A Modal Transcription of the Hausdorff Residue

Leo Esakia

Razmadze Mathematical Institute, Tbilisi, Georgia*
esakia@hotmail.com

Abstract. The modal system S4.Grz is the system that results when the axiom (Grz) $\Box(\Box(p \to \Box p) \to p) \to \Box p$ is added to the modal system S4, i. e. S4.Grz = S4 + Grz. The aim of the present note is to prove in a direct way, avoiding duality theory, that the modal system S4.Grz admits the following alternative definition: S4.Grz = S4 + R-Grz, where R-Grz is an additional inference rule:

(R-Grz) $$\frac{\vdash \Box(p \to \Box p) \to p}{\vdash p}$$

This rule is a modal counterpart of the following topological condition: If a subset A of a topological space X coincides with its Hausdorff residue $\rho(A)$ then A is empty. In other words the empty set is a unique "fixed" point of the residue operator $\rho(\cdot)$.

We also present some consequences of this alternative axiomatic definition.

1 The Modal System of Grzegorczyk

Grzegorczyk [6] axiomatically defines a Modal system S4.Grz (named after him), which is a proper normal extension of the system S4, and proves that HC (= Heyting's intuitionistic Calculus) could be embedded (via the Gödel translation) in the system S4.Grz. S4.Grz = S4 + Grz is the system that results when the axiom $\Box(\Box(p \to \Box p) \to p) \to \Box p$ (Grz) is added to the modal system S4. This modal system occupies a privileged position in modal logic: indeed, S4.Grz has the following significant properties.

(I) S4.Grz is the *largest* modal system in which HC can be embedded by the Gödel translation.

(II) Lat(HC) $\cong$ Lat(S4.Grz), i. e. the lattice Lat(HC) of all intermediate logics is isomorphic to the lattice Lat(S4.Grz) of all normal extensions of the system S4.Grz. (Blok-Esakia)

A sentence s of Peano Arithmetic PA is *demonstrable* if it is provable and true. Let us abbreviate Bew$(s) \wedge s$ as "Dem(s)", where Bew(s) is the arithmetization of the assertion that s is provable. An arithmetic realization * of modal formulas is an assignment to each atom p an arithmetic sentence p^* of PA which commutes with non-modal connectives and $(\Box p)^* = \text{Dem}(\langle p^* \rangle)$.

* The author was supported by the grant GNSF/ST08/3-397.

N. Bezhanishvili et al. (Eds.): TbiLLC 2009, LNAI 6618, pp. 46–52, 2011.

(III) *Arithmetical completeness.* The formulas of modal system all of whose arithmetical realizations are theorems of PA are precisely the theorems of S4.Grz ([2], [5]).

2 Topological Remarks

For various topological aspects of modal systems see [9] and the papers cited in [4]. Residues of sets as defined below were introduced by Hausdorff [7] (so called *Hausdorff's theory of Residues*).

Denote the residue $A \cap \mathrm{c}(\mathrm{c}A - A)$ of the set A by ρA and the *rest* $A - \rho A$ of A by πA. It is easily seen that $\pi A = A - \mathrm{c}(\mathrm{c}A - A)$ and $A = \rho A \cup \pi A$.

Let A be a subset of a topological space X and $x \in X$; A is *locally closed at* x (Bourbaki) if there exists an open neighborhood U of x such that $A \cap U = \mathrm{c}A \cap U$; A is *locally closed* if A is locally closed at each point $x \in A$. Thus the set of *all* locally closed points of A is πA.

Proposition. The following are equivalent:

1. A is locally closed.
2. $A = F \cap U$ for some open U and closed F.
3. $\rho(A) = A \cap \mathrm{c}(\mathrm{c}A - A) = \varnothing$;
4. $\pi(A) = A$, where $\pi(A) = A - \rho(A) = A - \mathrm{c}(\mathrm{c}A - A)$.

Definition (Hewitt [8]). A topological space X is *resolvable* if X contains two disjoint dense subsets. A space X is *irresolvable*, if it cannot be decomposed into two disjoint dense subsets. A space X is a *Hewitt space* (alias, *hereditarily irresolvable* [1]) if no nonempty set is resolvable (as a *subspace*).

Recall that a topological space is called *scattered* (Cantor) if it has no dense-in-itself non-empty subset. It is not hard to verify that every scattered space is a Hewitt space. An example of a Hewitt space which is not scattered has been constructed in [1]; in that paper there has been also shown that scatteredness and hereditary irresolvability coincide on a wide class of spaces, including all of the spectral, first countable, or locally compact Hausdorff spaces.

Proposition [1]. A topological space X is a Hewitt space iff for every subset A of X, $A \neq \varnothing$ implies $A - \rho A \neq \varnothing$.

It is well known that *every* topological space X has its *Hewitt decomposition*, i. e. $X = Y \cup Z$, where Y is closed and resolvable, Z is a Hewitt space and $Y \cap Z = \varnothing$.

An "equational" characterization of Hewitt spaces [3]:

The following conditions are equivalent:

1) A topological space X is a Hewitt space;
2) $\mathrm{c}A = A - \mathrm{c}(\mathrm{c}A - A)$ for every $A \subseteq X$;
3) The Closure of an arbitrary set $A \subseteq X$ coincides with the closure of the locally closed part of A, i. e. $\mathrm{c}A = \mathrm{c}\pi(A)$.