analogue of an observation due to Goldblatt:

Lemma 44. *An ultraproduct of topological spaces is homeomorphic to an open subspace of the ultrapower (over the same ultrafilter) of their topological sum.*

Proof. Suppose $(X_i)_{i\in I}$ is a family of topological spaces and $\mathfrak{D}$ is an ultrafilter over I. Denote by $X = \biguplus_{i\in I} X_i$ the topological sum of X_i and by $Y = \widehat{\prod_{\mathfrak{D}} X_i}$ their topological ultraproduct. Take arbitrary $a : I \to \biguplus_{i\in I} X_i$ such that $a(i) \in X_i$. Then a can be viewed both as an element of $\prod_{i\in I} X_i$ and as an element of $\prod_{i\in I} X$. This defines a natural embedding from Y into $\widehat{\prod_{\mathfrak{D}} X}$ which

is clearly injective. That this embedding is open is easily seen (recall that this suffices to be checked on the elements of the base). To show that it is also continuous, suppose $[a]_{\mathfrak{D}} \in \prod_{\mathfrak{D}} X$ is such that $A = \{i \mid a(i) \in X_i\} \in \mathfrak{D}$ (so $[a]_{\mathfrak{D}}$ comes from Y). Then any basic ultrabox neighborhood $\prod_{\mathfrak{D}} O_i$ of $[a]_{\mathfrak{D}}$ is such that $B = \{i \mid a(i) \in O_i \subseteq X\} \in \mathfrak{D}$. We clearly have $A \cap B \in \mathfrak{D}$, so $\prod_{\mathfrak{D}}(O_i \cap X_i)$ is another open neighborhood of $[a]_{\mathfrak{D}}$, now also in Y. The required continuity follows. Since we have established that Y can be embedded into $\widehat{\prod_{\mathfrak{D}} X}$ by an interior map, it follows that Y is homeomorphic to an open subspace of $\widehat{\prod_{\mathfrak{D}} X}$. $\dashv$

We are one step away from finding a nice criterion for a modally definable class to be $\mathcal{L}_t$-definable. It follows from Garavaglia's theorem [16] (the topological analogue of the Keisler–Shelah Theorem) that a class K of spaces is $\mathcal{L}_t$-definable iff K is closed under isomorphisms and ultraproducts and the complement of K is closed under ultrapowers.

Theorem 45. *A modally definable class* K *of spaces is* $\mathcal{L}_t$*-definable iff it is closed under ultrapowers.*

Proof. If K is $\mathcal{L}_t$-definable, then it is clearly closed under ultrapowers. For the converse direction take a modally definable class K that is closed under ultrapowers. Then K is closed under topological sums and open subspaces. It follows from Lemma 44 that K is closed under ultraproducts. It is easily seen that any modally definable class is closed under $\mathcal{L}_t$-isomorphisms. It follows from Theorems 16 and 24 that the complement of K is closed under ultrapowers. Hence K is $\mathcal{L}_t$-definable. $\dashv$

Since modally definable classes are closed under interior images and ultrapowers are interior images of box products via the canonical quotient map, we obtain

Corollary 46. *A modally definable class of spaces that is closed under box powers is* $\mathcal{L}_t$*-definable.*

3.4.0.2. Separating examples. To close this section we give examples separating $\mathcal{L}_t$-definability from modal definability. We have exhibited earlier $\mathcal{L}_t$-sentences defining the separation axioms $T_0 - T_2$. We have also shown in Corollary 37 that $T_0 - T_2$ are not definable in the basic modal language. Thus we have examples of $\mathcal{L}_t$-definable classes of spaces that are not modally definable.

To show that there are modally definable classes of spaces that are not $\mathcal{L}_t$-definable requires more work. Recall that the class of Hereditarily Irresolvable (HI) spaces is modally definable (Theorem 8). This class is not $\mathcal{L}_t$-definable. We will use the following lemma:

Lemma 47. *Any class* K *of spaces that is both modally definable and* $\mathcal{L}_t$*-definable is closed under Alexandroff extensions.*

Proof. Suppose $X \in \mathsf{K}$. By Theorem 42 there exists a topological ultrapower Y of X and an onto interior map $f : Y \to X^*$. Being an $\mathcal{L}_t$-definable class, K is closed under topological ultrapowers. Hence $Y \in \mathsf{K}$. Being a modally definable class, K is closed under interior images. Therefore $X^* \in \mathsf{K}$, as required. $\dashv$

Theorem 48. *The class of HI spaces is not* $\mathcal{L}_t$*-definable.*

Proof. By Theorem 8 and the above lemma, to prove that the class of HI spaces is not $\mathcal{L}_t$-definable it suffices to show that this class is not closed under Alexandroff extensions. In [5, Example 5.12] a space X is exhibited that is HI, but its Alexandroff extension is not HI. We reproduce this example for the reader's convenience.

Let $X = (\mathbb{N}, \tau)$ be a topological space with carrier $\mathbb{N} = \{1, 2, \dots\}$ and topology $\tau = \{[1, n) \mid n \in \mathbb{N}\} \cup \mathbb{N}$. This is the Alexandroff topology corresponding to the order $\geq$. To show that X is HI, observe first that for an arbitrary subset $A \subseteq \mathbb{N}$ we have $\mathbb{C}A = [\min A, \infty)$. Further, if $A, A' \subseteq B$ are such that $A \cap A' = \emptyset$ it is easily seen that either $\min A > \min B$ or $\min A' > \min B$. Hence either $B \not\subseteq \mathbb{C}A$ or $B \not\subseteq \mathbb{C}A'$. This shows that no subset of X can be decomposed into two disjoint dense in it sets, so X is HI.

Consider the Alexandroff extension X^*. Let $\mathfrak{F} \subseteq X^*$ denote the set of all the free ultrafilters over X. Fix two distinct free ultrafilters $\mathfrak{u}, \mathfrak{v} \in \mathfrak{F}$. We will show that both $\mathfrak{u}$ and $\mathfrak{v}$ belong to the least open neighborhood of any $\mathfrak{w} \in \mathfrak{F}$. To see this it suffices to check that for any non-empty $A \subseteq X$ we have $\mathbb{C}A \in \mathfrak{w}$. But if A is non-empty, then $\mathbb{C}A = [\min A, \infty)$ is cofinite and thus belongs to the free ultrafilter $\mathfrak{w}$. It follows that $\{\mathfrak{u}\}$ and $\{\mathfrak{v}\}$ are two disjoint dense in $\mathfrak{F}$ subsets. Hence X^* is not HI. $\dashv$

4. Interlude: an algebraic perspective

In this paper we have chosen to approach the question of definability from the *model-theoretic* perspective. While this approach is rather powerful and fruitful, it is not the only possible one. In this section we sketch an equally potent approach via *Universal Algebra*. We will outline how, using algebraic techniques, one can prove a slightly stronger version of Theorem 43. It should be noted however, that the algebraic techniques do not straightforwardly generalize to various extensions of the basic modal language. The model-theoretic approach provides more flexibility in this respect, as we will see in Section 5.

Most of the proofs that are missing in this section can be found in [15].

4.1. Algebraic semantics for modal logics

In a certain sense, the algebraic semantics for modal logic is most adequate, however it is also most abstract. Here we give a basic intuition of the universal algebraic approach to modal logic. More details can be found in standard textbooks [9,6].

Definition 49. A *modal algebra* is a tuple $(B, \Box)$ where B is a Boolean algebra and $\Box : B \to B$ is an operator such that for all $a, b \in B$ the following holds:

(i1) $\Box\top = \top$,
(i2) $\Box(a \wedge b) = \Box a \wedge \Box b$.

It is easily seen how $\mathcal{ML}$ can be interpreted on a modal algebra. Propositional letters designate elements of the Boolean algebra and the operations are interpreted by their algebraic counterparts. Every modal formula then becomes a polynomial that can be computed on tuples of elements of the algebra. The formulas that evaluate to $\top$ regardless of the assignment of the elements to the propositional letters are said to be *valid* in the algebra. It can be shown that every class of modal algebras determines a normal modal logic of the formulas that are valid on every algebra of the class [9, Chapter 7]. Conversely, a normal modal logic singles out a class (indeed, a *variety*) of modal algebras that validate all the formulas in the logic. This correspondence between varieties and logics is 1-1 [9, Chapter 7].

Modal algebras arising from topological spaces are called interior algebras. We discuss them next.

4.2. Interior algebras

Each non-empty set X gives rise to the Boolean algebra $\wp X$ of its subsets. Suppose in addition X is endowed with a topology τ. How is it possible to represent this additional structure algebraically? One natural possibility is to consider the Boolean algebra of all subsets with the corresponding interior operator $(\wp X, \mathbb{I})$. It is known that operation of interior satisfies the well-known Kuratowski axioms [11]. In fact, topological spaces can equivalently be described as sets endowed with operators satisfying the following:

(I1) $\mathbb{I}X = X$
(I2) $\mathbb{I}A \subseteq A$
(I3) $\mathbb{I}(A \cap B) = \mathbb{I}A \cap \mathbb{I}B$
(I4) $\mathbb{I}\mathbb{I}A = \mathbb{I}A$

Abstracting away from powerset Boolean algebras brings us to

Definition 50. An *interior algebra* is a modal algebra $(B, \Box)$ such that for all $a \in B$ the following holds:

(i3) $\Box a \leq a$,
(i4) $\Box\Box a = \Box a$.

Interior algebra homomorphisms are Boolean homomorphisms that commute with $\Box$.

More details on interior algebras are contained in [31,8].

4.3. Duality between spaces and interior algebras

It has been indicated above that each topological space X naturally gives rise to an interior algebra

$$X^+ = (\wp X, \mathbb{I}).$$

We call X^+ the *complex algebra* of X.

In fact the map $(\cdot)^+$ can be extended to do more than just producing an interior algebra from a topological space. Given an interior map $f : X \to Y$ between two topological spaces we can naturally manufacture an interior algebra homomorphism $f^+ : Y^+ \to X^+$ by putting $f^+ = f^{-1}$. Furthermore, it can easily be checked that the map $(\cdot)^+$ takes the topological sum of spaces into the algebraic product of the corresponding complex algebras. Thus $(\cdot)^+$ witnesses half of a duality going from topological spaces to interior algebras.

Another direction of the duality is provided by the construction of Alexandroff extensions of interior algebras. This is a straightforward generalization of the corresponding construction for topological spaces.

Definition 51 (*Alexandroff Extensions of Interior Algebars*). The *Alexandroff extension* of an interior algebra $(B, \Box)$ is the space $B_+ = (\mathtt{Uf}B, \tau_+)$, where $\mathtt{Uf}B$ is the set of ultrafilters of B, and τ_+ is the topology over $\mathtt{Uf}B$ generated by the sets of the form $\mathcal{F}^* = \{\mathfrak{u} \in \mathtt{Uf}B \mid \mathcal{F} \subseteq \mathfrak{u}\}$ for $\mathcal{F}$ an open filter over X.

Here by an open filter we mean a filter $\mathcal{F}$ of the Boolean algebra B such that if $a \in \mathcal{F}$ then $\Box a \in \mathcal{F}$.

Again it can easily be demonstrated that whenever $h : B \to C$ is an injective (surjective) interior algebra homomorphism, then the surjective (injective) interior map $h_+ : C_+ \to B_+$ can naturally be defined by putting $h_+(\mathfrak{u}) = \{a \in B \mid h(a) \in \mathfrak{u}\}$.

The maps $(\cdot)^+$ and $(\cdot)_+$ provide us with a link (duality) between interior algebras and homomorphisms on the one hand and topological spaces and interior maps on the other. With the help of this duality we can transfer the question *'which classes of topological spaces are modally definable?'* to the domain of interior algebras, where it obtains the following form: *'which classes of interior algebras are equationally definable?'* and is immediately answered by the fundamental theorem of Birkhoff—*'those and only those that are closed under products, subalgebras and homomorphic images'*.

We have just outlined the proof of the following.

Theorem 52. *The class* K *of topological spaces which is closed under the formation of Alexandroff extensions is modally definable iff it is closed under taking open subspaces, interior images, topological sums and it reflects Alexandroff extensions.*

Table 2
Characterization of definability in extended modal languages for $\mathcal{L}_t$-definable classes.

$\mathcal{ML}$	*closed under topological sums, open subspaces and interior images, reflecting Alexandroff extensions*	Theorem 43
$\mathcal{M}(E)$	*closed under interior images, reflecting Alexandroff extensions*	Theorem 56
$\mathcal{H}(@)$	*closed under open subspaces and u-morphic images*	Theorem 63
$\mathcal{H}(E)$	*closed under u-morphic images*	Theorem 62
$\mathcal{M}(D)$	*closed under u-morphic images*	Corollary 66

The proof of this theorem, as well as the details of the duality sketched above are presented in [15]. Another characterization of the modal definability for topological spaces that applies to *any* class of spaces is contained in [5, Theorem 5.10] and is also based on the duality outlined in this section. For the co-algebraic perspective on modal definability that encompasses both the relational and the topological cases, as well as more general semantical frameworks, we refer to [26].

Combining Theorem 52 with Theorem 42, we obtain our most general version of the definability theorem for the basic modal language:

Corollary 53. *Let* K *be a class of topological spaces satisfying at least one of the following conditions:*

(i) K *is* $\mathcal{L}_t$*-definable;*
(ii) K *is closed under box powers;*
(iii) K *is closed under ultrapowers;*
(iv) K *is closed under Alexandroff extensions.*

then K *is modally definable iff it is closed under taking open subspaces, interior images, topological sums and it reflects Alexandroff extensions.*

Proof. The easier 'only if' part follows from Theorems 32, 34, 36 and 40.

To prove the 'if' part suppose that K is closed under taking open subspaces, interior images, topological sums and it reflects Alexandroff extensions. Let us prove that under these conditions, if K satisfies any of the conditions (i)–(iii) above, then it also satisfies the condition (iv).

First we show that each of (i) and (ii) implies (iii). Indeed, if K is $\mathcal{L}_t$-definable, then it is closed under ultrapowers; also, if K is closed under box powers, since ultrapowers are interior images of box powers under the canonical quotient map and K is closed under interior images, we obtain that K is closed under ultrapowers.

Next we show that (iii) implies (iv). Indeed, it follows from Theorem 42 and the closure under interior images that if K is closed under taking ultrapowers, then K is closed under Alexandroff extensions.

Thus, in any of the cases (i)–(iv), K is closed under Alexandroff extensions. Now apply Theorem 52. ⊣

The analogue of Theorem 42 for relational semantics has a neat algebraic proof [18]. A similar proof for the topological case is lacking and we leave this as a challenge for the interested reader.

5. Extended modal languages

In order to increase the topological expressive power of the basic modal language, various extensions have been proposed. For instance, Shehtman [32] showed that *connectedness* becomes definable when the basic modal language is enriched with the *global modality*. Similarly, T_0, T_1 and *density-in-itself* become definable when we enrich the basic modal language either with nominals or with the difference modality. In this section, we show *exactly how much definable power* we gain by these additions, by giving analogues of Theorem 43 for these extended languages. Our findings are summarized in Tables 2 and 3 on page 43 and 44.

We believe Theorem 31 could also be generalized to the languages studied in this section, using appropriate analogues of topo-bisimulations. However, we have decided not to pursue this here suspecting the lack of many new insights.

5.1. The global modality

In the basic modal language with $\Diamond$ and $\Box$, one can only make statements about points that are arbitrarily close to the current point of evaluation. It appears impossible to say, for instance, that *there is a point satisfying p* (i.e., to express non-emptiness of p). The *global modality*, denoted by E, gives us the ability to make such global statements. For example, Ep expresses non-emptiness of the set p, and $A(p \to q)$ expresses that p is contained in q.

Table 3
Properties of topological spaces definable in various languages.

	$\mathcal{L}_t$	$\mathcal{M}$	$\mathcal{M}(E)$	$\mathcal{H}(@)$	$\mathcal{H}(E)$ (or $\mathcal{M}(D)$)
T_0	$\forall xy.(x \neq y \to \exists U_x.(y \notin U) \vee \exists V_y.(x \notin V))$	No	No	$@_i\Diamond j \wedge @_j\Diamond i \to @_i j$	Idem
T_1	$\forall xy.(x \neq y \to \exists U_x.(y \notin U))$	No	No	$\Diamond i \to i$	Idem
T_2	$\forall xy.(x \neq y \to \exists U_x.\exists V_y.\forall z.(z \notin U \vee z \notin V))$	No	No	No	No
Density-in-itself	$\forall x \forall U_x(\exists y \neq x(y \in U))$	No	No	$\Diamond\neg i$	Idem
Connectedness	No	No	$\mathsf{A}(\Box p \vee \Box\neg p) \to \mathsf{A}p \vee \mathsf{A}\neg p$	No	As in $\mathcal{M}(E)$
Hereditary irresolvability	No	$\Box(\Box(p \to \Box p) \to p) \to \Box p$	Idem	Idem	Idem

Formally, $\mathcal{M}(E)$ extends the basic modal language with an extra operator E that has the following semantics:

$$\mathfrak{M}, w \models E\phi \quad \text{iff } \exists v \in X.(\mathfrak{M}, v \models \phi).$$

The dual of E is denoted by A, i.e., $A\varphi$ is short for $\neg E\neg\varphi$. The standard translation can be extended in a straightforward way, by letting $ST_x(E\phi) = \exists x.(ST_x(\phi))$. In other words, $\mathcal{M}(E)$ is still a fragment of $\mathcal{L}_t$.

Shehtman [32] showed that connectedness can be defined using the global modality:

Proposition 54. *$A(\Box p \vee \Box\neg p) \to Ap \vee A\neg p$ defines connectedness.*

As connectedness is not definable in the basic modal language (Corollary 33), this shows that $\mathcal{M}(E)$ is more expressive than the basic modal language. As a consequence of this increased expressive power, certain operations on spaces do not preserve validity anymore.

Proposition 55. *Taking open subspaces, or taking topological sums, in general does not preserve validity of $\mathcal{M}(E)$-formulas.*

Proof. It suffices to show that connectedness is not preserved by these two operations. The real interval $(0, 1)$, with the usual topology, is connected, but its open subspace $(0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$ is not. Likewise, for any connected space X, the topological sum $X \biguplus X$ is no longer connected. $\dashv$

Taking interior images, on the other hand, *does* preserve validity of $\mathcal{M}(E)$-formulas, and taking Alexandroff extensions anti-preserves it. In fact, these two operations characterize definability in $\mathcal{M}(E)$, as the following analogue of Theorem 43 shows.

Theorem 56. *Let K be any $\mathcal{L}_t$-definable class of topological spaces. Then K is definable in the basic modal language with global modality iff it is closed under interior images and it reflects Alexandroff extensions.*

Proof. The 'only if' direction is a straightforward adaptation of Theorems 36 and 40. The proof of the 'if' direction is essentially a simplification of the proof of Theorem 43: suppose X is a topological space validating the $\mathcal{M}(E)$-theory of K, and let Δ be the following set of formulas, for all $B, C \subseteq X$:

$$\begin{aligned}
&Ep_B \text{ for non-empty } B\\
&A(p_{B\cap C} \leftrightarrow p_B \wedge p_C)\\
&A(p_{-B} \leftrightarrow \neg p_B)\\
&A(p_{\mathbb{I}B} \leftrightarrow \Box p_B)\\
&A(p_{\mathbb{C}B} \leftrightarrow \Diamond p_B).
\end{aligned}$$

Note that these formulas exactly correspond to conditions (1)–(5) from the proof of Theorem 42. As in the proof of Theorem 43, we can find a topological model $\mathfrak{N} = (Y, \mu)$ with $Y \in \mathsf{K}$, such that $\mathfrak{N} \models \Delta$. Finally, we proceed as in the proof of Theorem 42, and construct an interior map from an ultrapower of $\mathfrak{N}$ onto the Alexandroff extension X^* of X. Since K is closed under topological ultrapowers (Theorem 16) and images of interior maps, and reflects Alexandroff extensions, we conclude that $X \in \mathsf{K}$. $\dashv$

5.2. Nominals

Another natural extension of the basic modal language is with *nominals*. Nominals are propositional variables that denote singleton sets, i.e., they name points. In point-set topology one often finds definitions that involve both open sets and individual points. In the language $\mathcal{L}_t$, one can refer to the points in the space by means of point variables. The basic modal language lacks such means, and nominals can be seen as a way to solve this problem. Here are some examples of properties that can be defined using nominals[4]:

T_0	$@_i\Diamond j \wedge @_j\Diamond i \to @_i j$
T_1	$\Diamond i \to i$
Density-in-itself	$\Diamond\neg i$.

These properties are not definable in the basic modal language (Corollary 37). T_2-separation, on the other hand, remains undefinable even with nominals (Theorem 64).

Modal languages containing nominals are often called *hybrid languages*. In this section we investigate the topological expressive power of two hybrid languages, namely $\mathcal{H}(@)$ and $\mathcal{H}(E)$. Formally, fix a countably infinite set of nominals $\textsc{Nom} = \{i_1, i_2, \ldots\}$, disjoint from the set $\textsc{Prop}$ of proposition letters. Then the formulas of $\mathcal{H}(@)$ are given by the following recursive definition:

$$\mathcal{H}(@) \qquad \phi ::= \top \mid p \mid i \mid \phi \wedge \phi \mid \neg\phi \mid \Box\phi \mid @_i\phi$$

where $p \in \textsc{Prop}$ and $i \in \textsc{Nom}$. $\mathcal{H}(E)$ further extends $\mathcal{H}(@)$ with the global modality, which was described in the previous section. Thus, the formulas of $\mathcal{H}(E)$ are given by the following recursive definition:

$$\mathcal{H}(E) \qquad \phi ::= \top \mid p \mid i \mid \phi \wedge \phi \mid \neg\phi \mid \Box\phi \mid @_i\phi \mid E\phi.$$

As in the previous section, we will use $A\phi$ as an abbreviation for $\neg E\neg\phi$. We have introduced $@_i$ as a primitive operator, but it will become clear after introducing the semantics that $@_i$ can be defined in terms of the operator E.

Definition 57. A *hybrid topological model* $\mathfrak{M}$ is a topological space (X, τ) and a valuation $\nu : \textsc{Prop} \cup \textsc{Nom} \to \wp(X)$ which sends propositional letters to subsets of X and nominals to singleton sets of X.

The semantics for $\mathcal{H}(@)$ and $\mathcal{H}(E)$ is the same as for the basic modal language for the propositional letters, nominals, Boolean connectives, and the modality $\Box$. The semantics of $@$ and E is as follows:

$$\begin{array}{ll} \mathfrak{M}, w \models @_i\phi & \text{iff } \mathfrak{M}, v \models \phi \text{ for } \nu(i) = \{v\} \\ \mathfrak{M}, w \models E\phi & \text{iff } \exists v \in X.(\mathfrak{M}, v \models \phi). \end{array}$$

Validity and definability are defined as for the basic modal language, but considering only valuations that assign singleton sets to the nominals.

Proposition 58. *Taking topological sums or interior images in general does not preserve validity of $\mathcal{H}(@)$-formulas.*

Proof. The one-point space $X = \{0\}$ with the trivial topology validates $@_i\Diamond j$, but this formula is not valid on $X \biguplus X$. Thus topological sums do not preserve validity for $\mathcal{H}(@)$.

To see that $\mathcal{H}(@)$-validity is not preserved by interior maps, consider natural numbers with the topology induced by the ordering, i.e. the space $(\mathbb{N}, \tau)$ where $\tau = \{[a, \infty) \mid a \in \mathbb{N}\} \cup \{\emptyset\}$. The formula $\varphi = @_i\Box(\Diamond i \to i)$ (which defines anti-symmetry in the relational case) is easily seen to be valid in it. Then consider a topological space $X = \{0, 1\}$ with the trivial topology $\tau' = \{\emptyset, X\}$ and a map f that sends even numbers to 0 and odd numbers to 1. This is an interior map, however, φ is not even satisfiable on X. $\dashv$

On the other hand, the validity of $\mathcal{H}(@)$-formulas is preserved under taking open subspaces.

Lemma 59. *The validity of $\mathcal{H}(@)$ formulas is preserved under taking open subspaces.*

The proof is identical to that of Theorem 34. Recall from Section 3.2.2 that connectedness is not preserved under taking open subspaces. As a corollary, we obtain that connectedness is not definable in $\mathcal{H}(@)$.

Also, validity of $\mathcal{H}(E)$-formulas *is* reflected by Alexandroff extensions. We can in fact improve on this a bit, using the notion of a *topological ultrafilter morphic image*.

Definition 60. Let X and Y be topological spaces. Y is called a *topological ultrafilter morphic image* (or simply a *u-morphic image*) of X if there is a surjective interior map $f : X \to Y^*$ such that $|f^{-1}(\pi_y)| = 1$ for every principal ultrafilter $\pi_y \in Y^*$ (one can say figuratively 'f is injective on principal ultrafilters').

Clearly, every space is a u-morphic image of its Alexandroff extension.

Lemma 61. *The validity of $\mathcal{H}(E)$ formulas is preserved under taking u-morphic images.*

[4] The semantics of the $@_i$ operators is displayed on the next page.

Proof. Let X and Y be topological spaces and $f : X \to Y^*$ an interior map that is injective on principal ultrafilters. Suppose further that $Y \not\models \phi$. We will show that $X \not\models \phi$.

Since $Y \not\models \phi$, there is a valuation ν on Y such that $\nu(\phi) \neq Y$. Consider the valuation on Y^* defined by $\nu^*(p) = \{\mathfrak{u} \in Y^* \mid \nu(p) \in \mathfrak{u}\}$, where p can be a propositional letter or a nominal. It is not hard to see that ν^* assigns to each nominal a singleton set consisting of a principal ultrafilter. Next, we define the valuation ν' on X by $\nu'(p) = f^{-1}(\nu^*(p))$. Since f is injective on principal ultrafilters, ν' again assigns singleton sets to the nominals. Finally, a straightforward induction argument reveals that for all $a \in X$ and $\psi \in \mathcal{H}(E)$,

$$(X, \nu'), a \models \psi \quad \Leftrightarrow \quad (Y^*, \nu^*), f(a) \models \psi \quad \Leftrightarrow \quad \nu(\psi) \in f(a).$$

As $\nu(\neg\phi) \neq \emptyset$ there exists an ultrafilter $\mathfrak{u} \in Y^*$ which contains $\nu(\neg\phi)$. Since f is onto, there exists $a \in X$ such that $f(a) = \mathfrak{u}$. It follows that $(X, \nu'), a \models \neg\phi$ and therefore $X \not\models \phi$, as required. $\dashv$

The following two results characterize definability in $\mathcal{H}(E)$ and $\mathcal{H}(@)$ in terms of closure under taking u-morphic images. The proofs are inspired by relational results presented in [33].

Theorem 62. *Let* K *be any* $\mathcal{L}_t$*-definable class of topological spaces. Then* K *is definable in* $\mathcal{H}(E)$ *iff* K *is closed under u-morphic images.*

Proof. Lemma 61 constitutes the proof of the left-to-right direction. We will prove the right-to-left direction. Let $\mathrm{Log}(\mathsf{K})$ be the set of $\mathcal{H}(E)$-formulas valid on K. We will show that every space $X \models \mathrm{Log}(\mathsf{K})$ belongs to K, and hence $\mathrm{Log}(\mathsf{K})$ *defines* K.

Suppose $X \models \mathrm{Log}(\mathsf{K})$. We introduce a propositional letter p_A for every subset $A \subseteq X$, as well as a nominal i_a for every $a \in X$. These propositional letters and nominals are interpreted on X by the natural valuation. Let Δ be the following set of formulas, where B and C range over all subsets of X and a ranges over all points of X:

$$\begin{aligned}
&A(i_a \leftrightarrow p_{\{a\}})\\
&A(p_{-B} \leftrightarrow \neg p_B)\\
&A(p_{B\cap C} \leftrightarrow p_B \wedge p_C)\\
&A(p_{\mathbb{I}B} \leftrightarrow \Box p_B)\\
&A(p_{\mathbb{C}B} \leftrightarrow \Diamond p_B).
\end{aligned}$$

As in the proof of Theorem 56, we can find an $\mathcal{L}_t$-saturated (hybrid) topological model, based on a space $Y \in \mathsf{K}$, that makes Δ globally true. Note that conditions (1)–(5) from the proof of Theorem 42 hold for Y (the truth of $A(i_a \leftrightarrow p_{\{a\}})$ ensures that the predicates $P_{\{a\}}$ have non-empty interpretation). It follows, by the same argument as in the proof of Theorem 42, that the map $f : Y \to X^*$ defined by

$$f(a) = \{A \subseteq X \mid Y \models P_A(a)\}$$

is a surjective interior map. We will now show that f is injective on principal ultrafilters. Suppose there exist $w, v \in Y$ and $f(w) = f(v) = \pi_a$ where $a \in X$ and π_a is the principal ultrafilter containing $\{a\}$. By definition of f we get $Y \models P_{\{a\}}(w) \wedge P_{\{a\}}(v)$. By global truth of Δ we obtain $Y, w \models i_a$ and $Y, v \models i_a$, hence $w = v$.

It follows that X is a u-morphic image of Y. As K is closed under u-morphic images, we conclude that $X \in \mathsf{K}$ as required. $\dashv$

Theorem 63. *Let* K *be any* $\mathcal{L}_t$*-definable class of topological spaces. Then* K *is definable in* $\mathcal{H}(@)$ *iff it is closed under topological ultrafilter morphic images and under taking open subspaces.*

Proof. The 'only if' part is taken care of by Lemmata 59 and 61. The proof of the 'if' part proceeds as in Theorem 43, with some modifications.

The first difference is that the set of formulas Δ is augmented with formulas of the form $@_{i_a} p_A$, for all points a that belong to a non-empty set $A \subseteq X$.

A compactness argument similar to the one used in the proof of Theorem 43 shows that $\{@_{i_a}\Box\phi \mid \phi \in \Delta, a \in X\}$ is true in some $\mathcal{L}_t$-saturated topological model $\mathfrak{N} = (Y, \mu)$ with $Y \in \mathsf{K}$. For each $b \in Y$ named by a nominal, choose an open neighborhood O_b as described in Definition 18. Let O be the union of all these open neighborhoods. Note that by closure under open subspaces we obtain $O \in \mathsf{K}$. It is not hard to see that the submodel $\mathfrak{K}$ of $\mathfrak{N}$ based on the open subspace O globally satisfies Δ, and hence satisfies the conditions (1)–(5) described in the proof of Theorem 42.

Thus there exists an interior map f from O onto X^*. That f is injective on principal ultrafilters can be proved as in Theorem 62. Thus X is a u-morphic image of $O \in \mathsf{K}$. Since K is closed under u-morphic images, we obtain $X \in \mathsf{K}$ as required. $\dashv$

As an application, we will show that $\mathcal{H}(@)$ and $\mathcal{H}(E)$ are not expressive enough to be able to define the T_2 separation property. Recall the definition of irresolvability (Definition 7). We call a space X *α-resolvable* for a cardinal number α if X contains α-many pairwise disjoint dense subsets. In [13], an $2^{2^{\aleph_0}}$-resolvable T_2-space was constructed. We use this space to prove that

Theorem 64. *The class of* T_2 *topological spaces in not definable in* $\mathcal{H}(@)$ *and* $\mathcal{H}(E)$.

Proof. We employ an argument similar to, but more complicated than, the one used in Corollary 37. Our strategy is as follows: we construct spaces X and Y such that: Y is a T_2 space, X is a u-morphic image of Y, and X is not a T_2 space. Then we apply Theorem 62.

Take $X = (\mathbb{N}, \tau)$ where τ is the cofinite topology. That is

$$\tau = \{\emptyset\} \cup \{A \subseteq \mathbb{N} \mid \mathbb{N} - A \text{ is finite}\}.$$

Then X is T_1 since every singleton is closed, but not T_2 as any two non-empty opens necessarily meet. Denote by $\mathfrak{F}$ the set of all the free ultrafilters over $\mathbb{N}$. Then the following holds:

Claim 1: The topology τ^* of the Alexandroff extension X^* of X is described as follows:

$$O \in \tau^* \Leftrightarrow \mathfrak{F} \subseteq O.$$

Proof. Suppose $O \in \tau^*$. If $O = X^*$ the claim follows. Otherwise O contains a basic open set $\mathcal{G}^*$ which consists of all the ultrafilters extending a proper open filter $\mathcal{G}$. Note that if $A \in \mathcal{G}$ is not cofinite, then $\mathbb{I}A = \emptyset \notin \mathcal{G}$. Therefore, $\mathcal{G}$ consists of cofinite sets only. Since each free ultrafilter contains *all* cofinite sets, we obtain $\mathfrak{F} \subseteq \mathcal{G}^* \subseteq O$.

Now for the other direction. Suppose $\mathfrak{F} \subseteq O$. First note that $\mathfrak{F}$, being the extension of the open filter of all cofinite subsets of X, is a basic open in τ^*. Further, if $x \in X$, then the open filter $O_x = \{A \mid x \in A,\ A \text{ cofinite}\}$ is such that $O_x^* = \pi_x \cup F_x$ where π_x denotes the principal filter of x and $F_x \subseteq \mathfrak{F}$. It follows that

$$O = \mathfrak{F} \cup \bigcup_{\pi_x \in O} (\pi_x \cup F_x).$$

Since each $\pi_x \cup F_x = O_x^* \in \tau^*$ we obtain that $O \in \tau^*$. The claim is proved.

Next we will construct the space Y. Let $Z = (Z, \tau_1)$ be a $2^{2^{\aleph_0}}$-resolvable topological space which satisfies T_2 (according to [13] such a space exists). We will denote $2^{2^{\aleph_0}}$ many dense disjoint subsets of Z by Z_ι where $\iota \in \mathfrak{F}$. Here $\mathfrak{F}$ is again the set of all free ultrafilters over $\mathbb{N}$. Since the cardinality of $\mathfrak{F}$ is known to be $2^{2^{\aleph_0}}$ [11, Corollary 3.6.12], such indexing is possible. Let $\bar{Z} = Z - \bigcup_{\iota \in \mathfrak{F}} Z_\iota$. Thus

$$Z = \bar{Z} \cup \bigcup_{\iota \in \mathfrak{F}} Z_\iota.$$

Put $Y = (\mathbb{N} \cup Z, \tau')$ where τ' is as follows:

$$\tau' = \{\emptyset\} \cup \{O \subseteq Y \mid O \cap Z \in \tau_1, O \cap Z \neq \emptyset\}.$$

In words—the topology of Z as a subspace of Y is τ_1 and the neighborhoods of the points from $\mathbb{N}$ are the sets of the form $\{x\} \cup O$ where $x \in \mathbb{N}$ and $\emptyset \neq O \in \tau_1$.

Claim 2: Y is a T_2 space.

Proof. Indeed, any two points that belong to Z can be separated by two opens from τ_1, since (Z, τ_1) is a T_2 space. Any two points $x, y \in \mathbb{N}$ can be separated by open sets of the form $\{x\} \cup O_x$ and $\{y\} \cup O_y$ where $O_x, O_y \in \tau_1$ are non-empty open sets from Z such that $O_x \cap O_y = \emptyset$. Finally, two points x, y such that $x \in \mathbb{N}$ and $y \in Z$ can be separated by the sets $\{x\} \cup O_x$ and O_y where again O_x and O_y are disjoint non-empty open subsets of Z.

Now we construct the mapping $f : Y \to X^*$. Pick any $\zeta \in \mathfrak{F}$ and define $f : \mathbb{N} \cup Z \to X$ as follows:

$$f(x) = \begin{cases} \pi_x & \text{if } x \in \mathbb{N} \\ \iota & \text{if } x \in Z_\iota \\ \zeta & \text{if } x \in \bar{Z}. \end{cases}$$

Claim 3: The map f is a surjective interior map.

Proof. That f is surjective follows from the construction.

Let us show that f is continuous. Take $O \in \tau^*$. By Claim 1, we have $\mathfrak{F} \subseteq O$. It follows from the definition of f that $f^{-1}O$ is of the form $Z \cup A$ where $A \subseteq \mathbb{N}$. From the definition of τ' we obtain $f^{-1}(O) \in \tau'$.

To show that f is an open map, take an arbitrary open set $O \in \tau'$. It follows from the definition of τ' that $O \cap Z \in \tau_1$ and $O \cap Z \neq \emptyset$. Then, as each Z_ι is dense in Z, it follows that $O \cap Z_\iota \neq \emptyset$ for all $\iota \in \mathfrak{F}$. Hence, $f(O)$ contains $\mathfrak{F}$ and is open in X^* according to Claim 1.

Note that f is injective on principal ultrafilters, by construction. Therefore X is a u-morphic image of Y. Since Y is T_2 and X is not, it follows that the class of T_2 spaces is not closed under u-morphic images. Recall that the class of T_2 spaces is $\mathcal{L}_t$-definable. It follows by Theorems 62 and 63 that the class of T_2 spaces is not definable in $\mathcal{H}(E)$ and $\mathcal{H}(@)$. $\dashv$

5.3. The difference modality

In this section, we consider $\mathcal{M}(D)$, the extension of the basic modal language with the *difference modality D*. Recall that the global modality allows us to express that a formula holds *somewhere*. The *difference modality D* allows us to express that a formula holds *somewhere else*. For example, $p \wedge \neg Dp$ expresses that p is true at the current point and nowhere else. Formally,

$$\mathfrak{M}, w \models D\varphi \quad \text{iff } \exists v \neq w.(\mathfrak{M}, v \models \varphi).$$

The global modality is definable in terms of the difference modality: $E\phi$ is equivalent to $\phi \vee D\phi$. It follows that $\mathcal{M}(D)$ is at least as expressive as $\mathcal{M}(E)$. Furthermore, one can express in $\mathcal{M}(D)$ that a propositional letter p is true at a unique point (i.e., behaves as a nominal): this is expressed by the formula $E(p \wedge \neg Dp)$. Combining these two observations, it is not hard to show that every class of topological spaces definable in $\mathcal{H}(E)$ is also definable in $\mathcal{M}(D)$. The opposite also holds [17,27]:

Theorem 65. *$\mathcal{M}(D)$ can define exactly the same classes of topological spaces as $\mathcal{H}(E)$.*

Corollary 66. *An $\mathcal{L}_t$-definable class of topological spaces is definable in $\mathcal{M}(D)$ iff it is closed under u-morphic images.*

Recall that the separation axioms T_0 and T_1, as well as *density-in-itself*, are definable in the language $\mathcal{H}(E)$. They are definable in $\mathcal{M}(D)$ as follows, where $U\phi$ is short for $\phi \wedge \neg D\phi$:

T_0 :	$Up \wedge DUq \to \Box\neg q \vee D(q \wedge \Box\neg p)$
T_1 :	$Up \to A(p \leftrightarrow \Diamond p)$
Density-in-itself :	$p \to \Diamond Dp$.

For more on topological semantics of $\mathcal{M}(D)$ we refer to a recent study [25].

5.4. The ↓-binder

The last extension we will consider is the one with explicit point variables, and with the ↓-binder. The point variables are similar to nominals, but their interpretation is not fixed in the model. Instead, they can be bound to the current point of evaluation using the ↓-binder. For instance, $\downarrow x.\Box x$ expresses that the current point is an isolated point.

$\mathcal{H}(@, \downarrow)$ and $\mathcal{H}(E, \downarrow)$ are the extensions of $\mathcal{H}(@)$ and $\mathcal{H}(E)$, respectively, with state variables and the ↓-binder. Formally, let $\textsc{Var} = \{x_1, x_2, \ldots\}$ be a countably infinite set of point variables, disjoint from $\textsc{Prop}$ and $\textsc{Nom}$. The formulas of $\mathcal{H}(@, \downarrow)$ and $\mathcal{H}(E, \downarrow)$ are given by the following recursive definitions (where $p \in \textsc{Prop}$, $i \in \textsc{Nom}$, and $x \in \textsc{Var}$):

$$\mathcal{H}(@, \downarrow) \quad \phi ::= p \mid i \mid x \mid \neg\phi \mid \phi \wedge \psi \mid \Box\phi \mid @_i\phi \mid \downarrow x.\phi$$

$$\mathcal{H}(E, \downarrow) \quad \phi ::= p \mid i \mid x \mid \neg\phi \mid \phi \wedge \psi \mid \Box\phi \mid @_i\phi \mid E\phi \mid \downarrow x.\phi.$$

These formulas are interpreted, as usual, in topological models. However, the interpretation is now given relative to an *assignment* g of points to point variables (just as in $\mathcal{L}_t$). The semantics of the state variables and ↓-binder is as follows:

$$\begin{array}{lll} \mathfrak{M}, w, g \models x & \text{iff} & g(x) = w \\ \mathfrak{M}, w, g \models \downarrow x.\phi & \text{iff} & \mathfrak{M}, w, g^{[x \mapsto w]} \models \phi, \end{array}$$

where $g^{[x \mapsto w]}$ is the assignment that sends x to w and that agrees with g on all other variables. We will restrict attention to *sentences*, i.e., formulas in which all occurrences of point variables are bound. The interpretation of these formulas is independent of the assignment.

It turns out that $\mathcal{H}(E, \downarrow)$ is essentially a notational variant for a known fragment of $\mathcal{L}_t$, called $\mathcal{L}_I$. This is the fragment of $\mathcal{L}_t$ where quantification over opens is only allowed in the form, for U not occurring in α:

$$\begin{array}{ll} \exists U.(x \varepsilon U \wedge \forall y.(y \varepsilon U \to \alpha)), & \text{abbreviated as } [I_y\alpha](x), \text{ and, dually,} \\ \forall U.(x \varepsilon U \to \exists y.(y \varepsilon U \wedge \alpha)), & \text{abbreviated as } [C_y\alpha](x). \end{array}$$

Comparing the above with Definition 23 reveals that the formulas of the basic modal language translate inside $\mathcal{L}_I$ by the standard translation. So $\mathcal{ML}$ can be thought of as a fragment of $\mathcal{L}_I$. Apparently, adding nominals, ↓ and E to the language is just enough to get the whole of $\mathcal{L}_I$.

Theorem 67. *$\mathcal{H}(E, \downarrow)$ has the same expressive power as $\mathcal{L}_I$.*

Proof. The standard translation from modal logic to $\mathcal{L}_t$ can be naturally extended to $\mathcal{H}(E)$, treating nominals as first-order constants. The extra clauses are then

$$\begin{array}{lll} ST_x(t) & = & x = t \quad \text{for } t \in \textsc{Nom} \cup \textsc{Var} \\ ST_x(@_i\varphi) & = & \exists x.(x = c_i \wedge ST_x(\varphi)) \\ ST_x(E\varphi) & = & \exists x.ST_x(\varphi) \\ ST_x(\downarrow y.\varphi) & = & \exists y.(y = x \wedge ST_x(\varphi)). \end{array}$$

It is easily seen that this extended translation maps $\mathcal{H}(E, \downarrow)$-sentences to $\mathcal{L}_l$-formulas in one free variable. Conversely, the translation HT_x below maps $\mathcal{L}_l$-formulas $\alpha(x)$ to $\mathcal{H}(E, \downarrow)$-sentences:

$$\begin{array}{lcl} HT(s = t) & = & @_s t \\ HT(Pt) & = & @_t p \\ HT(\neg\alpha) & = & \neg HT(\alpha) \\ HT(\alpha \wedge \beta) & = & HT(\alpha) \wedge HT(\beta) \\ HT(\exists x.\alpha) & = & E \downarrow x.HT(\alpha) \\ HT([I_y\alpha](t)) & = & @_t \Box \downarrow y.HT(\alpha) \\ HT([C_y\alpha](t)) & = & @_t \Diamond \downarrow y.HT(\alpha) \\ \\ HT_x(\alpha(x)) & = & \downarrow x.HT(\alpha). \end{array}$$

It is not hard to see that both translations preserve truth, in the sense of Theorem 24. $\dashv$

This connection allows us to transfer a number of known results. For instance, $\mathcal{L}_l$ has a nice axiomatization, it is known to have interpolation, and the $\mathcal{L}_l$-theory of the class of T_1-spaces is decidable (see [28]). Hence, these results transfer to $\mathcal{H}(E, \downarrow)$. It is also known that $\mathcal{L}_l$ is strictly less expressive than $\mathcal{L}_t$. In particular, there is no $\mathcal{L}_l$-sentence that holds precisely on those topological models that are based on a T_2-space. Hence, the same holds for $\mathcal{H}(E, \downarrow)$. [5] Note that this does not imply *undefinability* of T_2 in $\mathcal{H}(E, \downarrow)$. Nevertheless, we conjecture that T_2 is not definable in $\mathcal{H}(E, \downarrow)$.

The precise expressive power of $\mathcal{L}_l$ on topological models can be characterized in terms of *potential homeomorphisms*.

Definition 68. A *potential homeomorphism* between topological models $\mathfrak{M} = (M, \tau, \nu)$ and $\mathfrak{N} = (N, \sigma, \mu)$ is a family F of partial bijections $f : M \to N$ satisfying the following conditions for each $f \in F$:

(1) f preserves truth of proposition letters and nominals (in both directions).
(2) - For each $m \in M$ there is a $g \in F$ extending f, such that $m \in dom(g)$.
- For each $n \in N$, there is a $g \in F$ extending f, such that $n \in rng(g)$.
(3) - For each $(m, n) \in f$ and open neighborhood $U \ni m$, there is an open neighborhood $V \ni n$ such that for all $n' \in V$ there is a $g \in F$ extending f and an $m' \in U$ such that $(m', n') \in g$.
- Likewise in the opposite direction.

The following characterization follows from the results in [28].

Theorem 69. *An $\mathcal{L}_t$-formula $\phi(x_1, \ldots, x_n)$ is equivalent to an $\mathcal{L}_l$-formula in the same free variables iff it is invariant for potential homeomorphisms.*

Substituting $\mathcal{H}(E, \downarrow)$ for $\mathcal{L}_l$, this gives us a Van Benthem-style characterization of $\mathcal{H}(E, \downarrow)$ as a fragment of $\mathcal{L}_t$. We leave it as an open problem to find a similar characterization of the expressive power of $\mathcal{H}(@, \downarrow)$. We also leave it as an open problem to characterize the classes of topological spaces definable in these languages.

Note that the union of the graphs of the partial bijections that constitute a potential homeomorphism gives rise to a total topo-bisimulation between the models in question. Thus a formula that is invariant for topo-bisimulations is also invariant for potential homeomorphisms. This is a semantical side of the fact that the basic modal language is a fragment of $\mathcal{L}_l$. In fact, we could have taken the language $\mathcal{L}_l$ as our first-order correspondence language from the very beginning. A feeling that $\mathcal{L}_l$ might be the 'right' candidate for the topological correspondence language might be strengthened by the fact that in its relational interpretation (i.e., on Kripke structures), $\mathcal{H}(E, \downarrow)$ has the full expressive power of the first-order correspondence language. We stand, however, by our choice of $\mathcal{L}_t$ since: (a) it provides stronger definability results (there are more $\mathcal{L}_t$-definable classes than $\mathcal{L}_l$-definable ones); (b) $\mathcal{L}_t$ is closer to both the usual first-order signature and the usual set-theoretic language used to formalize concepts in general topology.

6. Discussion

We have studied the expressive power of various (extended) modal languages interpreted on topological spaces. Tables 2 and 3 summarize and illustrate our main findings, concerning definability of classes of spaces. We also obtained a Van Benthem-style characterization of the basic modal language in terms of topo-bisimulations, thereby solving an open problem from [7].

Some of the key innovative elements in our story are (i) identifying the appropriate topological analogues of familiar operations on Kripke frames such as taking bounded morphic images, or, ultrafilter extensions (ii) identifying $\mathcal{L}_t$ as being the appropriate correspondence language on topological models (indeed, our result confirm once again that, as has been claimed before, $\mathcal{L}_t$ functions as the same sort of "landmark" in the landscape of topological languages as first-order logic is in the landscape of classical logics), and (iii) formulating the right notion of saturation for $\mathcal{L}_t$ (which many of our technical proofs depend on).

[5] In fact, Makowsky and Ziegler [28] showed that, in the absence of proposition letters and nominals, every two dense-in-itself T_1-spaces have the same $\mathcal{L}_t$-theory.

Our results on the hybrid language $\mathcal{H}(E, \downarrow)$ are remarkable. For example, they show that, while $\mathcal{H}(E, \downarrow)$ is expressively equivalent to the first-order correspondence language on relational structures, it is strictly less expressive than $\mathcal{L}_t$ on topological models. This seems one more instance of the more sensitive power of topological modeling.

Given that Alexandroffness is definable in $\mathcal{L}_t$, many of our results can be seen as generalizing known results for modal languages on (transitive reflexive) relational structures, and it is quite well possible that results on the topological semantics will yield new consequences for the relational semantics.

We finish by mentioning interesting directions for future research.

- **Correspondence theory for alternative semantics.** There are at least two other semantic paradigms where the approach taken in this paper might prove useful. We discuss them briefly.

  Diamond as derived set operator. For any subset S of a topological space, the derived set $\mathbf{d}S$ is the set of limit points of S, i.e., all points x of which each open neighborhood contains an element of S distinct from x itself. The closure operator can be defined in terms of the derived set operator: $\mathbb{C}S = S \cup \mathbf{d}S$. The converse does not hold, as $\mathbf{d}$ is strictly more expressive than $\mathbb{C}$ [32,3]. Indeed, if we interpret the $\blacklozenge$ as the derived set operator, then the modal formula $\blacklozenge\top$ defines density-in-itself. With the derived set operator we can also partially mimic nominals: $p \wedge \neg\blacklozenge p$ expresses that, within small enough neighborhoods, p acts as a nominal for the current point. Conversely, with nominals we can partially mimic the $\mathbf{d}$-operator: $@_i(\blacklozenge\phi \leftrightarrow \Diamond(\phi \wedge \neg i))$ is valid. The precise connection between $\mathbf{d}$ and nominals remains to be investigated. The standard translation should be modified in the following way to account for the new semantics:

  $$ST_x(\blacklozenge\phi) \ := \ \forall U.(x\varepsilon U \rightarrow \exists y.(y\varepsilon U \wedge x \neq y \wedge ST_y(\phi))).$$

  This is still a $\mathcal{L}_t$-formula. Whether or not the expressive power of the corresponding fragment of $\mathcal{L}_t$ can be characterized in a way we have presented here remains to be seen. The interested reader is referred to [12,32,3] for more details on this topological semantics.

  Neighborhood semantics. Neighborhood semantics is a generalization of the topological semantics that is not restricted to logics above S4. Formally, a neighborhood frame is a pair (W, n) where W is a non-empty set and $n \subseteq W \times \wp(W)$ a relation between elements of W and subsets of W. A neighborhood model consists of a neighborhood frame and a propositional valuation, as usual. Modal formulas can be interpreted in neighborhood models by letting $(W, n, V), w \models \Box\phi$ iff $(w, \{v \in W \mid (W, n, V), v \models \phi\}) \in n$. It is not hard to see that, under this interpretation of the modalities, neighborhood frames generalize topological spaces: a topological space (W, τ) can be identified with the neighborhood frame (W, n) where, for all $w \in W$ and $X \subseteq W$, $(w, X) \in n$ iff X contains some open neighborhood of w. In fact, the neighborhood frames obtained in this way are of a special kind: they are monotonic, in the sense that for all $w \in W$ and $X \subseteq Y \subseteq W$, $(w, X) \in n$ implies $(w, Y) \in n$.

  The correspondence theory for monotonic neighborhood frames has been explored in [20], where analogues of the Goldblatt–Thomason theorem and the Van Benthem theorem are presented (cf. also [30]). Analogues of the Van Benthem theorem and the Goldblatt–Thomason theorem for arbitrary neighborhood frames were obtained in [21] and [26], respectively, using a quite general co-algebraic approach. Still, we believe that a modification of the approach we follow in this paper could yield some further insights in this area. In particular, in order to develop the correspondence theory for other classes of neighborhood frames, it may help to identify an appropriate fragment of $\mathcal{L}^2$ that can play a similar role as $\mathcal{L}_t$ does in the topological case.
- **Further extensions of the language**. One could consider other extensions of the modal language, e.g., with propositional quantifiers [23] or fixed point operators [37]. It seems worthwhile to consider the extension of the signature of $\mathcal{L}_t$ with *function* symbols (to model continuous transformations of spaces) or a *binary relation* symbol (to model the time flow) and consider the applications to the domain of Dynamic Topological Logics of [2,24] or other structures for modal spatio-temporal logics.
- **Axiomatizations**. In this paper, we have investigated *expressive power* of extended modal languages interpreted on topological spaces. However, in order for these logics to be of practical use, their *proof theory* will have to be studied as well. In the case of the basic modal language, topological completeness has already been studied for a long time [29], but for more expressive modal languages, this is a new area of research. Some first results for $\mathcal{M}(D)$ and $\mathcal{H}(E)$ can be found in [25,34].

Appendix . $\mathcal{L}^2$ over topological models

In this appendix, we prove Theorem 11, here stated once again for reference:

Theorem 11. *$\mathcal{L}^2$ interpreted on topological models lacks Compactness, Löwenheim–Skolem and Interpolation, and is Π^1_1-hard for validity.*

This was already known for the more general case where $\mathcal{L}^2$-formulas can contain k-ary relation symbols with $k \geq 2$. The topological models we work with in this paper contain only unary predicates, but we will show that the bad properties of $\mathcal{L}^2$ already occur in this more restricted setting.

Proof. These facts can all be derived from the observation that $\mathcal{L}^2$ can define $(\mathbb{N}, \leq)$ up to isomorphism.

Table A.1
$\mathcal{L}^2$-formulas defining $(\mathbb{N}, \leq)$ up to isomorphism.

$\leq$ is a linear order
$\forall xy.(x \leq y \ \wedge \ y \leq x \ \rightarrow \ x = y)$
$\forall xy.(x \leq y \ \vee \ y \leq x)$
There is a least element
$\exists x.\forall y.(x \leq y)$
Each element has an immediate successor in the ordering
$\forall x.\exists y.(x < y \ \wedge \ \forall z.(x < z \ \rightarrow \ y \leq z))$
The space is Alexandroff (the down-sets are the closed sets)
$\forall x.\exists U_x.\forall V_x.\forall y.(y\varepsilon V \rightarrow y\varepsilon U)$
Each down-set other than X, $\emptyset$ has a least and a greatest element
$\forall U.(\exists x.(x \not\varepsilon U) \wedge \exists x.(x\varepsilon U) \rightarrow$
$\exists z_l, z_g.(z_l \not\varepsilon U \wedge z_g \not\varepsilon U \wedge \forall y.([y < z_l \vee z_g < y] \rightarrow y\varepsilon U)))$

- Definability of $(\mathbb{N}, \leq)$.

  Let $x \leq y$ stand for the $\mathcal{L}^2$-formula $\forall U.(x\varepsilon U \rightarrow y\varepsilon U)$, which defines the well-known *specialisation order* ($x \leq y$ iff $x \in \mathbb{C}\{y\}$). For each topological space (X, τ), $\leq$ defines a quasi-order on X. Conversely, every quasi-order on a set X is the specialisation order of some topology on X (in fact, of an Alexandroff topology on X).

  A special feature of $\leq$ is that every open set U is an *up-set* with respect to $\leq$ (i.e., whenever $x\varepsilon U$ and $x \leq y$ then also $y\varepsilon U$). Likewise, closed sets are *down-sets* with respect to $\leq$. If a space is Alexandroff, the converse holds as well: a set is open if and only if it is an up-set with respect to $\leq$, and it is closed if and only if it is a down-set.

  Now, let χ_N be the conjunction of the formulas in Table A.1, where we use $x < y$ as shorthand for $x \leq y \wedge x \neq y$.

  It is not hard to see that, if we take the open sets to be the up-sets, then $(\mathbb{N}, \leq)$ is a model for χ_N. In other words, χ_N is satisfiable. Now, suppose χ_N is true in some topological space (X, τ). We claim that $(X, \leq)$ is isomorphic to the natural numbers with their usual ordering. To prove this, it suffices to show that, for any $w \in X$, the set $\{v \mid v \leq w\}$ is finite (this property, together with the fact that $\leq$ is a linear order and each element has an immediate successor, characterizes the natural numbers up to isomorphism). In other words, we need to demonstrate that no infinite ascending or descending chains exist below an arbitrary point of X.

  Suppose that for some $w \in X$ the set $(w] = \{v \in X \mid v \leq w\}$ contains an infinite ascending chain $A = \{a_1, a_2, \ldots\}$ with $a_i < a_{i+1}$ for each $i \in \mathbb{N}$. Consider the down-set $(A] = \{w \in X \mid \exists a_i \in A.(w \leq a_i)\}$ generated by the set A. Since $(A] \subseteq (w]$, we know that $(A] \neq X$, and hence there is a greatest element $g \in (A]$. By the definition of $(A]$, we have $g \leq a_i$ for some $i \in \mathbb{N}$. By definition of A, we also have $a_i < a_{i+1}$ and so $g < a_{i+1}$, contradicting the maximality of g in $(A]$. Hence no infinite ascending chains exist below w.

  Next, suppose that for some $w \in X$ the set $(w] = \{v \in X \mid v \leq w\}$ contains an infinite descending chain $D = \{d_1, d_2, \ldots\}$, with $d_{i+1} < d_i$ for each $i \in \mathbb{N}$. Then the set $X - [D) = \{w \in X \mid \forall d_i \in D.(w < d_i)\}$ is a non-empty down-set (for non-emptiness, note that the least element of X cannot belong to D, and hence belongs to $X - [D)$). But then, there must be a greatest element $g \in X - [D)$. Let g' be the immediate successor of g. Note that by maximality of g we must have $d_i \leq g'$ for some $i \in \mathbb{N}$. By definition of D, $d_{i+1} < d_i$ and we obtain $d_{i+1} \leq g$, hence $g \in [D)$, a contradiction. Thus no infinite descending chains exist below w.
- Failure of Compactness

  Consider the following set of $\mathcal{L}^2$-sentences with one unary predicate P:

  $$\Gamma \equiv \left\{\chi_N, \exists x.P(x)\right\} \cup \{\varphi_n \mid n \in \mathbb{N}\}$$

  where $\varphi_k \equiv \forall x.(P(x) \rightarrow \exists y_1, \ldots, y_k.(y_1 < y_2 < \ldots < y_k < x)$ express that every point in P has at least k predecessors. Every finite subset of Γ is satisfiable but Γ itself is not.

  In fact, it is possible to show failure of compactness even without using any unary predicates.
- Failure of upward and downward Löwenheim–Skolem

  Since χ_N characterizes $(\mathbb{N}, \leq)$ up to isomorphism, clearly, it has only countable models. Thus, the upward Löwenheim–Skolem theorem fails for $\mathcal{L}^2$. The downwards Löwenheim–Skolem theorem fails as well: we can easily express in $\mathcal{L}^2_=$ that the specialisation order $\leq$ is a dense linear ordering without endpoints. Further, we can express (on Alexandroff spaces) that each non-empty up-set has an infimum:

  $$\text{Inf} \quad \forall U.\Big(\exists x.(x\varepsilon U) \rightarrow \exists y.\forall z.\,((y < z \rightarrow z\varepsilon U) \wedge (z\varepsilon U \rightarrow y \leq z))\Big).$$

  Combining these formulas, we can enforce a *complete dense linear order without endpoints*. An example of an infinite model satisfying this is $\mathbb{R}$ with its usual ordering. Any *countable* model, on the other hand, would have to be isomorphic to $\mathbb{Q}$, as a countable dense linear order without endpoints, which contradicts the conjunct Inf (e.g., the up-set $\left\{w \in \mathbb{Q} \mid w^2 > 2\right\}$ has no infimum).
- Failure of Interpolation

  Let P, Q, R be distinct unary predicates. Let $\phi_{\text{even}}(P)$ be the $\mathcal{L}^2$-sentences expressing that, on the natural numbers, P is true exactly of the even numbers, and $\phi_{\text{even}}(Q)$ likewise (it is not hard to see that there are such formulas). Then the following implication is valid:

  $$\chi_N \wedge \phi_{\text{even}}(P) \wedge \exists x.(Px \wedge Rx) \quad \rightarrow \quad (\phi_{\text{even}}(Q) \rightarrow \exists x.(Qx \wedge Rx))$$

Any interpolant for this implication has to express that R is true of some even number, without the help of additional predicates. Using an Ehrenfeucht–Fraïsse-style argument, one can show that this is impossible (note that we are essentially in first-order logic: quantification over open sets does provide any help, as the only open sets are the up-sets).

- Σ^1_1-hard satisfiability problem.

  Using χ_N, we can reduce the problem of deciding whether an existential second-order (ESO) formulas is true on $(\mathbb{N}, \leq)$ —a well-known Σ^1_1-complete problem — to the satisfiability problem of $\mathcal{L}^2$. For simplicity we will discuss here only the case for ESO sentences of the form $\exists R.\phi(R, \leq)$, where R is a single binary relation. The argument generalizes to more relations, and relations of other arities.

  Let an ESO sentence $\exists R.\phi$ be given. Let N, P_1 and P_2 be distinct unary predicates. Intuitively, the elements of the model satisfying N will stand for natural numbers, while the other elements only play a technical role for coding up the binary relation R. Let $x <^+ y$ be short for $x < y \ \wedge\ Ny \ \wedge\ \forall z.(x < z \ \wedge\ Nz \ \rightarrow\ y \leq z)$, expressing that y is the least N-element greater than x. By induction, we define an $\mathcal{L}^2$-formula ϕ^* as follows:

$$\begin{array}{lcl}
(x = y)^* & = & Nx \wedge Ny \wedge x = y \\
(x \leq y)^* & = & Nx \wedge Ny \wedge x \leq y \\
(Rxy)^* & = & \exists x'y'z.(z{<}x'{<}^+x \ \wedge\ z{<}y'{<}^+y \ \wedge\ P_1x' \ \wedge\ P_2y') \\
(\phi \wedge \psi)^* & = & \phi^* \wedge \psi^* \\
(\neg\phi)^* & = & \neg\phi^* \\
(\exists x.\phi)^* & = & \exists x.(Nx \wedge \phi^*)
\end{array}$$

  We claim that $(\mathbb{N}, \leq) \models \exists R.\phi$ iff $\phi^* \wedge \chi_N^\circ$ is satisfiable, where χ_N° is the relativisation of χ_N to N (i.e., the formula obtained from χ_N by relativising all quantifiers by N, thus expressing that the subspace defined by N with its specialisation order is isomorphic to $(\mathbb{N}, \leq)$).

  The difficult direction is *left-to-right*. We give a rough sketch. Suppose that $(\mathbb{N}, \leq) \models \exists R.\phi$. Let $\mathcal{R} \subseteq \mathbb{N} \times \mathbb{N}$ be a witnessing binary relation. Now, we define our model for $\phi^* \wedge \chi_N^\circ$ as follows: the subspace defined by N is simply the Alexandroff topology generated by $(\mathbb{N}, \leq)$. For each pair $(m, n) \in \mathcal{R}$, we create three distinct $\neg N$-elements, $(m, n)_0$, $(m, n)_1$ and $(m, n)_2$. Then we make sure that m is the least N-successor of $(m, n)_1$ and P_1 holds at $(m, n)_1$, n is the least N-successor of $(m, n)_2$ and P_2 holds at $(m, n)_2$, $(m, n)_0 < (m, n)_1$ and $(m, n)_0 < (m, n)_2$. In this way, we ensure that, for any pair of natural numbers m, n, $(m, n) \in \mathcal{R}$ iff the $\mathcal{L}^2$-formula $(Rxy)^*$ is true of (m, n) in the constructed model. Once this observation is made, the claim becomes easy to prove. $\dashv$

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