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Around provability logic☆

Leo Esakia

Razmadze Mathematical Institute, Tbilisi 0193, Georgia

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ABSTRACT

We present some results on algebraic and modal analysis of polynomial (intrinsically definable) distortions of the standard provability predicate in Peano Arithmetic PA, and investigate three provability-like modal systems related to the Gödel–Löb modal system GL. We also present a short review of relational and topological semantics for these systems, and describe the dual category of algebraic models of our main modal system.

1. Introduction and motivation

We are going to discuss certain intrinsic reincarnations of the standard provability predicate in Peano Arithmetic PA that are of special interest in connection with the study of Provability Logic. All special modifications of the standard provability predicate proposed for consideration here are metamathematical predicates distinct from the standard provability, yet strong enough to satisfy the Hilbert–Bernays derivability conditions. Our modifications are internally definable and need not be introduced as an additional structure. The purpose of our paper is to discuss certain observations arising from a study of such reincarnations of the standard provability predicate. Namely:

- What are the implications of such reincarnations?
- What can we say about the modal logics of such "non-standard" provability predicates?

Note that our study has opened more questions than it has answered.

We assume that the reader is familiar with the concept of provability as a modality, i.e., as a modal operator $\Box$ acting on propositional formulas. Details of Provability Logic can be found in [4].

The Gödel–Löb modal system GL is the result of adding the Löb Axiom $\Box(\Box p \to p) \to \Box p$ to K4. *Arithmetical completeness* of GL, i.e., the fact that GL adequately reflects the behavior of the standard provability predicate in PA, was proved by Solovay [18] in 1976. He defined an *arithmetical realization* of modal formulas of the system GL and proved its arithmetical completeness. Using more technical terminology, we say that an arithmetical realization of modal formulas is an assignment * to each atom p of an arithmetical sentence p^* which commutes with non-modal connectives and $(\Box p)^* = \mathrm{Pr}(\ulcorner p^* \urcorner)$, where $\mathrm{Pr}(\cdot)$ is the standard provability predicate for PA and $\ulcorner p^* \urcorner$ is the code numeral of p^*.

Arithmetical completeness of GL (Solovay [18]): GL $\vdash p$ iff under all arithmetical realizations * the sentence p^* is provable in PA.

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E-mail addresses: esakia@hotmail.com, esakia@rmi.acnet.ge.

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Suppose we modify the notion of arithmetical realization by amending the recursive clause for the box $\Box$. It should be emphasized that we only offer modal analyses of some special type of modification, namely:
A-distortion: $(\Box p)^* = (A \to p^*) \wedge \mathrm{Pr}(\ulcorner p^* \urcorner)$, where the parameter A is a given sentence in the language of PA.

With algebraic nomenclature at hand, our definition of distortions is easily translatable into the language of diagonalizable (alias, GL-) algebras. We refer the reader to the important paper "On the Diagonalizable Algebra of Peano Arithmetic" by Franco Montagna [16]; there Montagna finds a very elegant algebraic proof of Solovay's Completeness Theorem: Montagna has shown, among other things, that the free diagonalizable algebra on a denumerable set of generators is a subalgebra of the diagonalizable algebra of Peano arithmetic PA.

Following Montagna [16], we will denote operators corresponding to modalities $\Box$ and $\Diamond$ by τ and δ.

Let $(B, \wedge, \vee, \to, \bot, \tau)$ be an arbitrary diagonalizable algebra (for example, the diagonalizable algebra of Peano Arithmetic PA) and $e \in B$; we define a new (polynomially definable) modal operator τ_e on the Boolean algebra B (the notion of polynomial used here is simply that from universal algebra: polynomials are functions arising from constant functions and the identity function by means of the Boolean operations and τ):
e-distortion: $\tau_e p := (e \to p) \wedge \tau p$ for every $p \in B$.

We present some observations regarding these distortions. Denote by $(B; \tau_e)$ the Boolean algebra B endowed with the operator τ_e.

Let us note here that one can define the e-distortion of the dual operator δ as follows: $\delta_e p = \delta p \vee (e \wedge p)$ for every $p \in B$.
Main observation: *some particular cases which illustrate the general picture:*

I. for every $e \in B$, the algebra $(B; \tau_e)$ is a K4-algebra, satisfying the additional condition $\tau_e(\tau_e(p \to \tau_e p) \to p) \leqslant \tau_e p$, i.e., $(B; \tau_e) \in$ K4.Grz;
II. if $e = \bot$, then the modal operator τ_e coincides with τ;
III. if $e = \neg\bot$, then $\tau_e p$ represents the "demonstrability" predicate $\mathrm{Dem}(\ulcorner p^* \urcorner) := p^* \wedge \mathrm{Pr}(\ulcorner p^* \urcorner)$. We recall that a sentence s of PA is demonstrable if it is provable and true. Let us abbreviate $s \wedge \mathrm{Pr}(\ulcorner s \urcorner)$ as "Dem(s)" ([3]);
IV. if $e \neq \bot$ and $e \leqslant \neg\tau\bot$, then a modal version $\neg\tau_e\bot \leqslant \neg\tau_e\neg\tau_e\bot$ of Gödel's Second Incompleteness Theorem is still *valid* in the algebra $(B; \tau_e)$ while the Löb Axiom *is refutable*;
V. if $e \neq \bot$ and $e \leqslant \tau\bot$, then a modal version of Gödel's Second Incompleteness Theorem *is refutable* in the algebra $(B; \tau_e)$ and a weaker form (some *instance*) of the Löb Axiom, namely,

$$\tau_e(\tau_e(p \to \tau_e p) \to (p \to \tau_e p)) \leqslant \tau_e(p \to \tau_e p),$$

is valid.

The paper is organized as follows.

In Section 2 we introduce provability-like modal systems including the central modal system K4.Grz.

Section 3 contains the definition of the basic equational class of derivative algebras, i.e., Boolean algebras with an operator, which capture the algebraic properties of the topological derivative. In these terms we next define varieties of derivative algebras corresponding to our main modal systems. We prove a lemma concerning intimate connection between derivative algebras and the corresponding closure algebras.

The central theme of Section 4 is the polynomial generation of new derivative operations from old in arbitrary diagonalizable algebras. In 4.1 we present algebraic reasons for the above I, II, and III, and show that the algebra (B, δ_e), where δ_e is an arbitrary distortion of δ in a diagonalizable algebra (B, δ), is a K4.Grz-algebra. Moreover we show that the lattice of all distortions of the derivative operator δ is Boolean. Next we present algebraic reasons for IV in 4.2 and for V in 4.3.

In Section 5 we define the modal system K4.Grz + (g) obtained by postulating the modal version (g) of Gödel's Second Incompleteness Theorem and the modal system K4.Grz + (wL) obtained by postulating a special instance (wL) of the Löb Axiom. It is proved that the Gödel–Löb modal system GL can be axiomatized as K4.Grz + (g) + (wL).

In the Appendix of the paper we present a review (that, nevertheless, is likely to be incomplete) of some old and not-so-old results on topological semantics for the modal systems considered in this paper and provide the reader with some topological and relational completeness results which were obtained jointly with Gabelaia. We also describe a dual category for the equational class of K4.Grz-algebras.

2. Provability-like modal systems

The modal system K (named after Kripke) is the basic normal modal logic whose axioms are all Boolean tautologies and all expressions of the form $\Box(p \to q) \to (\Box p \to \Box q)$ and whose rules are modus ponens and necessitation. The diamond $\Diamond$ as usual means the dual $\neg\Box\neg$ of $\Box$. Recall that the system K4 is obtained by adding $\Box p \to \Box\Box p$ to K as a new axiom schema. It is appropriate to mention here that axioms and the necessitation rule of the system K4 are a modal simulation of the Hilbert–Bernays derivability conditions; they are derivability conditions because they are formalizations of key properties of the provability predicate $\mathrm{Pr}(\cdot)$.

A slightly weakened version of K4 is the modal system wK4 = K $+(p \wedge \Box p) \to \Box\Box p$. Finally, the *central* modal system in our paper is K4.Grz, a normal extension of K4, which is obtained by adding the formula Grz to K4 as an additional axiom, i.e., K4.Grz = K4 $+\Box(\Box(p \to \Box p) \to p) \to \Box p$.

What makes K4.Grz of interest is that the system K4.Grz (a special intermediate system: K4 $\subset$ K4.Grz $\subset$ GL) captures additional provability properties of $\mathrm{Pr}(\cdot)$ that do not depend on Gödel's Diagonal Lemma and Löb's Principle. Note that adding the "reflexive" axiom $\Box p \to p$ to the system K4.Grz gives the well-known system S4.Grz, i.e., S4.Grz = K4.Grz $+\Box p \to p$.

3. Algebraic background

Derivative algebras are Boolean algebras with a unary operation δ, which captures the algebraic properties of the topological derivation. Recall that the *derivative* δA of A is, by definition, the set of all accumulation (alias, *limit*) points of a subset A of a topological space X. A point x is said to be a limit point of a set A if every neighborhood of x contains a point of A other than x.

Definition 1. We say that a Boolean algebra B is a *derivative algebra* with respect to the operation δ, if (1) $\delta\bot = \bot$, (2) $\delta(a \vee b) = \delta a \vee \delta b$, (3*) $\delta\delta a \leqslant a \vee \delta a$.

Remark: It must be pointed out that we weaken the definition of derivative algebra (McKinsey–Tarski [15]) slightly; namely, we postulate condition (3*) instead of (3) $\delta\delta a \leqslant \delta a$. We justify this weakening by noting that there are topological spaces in which condition (3) is not valid (for example, spaces with anti-discrete topology, i.e., having the empty set and the whole space as the only open sets). Note that the system of postulates for derivative algebras (or for topological spaces in terms of derivative) has a certain completeness property: every "topological" equation, which is identically true in *all* topological spaces, can be derived from these postulates [8].

With the operator δ is associated a dual operator τ (co-derivative) defined by $\tau a := \neg\delta\neg a$. Note that the dual of the above condition (3*) has the form $a \wedge \tau a \leqslant \tau\tau a$, which is the algebraic version of the axiom of wK4 appearing above. Using the usual intuitively obvious relations between closure and derivative operations in topological spaces the *closure* of a set can be defined in terms of the *derivative*, namely, $\mathrm{c}A := A \cup \delta A$. If we introduce a corresponding definition into derivative algebra (namely, $\mathrm{C}a := a \vee \delta a$), we can easily show that the derivative algebra (B, δ) becomes a *closure* algebra (B, C) with respect to the operation C just defined. We remark that the *interior* operator I can be defined as follows: $\mathrm{I}a := a \wedge \tau a$. We will use whichever of δ (resp., C) and τ (resp., I) is rhetorically the most convenient. As an immediate consequence of the definition we have a corollary: in any derivative algebra (B, δ) the operator C satisfies the well-known Kuratowski Axioms: (1) $a \leqslant \mathrm{C}a$, (2) $\mathrm{C}a = \mathrm{CC}a$, (3) $\mathrm{C}(a \vee b) = \mathrm{C}a \vee \mathrm{C}b$, (4) $\mathrm{C}\bot = \bot$.

With these remarks in mind we can (and henceforth will) use topological notions and terminology in algebraic contexts.

We say that an element $a \in B$ is *open* if $a \leqslant \tau a$ (i.e., $\mathrm{I}a = a$) and *closed* if $\delta a \leqslant a$ (i.e., $\mathrm{C}a = a$).

It is also very convenient to have special notations for the following topological notions.

Let X be an arbitrary topological space and A a subset of X. The derived set δA of the set A includes some points of A and some points of its complement. Any point of A not in the derived set is called an *isolated* point since it must be contained in an open set containing no other point of A. We therefore have: the set μA of isolated points of A is equal to $A - \delta A$. Given $x \in A \subseteq X$, according to [5] A is called *locally closed* at x if there exists an open neighborhood U of x such that $A \cap U = \mathrm{c}A \cap U$ (i.e., $A \cap U$ is closed in U). A is *locally closed* if A is locally closed at each point $x \in A$. It is known ([5], Ch. I, §3, Prop. 5) that a set A is locally closed iff $A = F \cap U$ for some closed set F and open set U. Denote the *residue* [11] $A \cap \mathrm{c}(\mathrm{c}A - A)$ of the set A by ρA and the *rest* $A - \rho A$ of A by πA. It is easily seen $\pi A = A - \mathrm{c}(\mathrm{c}A - A)$ and $A = \rho A \cup \pi A$.

Using those topological notions we introduce the following *"point-free"* definition.

Definition 2. If (B, δ) is an arbitrary derivative algebra and $a \in B$ we say that $\mu a = a - \delta a$ is the *isolated part* of a, and $\pi a = a - \delta(\delta a - a)$ is the *locally closed part* of a.

It is not hard to see that

Lemma 1. *The following conditions are equivalent:*

1. *an element $a \in B$ is* locally closed, *i.e., $\pi a = a$;*
2. *$a = f \wedge g$ for some closed element f and open element g;*
3. *$a = \pi b$ for a suitable element $b \in B$.*

Proof. To begin with, let us note that the operator π is definable in terms of closure, namely,

$$a - \mathrm{C}(\mathrm{C}a - a) = a - \delta(\delta a - a) = \pi a. \tag{A}$$

Let us transform the left-hand side as follows: $a - \mathrm{C}(\mathrm{C}a - a) = a - ((\mathrm{C}a - a) \vee \delta(\mathrm{C}a - a)) = a \wedge \neg(\mathrm{C}a - a) \wedge \neg\delta(\mathrm{C}a - a) = a - \delta(\mathrm{C}a - a) = a - \delta((a \vee \delta a) - a) = a \wedge \neg\delta((a \wedge \neg a) \vee (\delta a - a)) = a - \delta(\delta a - a)$. Note also that

$$\mathrm{C}a - \mathrm{C}(\mathrm{C}a - a) = a - \mathrm{C}(\mathrm{C}a - a). \tag{B}$$

Indeed, $a - \mathrm{C}(\mathrm{C}a - a) \leqslant \mathrm{C}a - \mathrm{C}(\mathrm{C}a - a)$ is obvious. It remains to show that $\mathrm{C}a - \mathrm{C}(\mathrm{C}a - a) \leqslant a - \mathrm{C}(\mathrm{C}a - a)$; since $\mathrm{C}a - \mathrm{C}(\mathrm{C}a - a) \leqslant -\mathrm{C}(\mathrm{C}a - a)$ and $(\mathrm{C}a - a) - \mathrm{C}(\mathrm{C}a - a) = 0$, i.e., $\mathrm{C}a - \mathrm{C}(\mathrm{C}a - a) \leqslant a$, one has $\mathrm{C}a - \mathrm{C}(\mathrm{C}a - a) \leqslant a - \mathrm{C}(\mathrm{C}a - a)$.

(2) $\Rightarrow$ (1). Let $a = f \wedge g$, where f is a closed element and g is an open element. Then $\mathrm{C}a - a = \mathrm{C}(f \wedge g) - (f \wedge g) = \mathrm{C}(f \wedge g) \wedge (-f \vee -g) = (\mathrm{C}(f \wedge g) - f) \vee (\mathrm{C}(f \wedge g) - g)$. But $\mathrm{C}(f \wedge g) - f \leqslant (\mathrm{C}f \wedge \mathrm{C}g) - f = f \wedge \mathrm{C}g \wedge \neg f = 0$.

Consequently $Ca - a = C(g \wedge f) \wedge \neg g$ and, thus, $Ca - a$ is a closed element, that is, $C(Ca - a) = Ca - a$; hence $\pi a = Ca - C(Ca - a) = Ca - (Ca - a) = a$, that is, $\pi a = a$.

(1) $\Rightarrow$ (2). If $a = \pi(a)$, then $a = \pi a = Ca - C(Ca - a) = Ca \wedge I(-Ca \vee a)$.

Thus (1) and (2) are equivalent. Moreover (1) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (2). Suppose, by (3), $a = \pi b$; since $\pi b = Cb - C(Cb - b)$, (2) holds. □

In the terminology of the derivative δ we define the variety of algebras corresponding to our main modal systems.

Definition 3. Let B be a Boolean algebra and δ be a unary function on B. We say that (B, δ) is:

1. a *K4-algebra* if (B, δ) is a derivative algebra and $\delta\delta a \leqslant \delta a$ for each $a \in B$;
2. a *K4.Grz-algebra* if (B, δ) is a K4-algebra satisfying the following condition for each $a \in B$:

$$\delta a \leqslant \delta(a - \delta(\delta a - a)), \tag{grz}$$

or, in our terminology, $\delta a \leqslant \delta\pi a$;

3. a *GL-algebra* (diagonalizable algebra) if (B, δ) is a K4-algebra and

$$\delta a \leqslant \delta(a - \delta a), \tag{L}$$

or, in our terminology, $\delta a \leqslant \delta\mu a$.

Since δ is monotone, in (grz) and (L) we can replace "$\leqslant$" by "=". We discuss the topological significance of the conditions $\delta a \leqslant \delta\pi a$, and $\delta a \leqslant \delta\mu a$ in the Appendix.

Using the de Morgan laws and $\delta a = \neg\tau\neg a$ ($a \in B$), it is not difficult to see that (grz) and (L) are equivalent to

$$\tau(\tau(a \to \tau a) \to a) \leqslant \tau a \tag{grz'}$$

and

$$\tau(\tau a \to a) \leqslant \tau a. \tag{L'}$$

It is known (see, for example, [4]) that Löb's Axiom is equivalent to the Löb Rule, whose algebraic version can be written as follows:

$$a \neq 0 \Rightarrow \mu a \neq 0. \tag{RL}$$

Similarly, axiom (Grz) is equivalent to the Grz Rule [7], whose algebraic version is as follows:

$$a \neq 0 \Rightarrow \pi a \neq 0. \tag{RGrz}$$

We will require the following Main Lemma (see [9]):

Lemma 2 (*Main Lemma*). *A* K4-*algebra* (B, δ) *satisfies the equation*

(a) $\delta a = \delta(a - \delta(\delta a - a))$, *i.e.,* $(B, \delta) \in$ K4.Grz, *iff the corresponding* S4-*algebra* (B, C) *satisfies the equation*
(b) $Ca = C(a - C(Ca - a))$, *i.e.,* $(B; C) \in$ S4.Grz *(where, as above,* $Ca := a \vee \delta a$*).*

In the interest of completeness we present a proof.

Proof. Taking into account the monotonicity of operators δ and C it is only necessary to verify that the following conditions are equivalent: (c) $\delta a \leqslant \delta(a \wedge \neg\delta(\delta a \wedge \neg a))$ and (d) $Ca \leqslant C(a \wedge \neg C(Ca \wedge \neg a))$

($\Leftarrow$) We eliminate step by step the closure operator C in condition (d).

(1) $Ca \wedge \neg a = \neg a \wedge (a \vee \delta a) = \neg a \wedge \delta a$
(2) $a \wedge \neg C(Ca \wedge \neg a) = a \wedge \neg((\delta a \wedge \neg a) \vee \delta(\delta a \wedge \neg a)) = a \wedge \neg(\delta a \wedge \neg a) \wedge \neg\delta(\delta a \wedge \neg a)) = a \wedge \neg\delta(\delta a \wedge \neg a) \wedge (\neg\delta a \vee a) = a \wedge \neg\delta(\delta a \wedge \neg a)$.

Thus condition (d) is equivalent to the condition

$$a \vee \delta a \leqslant (a \wedge \neg\delta(\delta a \wedge \neg a)) \vee \delta(a \wedge \neg\delta(\delta a \wedge \neg a)). \tag{d*}$$

Applying the monotonicity and additivity of the derivative operator δ, we obtain

$$\delta a \vee \delta\delta a \leqslant \delta(a \wedge \neg\delta(\delta a \wedge \neg a)) \vee \delta\delta(a \wedge \neg\delta(\delta a \wedge \neg a)).$$

From the K4-axiom $\delta\delta a \leqslant \delta a$, we see that $\delta a \leqslant \delta(a \wedge \neg\delta(\delta a \wedge \neg a))$.

($\Rightarrow$) We notice that $\delta a \wedge \neg a \leqslant \delta a$ implies $\delta(\delta a \wedge \neg a) \leqslant \delta\delta a \leqslant \delta a$. Thus we have $\delta(\delta a \wedge \neg a) \leqslant \delta a$ and $\neg\delta a \leqslant \neg\delta(\delta a \wedge \neg a)$. Taking conjunction with a on both sides, we obtain

(e) $a \wedge \neg\delta a \leqslant a \wedge \neg\delta(\delta a \wedge \neg a)$.

Formula (e) together with (c) implies

$$\delta a \vee (a \wedge \neg\delta a) \leqslant (a \wedge \neg\delta(\delta a \wedge \neg a)) \vee \delta(a \wedge \neg\delta(\delta a \wedge \neg a)).$$

Using the equation

$$\delta a \vee (a \wedge \neg\delta a) = a \vee \delta a, \quad \text{we have } a \vee \delta a \leqslant (a \wedge \neg\delta(\delta a \wedge \neg a)) \vee \delta(a \wedge \neg\delta(\delta a \wedge \neg a)),$$

i.e., condition (d*), which is equivalent to (d). □

4. Distortion algebras

The central theme in this section is the polynomial generation of new derivative operations from old in arbitrary diagonalizable algebras.

4.1. Algebraic analysis of distortions and some special cases

We start by observing that for each GL-algebra (B, δ) and $e \in B$, the following hold:

Proposition 1. (a) *$Ca = C_e a$; that is, $a \vee \delta a = a \vee \delta_e a$;*
(b) *$\delta a \leqslant \delta_e a$ $(a \in B)$;*
(c) *$e \leqslant e' \Rightarrow \delta_e a \leqslant \delta_{e'} a$;*
(d) *If $e = 0$, then $\delta_e a = \delta a$ $(a \in B)$;*
(e) *If $e = 1$, then $\delta_e a = a \vee \delta a = Ca$.*

Proof is immediate from the definition of distortions of δ.

Theorem 1. *Let (B, δ) be an arbitrary diagonalizable algebra and $e \in B$. Then the algebra (B, δ_e), where δ_e is a distortion of δ, is a* K4.Grz*-algebra.*

Proof. (1) We show that the operator δ_e is monotone; that is, $a \leqslant b \Rightarrow \delta_e a \leqslant \delta_c b$. Suppose $a \leqslant b$; then, by the monotonicity of δ, we obtain $\delta_e a = (a \wedge e) \vee \delta a \leqslant (b \wedge e) \vee \delta b = \delta_e b$.

(2) We show that δ_e is additive; that is, $\delta_e(a \vee b) = \delta_e a \vee \delta_e b$; $\delta_e(a \vee b) = ((a \vee b) \wedge e) \vee \delta(a \vee b) = (a \wedge e) \vee (b \wedge e) \vee \delta a \vee \delta b = (a \wedge e) \vee \delta a \vee (b \wedge e) \vee \delta b = \delta_e a \vee \delta_e b$.

(3) We show that $\delta_e \delta_e a \leqslant \delta_e a$. The following equalities hold: $\delta_e \delta_e a = (((a \wedge e) \vee \delta a) \wedge e) \vee \delta((a \wedge e) \vee \delta a) = (((a \wedge e) \vee \delta a) \wedge e) \vee \delta(a \wedge e) \vee \delta\delta a = (a \wedge e) \vee (\delta a \wedge e) \vee \delta(a \wedge e) \vee \delta\delta a$. But

(a) $a \wedge e \leqslant (a \wedge e) \vee \delta a$,
(b) $\delta a \wedge e \leqslant (a \vee \delta a) \wedge (e \vee \delta a) = (a \wedge e) \vee \delta a$,
(c) $\delta(a \wedge e) \leqslant (a \wedge e) \vee \delta a$, and lastly,
(d) $\delta\delta a \leqslant (a \wedge e) \vee \delta a$, as $\delta\delta a \leqslant \delta a$.

Therefore, (B, δ_e) is a K4-algebra. It is left to be shown that $(B, \delta) \in$ K4.Grz; that is, $\delta_e a \leqslant \delta_e(a - \delta_e(\delta_e a - a))$ $(a \in B)$. Since $(B, \delta) \in GL$, it is known (see, for example, [4]) that $(B, C) \in$ S4.Grz; that is, $a \leqslant C(a - C(Ca - a))$. As $(\forall b)(Cb = C_e b)$ for each $e \in B$, by the Main Lemma one has $\forall a \in B$ $\delta_e a \leqslant \delta_e(a - \delta_e(\delta_e a - a))$. □

Let (B, δ) be an arbitrary diagonalizable algebra.

Lemma 3. *If $e_1, e_2 \in B$ and $e_1 \neq e_2$, then $\delta_{e_1} \neq \delta_{e_2}$; that is, $\delta_{e_1}(a) \neq \delta_{e_2}(a)$ for some $a \in B$.*

Proof. Let $e_1 \neq e_2$. Without loss of generality, we may assume that $e_1 \nleqslant e_2$. Then $e_1 - e_2 \neq 0$. Let $a = e_1 - e_2$. Then $\delta_{e_1}(e_1 - e_2) = ((e_1 - e_2) \wedge e_1) \vee \delta(e_1 - e_2) = (e_1 - e_2) \vee \delta(e_1 - e_2)$. On the other hand, $\delta_{e_2}(e_1 - e_2) = ((e_1 - e_2) \wedge e_2) \vee \delta(e_1 - e_2) = \delta(e_1 - e_2)$. Suppose that $\delta_{e_1}(e_1 - e_2) = \delta_{e_2}(e_1 - e_2)$; that is, $(e_1 - e_2) \vee \delta(e_1 - e_2) = \delta(e_1 - e_2)$, which means that $e_1 - e_2 \leqslant \delta(e_1 - e_2)$. By Löb's Rule, $e_1 - e_2 = 0$, which contradicts our assumption that $e_1 \nleqslant e_2$. Thus, $\delta_{e_1}(e_1 - e_2) \neq \delta_{e_2}(e_1 - e_2)$, which means that $\delta_{e_1} \neq \delta_{e_2}$. □

Basing on the Main Lemma, note that in any K4.Grz-algebra (in particular, in any diagonalizable algebra) (B, τ) for $e = 1$ the operator τ_e satisfies the axiom (grz). Note that in case $e = 1$ one has $\tau_e a = a \wedge \tau a$ $(\forall a \in B)$.

We will use (following Boolos [4]) the following abbreviation: $\boxdot\alpha := \alpha \wedge \Box\alpha$ (dotted box). We will say that a formula α is "dotting" if all occurrences of boxes in the formula α are "dotted".

Recall that an arithmetical realization of modal formulas is an assignment * to each atom p of an arithmetical sentence p^* which commutes with non-modal connectives and $(\Box p)^* = \mathrm{Pr}(\ulcorner p^* \urcorner)$, where $\mathrm{Pr}(\cdot)$ is the standard provability predicate for PA.

Note that arithmetical realization of the dotted box, i.e., $(p \wedge \Box p)^*$, is expressible in terms of the demonstrability predicate, namely, $(\boxdot p)^* = \mathrm{Dem}(\ulcorner p^* \urcorner)$.

A somewhat refined version of the main result of Chapter 12 ([4], p. 156) is valid.

Theorem 2. *For every dotted formula p the following assertions are equivalent:*

1. K4.Grz $\vdash p$;
2. *for all realizations *, the sentence p^* is provable in Peano Arithmetic* PA;
3. *for all realizations *, the sentence p^* is true in the standard model of* PA.

Sketch of proof. Since K4.Grz is a subsystem of S4.Grz, we obtain K4.Grz $\vdash p$ implies S4.Grz $\vdash p$. If K4.Grz $\nvdash p$, then there is a K4.Grz-algebra (B, δ) such that $p = 1$ is not true in (B, δ). We consider the algebra (B, C). By the Main Lemma, (B, C) is a S4.Grz-algebra. Because p is dotted, $p = 1$ is also not true in (B, C). Therefore, S4.Grz $\nvdash p$. Thus, for a dotted formula p, we have K4.Grz $\vdash p$ iff S4.Grz $\vdash p$. The rest follows from the main result of Chapter 12 ([4], p. 156). □

Let (B, δ) be a fixed diagonalizable algebra. Let us denote by $\mathrm{D}(B, \delta)$ the set of all distortions of the operator δ, i.e., $\mathrm{D}(B, \delta) = \{\delta_e : e \in B\}$. Let us define a partial order on the set $\mathrm{D}(B, \delta)$ in a component-wise manner: for any $\delta_1, \delta_2 \in \mathrm{D}(B, \delta)$ put

$$\delta_1 \leqslant \delta_2 \Leftrightarrow (\forall a \in B)(\delta_1 a \leqslant \delta_2 a).$$

It is easy to see that with respect to this partial order in $\mathrm{D}(B, \delta)$ there exists the smallest element $\delta = \delta_e$ when $e = 0$ and the largest element $c = \delta_e$ for $e = 1$.

Let us equip the set $\mathrm{D}(B, \delta)$ with the following operations. For $\delta_1, \delta_2 \in \mathrm{D}(B, \delta)$,

$$(\delta_1 \circ \delta_2)(a) = \delta_1(a) \wedge \delta_2(a) \quad (\text{for } a \in B)$$
$$(\delta_1 + \delta_2)(a) = \delta_1(a) \vee \delta_2(a) \quad (\text{for } a \in B)$$
$$(-\delta_1)(a) = (\mathrm{C}a - \delta_1 a) \vee \delta a \quad (\text{for } a \in B).$$

Let us define the map $F : B \to \mathrm{D}(B, \delta)$ putting $F(e) = \delta_e$ for $e \in B$.

We already know that the map F is injective and preserves Boolean order (see Proposition 1 and Lemma 3).

Theorem 3. *The map $F_\delta : B \to \mathrm{D}(B, \delta)$ commutes with Boolean operations on the Boolean algebra B, i.e.,*

1. $F(e_1 \wedge e_2) = F(e_1) \circ F(e_2)$;
2. $F(e_1 \vee e_2) = F(e_1) + F(e_2)$;
3. $F(\neg e) = -F(e)$.

Proof. (1) $\delta_{e_1}(a) \wedge \delta_{e_2}(a) = [(e_1 \wedge a) \vee \delta a] \wedge [(e_2 \wedge a) \vee \delta a] = \delta a \vee (e_1 \wedge a \wedge e_2 \wedge a) = \delta a \vee (e_1 \wedge e_2 \wedge a) = \delta_{e_1 \wedge e_2} a$.

(2) $\delta_{e_1} a \vee \delta_{e_2} a = [(e_1 \wedge a) \vee \delta a] \vee [(e_2 \wedge a) \vee \delta a] = \delta a \vee ((e_1 \wedge a) \vee (e_2 \wedge a)) = \delta a \vee (a \wedge (e_1 \vee e_2)) = \delta_{e_1 \vee e_2} a$.

(3) $-\delta_e a = (\mathrm{C}a - \delta_e a) \vee \delta a$. Let us transform the polynomial $\mathrm{C}a - \delta_e a = (a \vee \delta a) \wedge \neg((a \wedge e) \vee \delta a) = (a \vee \delta a) \wedge \neg(a \wedge e) \wedge \neg \delta a = (a \wedge \neg(a \wedge e) \wedge \neg \delta a) \vee (\delta a \wedge \neg(a \wedge e) \wedge \neg \delta a) = ((a \wedge \neg(a \wedge e) \wedge \neg \delta a) = a \wedge (\neg a \wedge \neg e) \wedge \neg \delta a = a \wedge \neg(a \wedge e) \wedge \neg \delta a = a \wedge (\neg a \vee \neg e) \wedge \neg \delta a = (a \wedge \neg a \wedge \neg \delta a) \vee (a \wedge \neg e \wedge \neg \delta a) = a \wedge \neg e \wedge \neg \delta a$. Thus $(\mathrm{C}a - \delta_e a) \vee \delta a = (a \wedge \neg e \wedge \neg \delta a) \vee \delta a = ((a \wedge \neg e) \vee \delta a) \wedge (\neg \delta a \vee \delta a) = (a \wedge \neg e) \vee \delta a = \delta_{\neg e} a$. □

Corollary 1. *The set $\mathrm{D}(B, \delta)$ of all distortions forms a Boolean algebra isomorphic to the Boolean algebra $(B, \wedge, \vee, \neg)$.*

For any finite Boolean algebra B let us denote by $\mathrm{At}(B)$ the set of its atoms.

Theorem 4. *Let (B, δ) be a finite diagonalizable algebra and let δ' be a normal ($\delta' 0 = 0$) and additive ($\delta'(a \vee b) = \delta' a \vee \delta' b$, $a, b \in B$) operator on B such that $a \vee \delta a = a \vee \delta' a$. Then δ' coincides with a distortion δ_e of the operator δ for $e = \bigvee\{a \in \mathrm{At}(B) : a - \delta' a = 0\}$.*

Proof. In view of additivity of the operators δ and δ' it suffices to ensure the equality $\delta' a = \delta_e a$ for $a \in \mathrm{At}(B)$.
Case 1. Let $a \in \mathrm{At}(B)$ and $a - \delta' a = 0$, i.e., $a \leqslant \delta' a$. Then $\delta_e a = (a \wedge e) \vee \delta a = a \vee \delta a$. Since $a \leqslant \delta' a$, one has $a \vee \delta' a = \delta' a$. But by the hypothesis of the theorem we have $a \vee \delta' a = a \vee \delta a$; hence $\delta' a = a \vee \delta a = \delta_e a$.
Case 2. $a \in \mathrm{At}(B)$ and $a - \delta' \neq 0$. Let us show $\delta_e a = \delta' a$, i.e., $(a \wedge e) \vee \delta a = \delta' a$. But $a \wedge e = a \wedge \bigvee\{b \in \mathrm{At}B : b - \delta' b = 0\}$. Since $a - \delta' a \neq 0$, for $b \in \mathrm{At}B$ and $b - \delta' b = 0$ we have $a \neq b$ and, since a is an atom, $a \wedge b = 0$. Hence $a \wedge \bigvee\{b \in \mathrm{At}B : b - \delta' b = 0\} = 0$, and therefore $\delta_e a = (a \wedge e) \vee \delta a = \delta a$. Let us show that $\delta' a = \delta a$. From the condition $a \vee \delta' a = a \vee \delta a$ it follows that $\forall b \in \mathrm{At}(B)\ b \leqslant a \vee \delta' a \Leftrightarrow b \leqslant a \vee \delta a$, i.e., $\forall b \in \mathrm{At}B\ (b - a \leqslant \delta' a \Leftrightarrow b - a \leqslant \delta a)$, i.e., for $b \neq a$, $b \leqslant \delta' a \Leftrightarrow b \leqslant \delta a$. □

4.2. Distortion algebras in which a modal version of Gödel's Second Incompleteness Theorem is valid, while the Löb Axiom is refutable

Let (B, τ) be a diagonalizable algebra, $e \neq 0$, and $e \leqslant \neg \tau 0 = \delta 1$. Let us show that in the K4.Grz-algebra (B, τ_e) the modal version of the Gödel Second Incompleteness Theorem holds; that is,

$$\neg \tau_e 0 \leqslant \neg \tau_e(\neg \tau_e 0). \tag{a}$$

In terms of the derivative operation, condition (a) is written as

$$\delta_e 1 \leqslant \delta_e \neg \delta_e 1. \tag{b}$$

Proof. $\delta_e 1 = (e \wedge 1) \vee \delta 1 = e \vee \delta 1$.

$\neg \delta_e 1 = \neg(e \vee \delta 1) = \neg e \wedge \neg \delta 1$.

$\delta_e(\neg \delta_e 1) = \delta_e(\neg e \wedge \neg \delta 1) = (e \wedge \neg e \wedge \neg \delta 1) \vee \delta(\neg(e \vee \delta 1))$. Therefore, (b) reduces to

$$e \vee \delta 1 \leqslant \delta \neg(e \vee \delta 1); \tag{c}$$

$e \leqslant \delta 1$ implies

$$\delta 1 \leqslant \delta \neg \delta 1. \tag{d}$$

But (d) is a substitution instance of the Löb Axiom $\delta a \leqslant \delta(a - \delta a)$, when $a = 1$. Let (B, δ) be a diagonalizable algebra and $e = \neg \tau 0 = \delta 1 \neq 0$. We show that Löb's Axiom is falsified in (B, δ_e); that is,

$$\delta_e a \leqslant \delta_e(a - \delta_e a) \tag{e}$$

is false for some $a \in B$. Set $a = e = \delta 1$ and show that (b) is false, which in this case reads as

$$\delta 1 \leqslant (\delta 1 \vee \delta \neg \delta 1) \wedge (\neg \delta 1 \wedge \delta(\neg \delta 1)). \tag{f}$$

By simplifying (f), we obtain

$$\delta 1 \leqslant \neg \delta 1 \wedge \delta(\neg \delta 1). \tag{g}$$

From (g) it follows that $\delta 1 \leqslant \neg \delta 1$; that is, $\delta 1 = 0$. But this contradicts the condition $\delta 1 \neq 0$. Therefore, condition (f) is false, and hence so is (e). □

4.3. Distortion algebras in which a modal version of Gödel's Second Incompleteness Theorem is refutable but a weaker form of the Löb Axiom is valid

Let (B, τ) be a diagonalizable algebra. We will need the following substitution instance of the Löb Axiom: $\tau(\tau b \to b) \leqslant \tau b$ for $b = a \to \tau a$.

$$\tau(\tau(a \to \tau a) \to (a \to \tau a)) \leqslant \tau(a \to \tau a) \tag{a}$$

or, in terms of the derivative operator,

$$\delta(a \wedge \delta \neg a) \leqslant \delta((a \wedge \delta \neg a) - \delta(a \wedge \delta \neg a)). \tag{b}$$

We show that in (B, δ_e), if $e \leqslant \tau 0 = \neg \delta 1$, the weakened formulation (b) of the Löb Axiom is satisfied; that is,

$$\delta_e(a \wedge \delta_e \neg a) \leqslant \delta_e((a \wedge \delta \neg a) - \delta_e(a \wedge \delta \neg a)). \tag{c}$$

Let us transform the left-hand side: $\delta_e(a \wedge \delta_e \neg a) = \delta_e(a \wedge ((\neg a \wedge e) \vee \delta \neg a)) = \delta_e((a \wedge \neg a \wedge e) \vee (a \wedge \delta \neg a)) = \delta_e(a \wedge \delta \neg a) = (a \wedge \delta \neg a \wedge e) \vee \delta(a \wedge \delta \neg a)$. Since $\delta \neg a \leqslant \delta 1$, one has $\delta \neg a \wedge \neg \delta 1 = 0$ and, since $e \leqslant \neg \delta 1$, a fortiori $\delta \neg a \wedge e = 0$. Hence $(a \wedge \delta \neg a \wedge e) \vee \delta(a \wedge \delta \neg a) = \delta(a \wedge \delta \neg a)$. Thus $\delta_e(a \wedge \delta_e \neg a) = \delta(a \wedge \delta \neg a)$.

Right-hand side: $\delta_e((a \wedge \delta \neg a) - \delta_e(a \wedge \delta \neg a)) = \delta_e((a \wedge \delta \neg a) - \delta(a \wedge \delta \neg a)) = ((a \wedge \delta \neg a) \wedge \neg \delta(a \wedge \delta \neg a) \wedge e) \vee \delta(a \wedge \delta \neg a \wedge \neg \delta(a \wedge \delta \neg a))$. Since $\neg \delta 1 \leqslant \neg \delta(a \wedge \delta \neg a)$ and $e \leqslant \neg \delta 1$, one has $e \leqslant \neg \delta(a \wedge \delta \neg a)$. Hence the last expression is equal to $(a \wedge \delta \neg a \wedge e) \vee \delta(a \wedge \delta \neg a \wedge \neg \delta(a \wedge \delta \neg a))$ which, since $\delta \neg a \leqslant \delta 1$ implies $\delta \neg a \wedge \neg \delta 1 = 0$ and a fortiori $\delta \neg a \wedge e = 0$, is equal to $\delta((a \wedge \delta \neg a) - \delta(a \wedge \delta \neg a)$. Thus the condition (c) takes shape:

$$\delta(a \wedge \delta \neg a) \leqslant \delta((a \wedge \delta \neg a) - \delta(a \wedge \delta \neg a)). \tag{d}$$

But (d) holds, being a substitution instance of the Löb Axiom (see (b)).

Let (B, δ) be a diagonalizable algebra with non-degenerate Boolean part, i.e., $1 \neq 0$. Then the modal version of Gödel's Second Incompleteness Theorem is refutable in the algebra (B, δ_e) for $e = \neg \delta 1$, i.e.,

$$\delta_e 1 \leqslant \delta_e \neg \delta_e 1 \tag{e}$$

is false. Indeed, $\delta_e 1 = (\neg \delta 1 \wedge 1) \vee \delta 1 = 1$ so $\neg \delta_e 1 = \neg 1 = 0$; whereas $\delta_e \neg \delta_e 1 = \delta_e 0 = 0$. Thus (e) is equivalent to

$$1 \leqslant 0, \text{ i.e., } 1 = 0. \tag{f}$$

5. Three modal satellites of the Gödel–Löb modal system

Our analysis of special reincarnations of the standard provability predicate in PA led us to the modal system K4.Grz and two of its normal extensions. Namely, the modal system K4.Grz + (g) obtained by postulating the modal version of Gödel's Second Incompleteness Theorem and the modal system K4.Grz + (wL) obtained by postulating a special instance of the Löb Axiom

$$\tau(\tau p \to p) \to \tau p \tag{l}$$

consisting of two "independent" parts, namely

$$\neg \tau \bot \to \neg \tau(\neg \tau \bot) \tag{g}$$

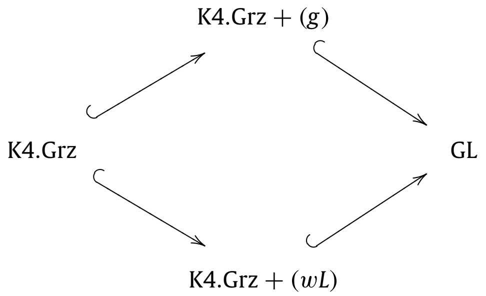
and

$$\tau(\tau(p \to \tau p) \to (p \to \tau p)) \to \tau(p \to \tau p). \tag{wL}$$

Obviously (wL), being an instance of (L), formalizes the Löb Principle *not for arbitrary propositions p*, but only for propositions of the form $p \to \tau p$, expressing the local principle of semantic completeness: "truth of p implies provability of p".

As we will show below, GL = K4.Grz + (g) + (wL).

Before that, let us present a simple (four-element) diagonalizable algebra (B, δ) and its two different distortions that will show us that all four modal systems are *different* in that the inclusions between them are *proper*, i.e.,



Let $B = \{0, a, b, 1\}$ be the four-element Boolean algebra and δ be the operator on B defined as follows: $\delta 0 = 0$, $\delta a = 0$, $\delta b = a$, $\delta 1 = a$. It is easy to verify that (B, δ) is a diagonalizable algebra; let δ_e be the distortion of the operator δ with $e = a$ and $e = b$. Denote by (B, δ_1) and (B, δ_2) the algebras with $\delta_1 = \delta_e$ for $e = a$ and $\delta_2 = \delta_e$ for $e = b$.

Simple checking shows that

(a) $(B, \delta), (B, \delta_1), (B, \delta_2) \in$ K4.Grz;
(b) $(B, \delta_1) \in$ K4.Grz + (g) and $(B, \delta_1) \notin$ K4.Grz + (wL);
(c) $(B, \delta_2) \in$ K4.Grz + (wL) and $(B, \delta_2) \notin$ K4.Grz + (g);
(d) $(B, \delta_1), (B, \delta_2) \notin$ GL.

We will need the following lemma having certain independent significance.

Lemma 4. *Let (B, τ) be any* K4*-algebra. The following conditions are equivalent:*

(g) $\neg\tau 0 \leqslant \neg\tau(\neg\tau 0)$ (modal version of Gödel's Second Incompleteness Theorem);
(g') $\tau(\tau p \to p) \leqslant \tau(\tau\neg p \to p)$ $(\forall p \in B)$.

Proof. As we already know, in terms of the derivative operator δ condition (g) is equivalent to

$$\delta 1 \leqslant \delta\neg\delta 1 \quad \text{or, more shortly, } \delta 1 \leqslant \delta\mu 1. \tag{1}$$

It is easy to check that the δ-version of the condition (g′) looks as follows:

$$\delta(p \wedge \neg\delta\neg p) \leqslant \delta(p - \delta p). \tag{2}$$

Let us prove the equivalence of the conditions (1) and (2).

(2) ⇒ (1). Substituting 1 in place of p in (2), we obtain $\delta(1 \wedge \neg\delta\neg 1) \leqslant \delta(1 \wedge \neg\delta 1)$, i.e., $\delta 1 \leqslant \delta\neg\delta 1$, i.e., $\neg\delta 1 \vee \delta\neg\delta 1 = 1$; i.e., $\mathrm{C}\mu 1 = 1$.

(1) ⇒ (2). Note that $p \wedge \mu 1 \leqslant \mu p$; indeed, $p \wedge \neg\delta 1 \leqslant p \wedge \neg\delta p$ (since $\delta p \leqslant \delta 1$). Furthermore, we have $p \wedge \neg\delta\neg p = \mathrm{I}p \leqslant \mathrm{C}\mu 1$ (using $\mathrm{C}\mu 1 = 1$). Let us use the following property of arbitrary closure algebras (see [15], Cor. 1.8, p. 146). If an element a is open, and b is arbitrary, then $a \wedge \mathrm{C}(a \wedge b) = a \wedge \mathrm{C}b$. Putting in this equality $a = \mathrm{I}p$ and $b = \mu 1$, we obtain

$$\mathrm{I}p \wedge \mathrm{C}(\mathrm{I}p \wedge \mu 1) = \mathrm{I}p \wedge \mathrm{C}\mu 1; \tag{a}$$

and using the density of $\mu 1$, we obtain

$$\mathrm{I}p \wedge \mathrm{C}(\mathrm{I}p \wedge \mu 1) = \mathrm{I}p, \tag{b}$$

i.e.,

$$\mathrm{I}p \leqslant \mathrm{C}(\mathrm{I}p \wedge \mu 1); \tag{c}$$

but $p \wedge \mu 1 \leqslant \mu p$, and a fortiori $\mathrm{I}p \wedge \mu 1 \leqslant \mu p$. Hence $\mathrm{C}(\mathrm{I}p \wedge \mu 1) \leqslant \mathrm{C}\mu p$, and by (b) we obtain

$$\mathrm{I}p \leqslant \mathrm{C}\mu p; \tag{d}$$

then

$$\delta\mathrm{I}p \leqslant \delta\mathrm{C}\mu p.$$

But $\delta\mathrm{C}\mu p = \delta(\mu p \vee \delta\mu p) = \delta\mu p \vee \delta\delta\mu p = \delta\mu p$. Consequently $\delta\mathrm{I}p \leqslant \delta\mu p$. Thus

$$\delta(p \wedge \neg\delta\neg p) \leqslant \delta(p - \delta p). \quad \square$$

Theorem 5. *The modal system* GL *can be axiomatized as* K4.Grz + *(g)* + *(wL).*

Proof. (g) + (wL) ⇒ (L), i.e.,

$$\delta(p \wedge \delta\neg p) \leqslant \delta((p \wedge \delta\neg p) - \delta(p \wedge \delta\neg p)) \tag{wL}$$

and

$$\delta(p \wedge \neg\delta\neg p) \leqslant \delta(p \wedge \neg\delta p) \tag{g}$$

imply

$$\delta p \leqslant \delta(p - \delta p). \tag{L}$$

Representing p in the form $p = (p \wedge \delta\neg p) \vee (p \wedge \neg\delta\neg p)$, we see that (L) is equivalent to the conjunction of

$$\delta(p \wedge \delta\neg p) \leqslant \delta(p \wedge \neg\delta p) \tag{L$_1$}$$

and

$$\delta(p \wedge \neg\delta\neg p) \leqslant \delta(p \wedge \neg\delta p). \tag{L_2}$$

We have (g) = (L_2). Let us show that (g) + (wL) ⇒ (L_1). Clearly $p \wedge \delta\neg p \wedge \neg\delta p \leqslant p \wedge \neg\delta p$, i.e., $p \wedge \delta\neg p \wedge \neg\delta((p \wedge \delta\neg p) \vee (p \wedge \neg\delta\neg p)) \leqslant p \wedge \neg\delta p$, i.e., $p \wedge \delta\neg p \wedge \neg\delta(p \wedge \delta\neg p) \wedge \neg\delta(p \wedge \neg\delta\neg p) \leqslant p \wedge \neg\delta p$, i.e., $\mu(p \wedge \delta\neg p) \wedge \neg\delta(p \wedge \neg\delta\neg p) \leqslant \mu p$, i.e., $\mu(p \wedge \delta\neg p) \leqslant \mu p \vee \delta(p \wedge \neg\delta\neg p)$; consequently $\delta\mu(p \wedge \delta\neg p) \leqslant \delta\mu p \vee \delta\delta(p \wedge \neg\delta\neg p) \leqslant \delta\mu p \vee \delta(p \wedge \neg\delta\neg p)$, i.e.,

$$\delta\mu(p \wedge \delta\neg p) - \delta(p \wedge \neg\delta\neg p) \leqslant \delta\mu p. \tag{1}$$

But by (g) we have $\delta(p \wedge \neg\delta\neg p) \leqslant \delta(p \wedge \neg\delta p)$; hence

$$\delta\mu(p \wedge \delta\neg p) - \delta(p \wedge \neg\delta p) \leqslant \delta\mu(p \wedge \delta\neg p) - \delta(p \wedge \neg\delta\neg p). \tag{2}$$

(1) and (2) by transitivity of $\leqslant$ give

$$\delta\mu(p \wedge \delta\neg p) \leqslant \delta\mu p \vee \delta\mu p = \delta\mu p. \tag{3}$$

(wL) $\delta(p \wedge \delta\neg p) \leqslant \delta\mu(p \wedge \delta\neg p)$ and (3) by transitivity give (L_1), i.e., $\delta(p \wedge \delta\neg p) \leqslant \delta\mu p$.

Thus (g) & (wL) ⇒ (L). □

Final remarks

Note that every distortion τ_e of τ in an arbitrary diagonalizable algebra (B, τ) is an operator *weaker than* the original operator τ; that is, $\tau_e p \leqslant \tau p$ for every $p \in B$.

To conclude, let us present a certain observation arising from consideration of an internally definable *strengthening* of the "standard" provability operator. More precisely, we introduce the following definition.

Let e be a fixed element of a diagonalizable algebra (B, τ).

e-modest strengthening: for every $p \in B$, $\tau^e p := \tau p \vee (e \wedge \tau(e \to p))$.

The e-modest strengthening is of special interest in connection with incompleteness of Peano Arithmetic.

Recall that Gödel's First Incompleteness Theorem asserts that if PA is consistent then it is incomplete. Nevertheless there are consistent principles asserting completeness.

We consider the local principle of semantic completeness: $p \to \mathrm{Pr}(\ulcorner p \urcorner)$ (somewhat loosely, "if p is true then p is provable"). This principle is equivalent to ¬Consis, i.e., to $\mathrm{Pr}(\ulcorner \bot \urcorner)$ ("inconsistency"), and hence is not in general derivable. One half of the relation between p and $\mathrm{Pr}(\ulcorner p \urcorner)$ in PA is provided by a theorem of Löb which states that $\mathrm{PA} \vdash \mathrm{Pr}(\ulcorner p \urcorner) \to p$ implies $\mathrm{PA} \vdash p$.

However, there is a large and interesting set (see [14]) of sentences for which $\mathrm{PA} \vdash p \to \mathrm{Pr}(\ulcorner p \urcorner)$, i.e., $p \leqslant \tau p$ in the diagonalizable algebra $D(\mathrm{PA})$ of Peano Arithmetic. The set $H = \{p \in D(\mathrm{PA}) : p \leqslant \tau p\}$ can be shown to be a sublattice of $D(\mathrm{PA})$ which is a Heyting algebra.

It may happen that an element e of a diagonalizable algebra (B, τ) (for example, of $D(\mathrm{PA})$) is a "violator" of the local principle of semantic completeness, i.e., $e \nleqslant \tau e$. The latter condition of course means that the "perplexus" $e \wedge \neg\tau e$ ("e is true but not provable") is non-zero.

The e-modest strengthening τ^e is the least among all those strengthenings of τ which restore the local principle of semantic completeness for e. More precisely,

Observation. Let (B, τ) be a diagonalizable algebra and $e \in B$. Then

(i) The algebra (B, τ^e) is a diagonalizable algebra.
(ii) If a K4-operator τ' is a strengthening of τ, i.e., $\tau p \leqslant \tau' p$ for all $p \in B$, and $e \leqslant \tau' e$, then $\tau^e p \leqslant \tau' p$ for every $p \in B$.

Finally, let us indicate a simple but important special case. Let $e = \neg\tau 0$ ("consistency" in the case of $D(\mathrm{PA})$). Then $\tau^e p = \tau((\neg\tau 0) \to p)$. Roughly speaking, in this case $\tau^e p$ expresses not the "absolute" provability of p but rather the provability of p under the assumption of consistency of PA. The above observation implies that here τ^e is the least strengthening of the "standard" operator τ under which e ("consistency of PA") satisfies the local principle of semantical completeness.

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Appendix

We start by recalling the necessary definitions to state topological completeness theorems for the modal systems discussed in the paper.

Definition 4. A topological space is called a *scattered space* if it has no dense-in-itself non-empty subset.

An "equational" (à la Kuratowski) characterization of scattered spaces using the derivation operation as a primitive notion is contained in the following proposition.

Proposition 2 ([7]). *A scattered space is a set X equipped with an operator δ, satisfying the equations:*

1. $\delta\emptyset = \emptyset$;
2. $\delta(A \cup B) = \delta A \cup \delta B$;
3. $\delta A = \delta(A - \delta A)$; *i.e.*, $\delta A = \delta\mu A$ *(the dual form of the Löb Axiom).*

Proposition 3 (*Topological Completeness of GL [8]*). *GL $\models p$ iff p is valid in every scattered space.*

Reducible sets as defined below were introduced by Hausdorff [11], p. 192 (so-called Hausdorff's Theory of Residues). A set A is called reducible if it can be obtained as a chain of differences of a well-ordered decreasing sequence of non-empty closed sets whose intersection is empty.

Definition 5 ([8]). A topological space X is called well reducible if every subset A of X is reducible.

Proposition 4 ([8]). *A topological space X is well reducible iff for every subset A of X $A \neq \emptyset$ implies $A - \rho(A) \neq \emptyset$.*

Theorem 6 (*An "Equational" Characterization of Well-reducible Spaces [8]*). *The following conditions on a topological space X are equivalent:*

(1) *The space X is well reducible.*
(2) $\mathrm{C}A = A - \mathrm{C}(\mathrm{C}A - A)$ *for every* $A \subseteq X$; *i.e.*, $\mathrm{C}A = \mathrm{C}\pi(A)$.
(3) $dA = A - d(dA - A)$ *for every* $A \subseteq X$; *i.e.*, $dA = d\pi(A)$.

It is not hard to verify that every scattered space is well reducible. An example of a well-reducible space which is not scattered has been constructed in [2]:
Example: If X is an infinite set, $\mathcal{F}$ is a free ultrafilter on X and the topology $T = \mathcal{F} \cup \emptyset$, then (X, T) is a dense-in-itself (so, not scattered) well-reducible space.
In [2] there has also been shown that scatteredness and well-reducibility coincide on a wide class of spaces, including all of the spectral, first countable, or locally compact Hausdorff spaces (see, e.g., [13] for definitions of these notions).

Proposition 5 (*Topological Completeness of* S4.Grz *[8]*). S4.Grz $\models$ *p iff p is valid in every well-reducible topological space (under reading the diamond $\Diamond$ as closure operation* C*).*

Recently, in [2], the authors have established the equivalence of this notion with that of hereditary irresolvability.

Definition 6 (*Hewitt [12]*). A space X is *resolvable* iff X contains two disjoint dense subsets. A space X is *hereditarily irresolvable* if no non-empty subspace is resolvable.

Proposition 6 ([2]). *A space X is well reducible iff X is hereditarily irresolvable.*

So S4.Grz is the modal logic of well-reducible = hereditarily irresolvable spaces when the modal diamond is interpreted as the topological closure operator.

Proposition 7 (*Topological Completeness of* K4.Grz *[10]*). *The modal system* K4.Grz *is sound and complete with respect to the class of all hereditarily irresolvable spaces (when the diamond is read as the derivation operator).*

Definition 7. Call a topological space X *weakly scattered* if the set of isolated points of X is everywhere dense in X.

Theorem 7 (*[10] Topological Completeness of* K4.Grz + (g):). K4.Grz + (g) $\models$ *p iff p is valid in every weakly scattered space; (under reading the diamond-modality as the derivative operation).*

Definition 8. A topological space is called *almost scattered* if it has no dense-in-itself non-empty sets with empty interior.

Theorem 8 (*[10] Topological Completeness of* K4.Grz + (wL)). K4.Grz + (wL) $\models$ *p iff p is valid in every topological almost-scattered space (under reading the diamond-modality $\Diamond$ as the derivative operation).*

Now we consider the Kripke semantics for the considered modal systems. Recall that a Kripke frame $F = (X, R)$ is a non-empty set X together with a binary relation $R \subseteq X \times X$.

We say that R is transitive, if

(tr) $(\forall x, y, z \in X)(xRy \wedge yRz \to xRz)$

irreflexive, if

(irref) $(\forall x \in X)(\neg xRx)$

and antisymmetric, if

(asymm) $(\forall x, y \in X)(xRy \wedge yRx \to x = y)$.

A point $x \in A$ is called a *maximal* (resp., *strongly maximal*) point of A if, for every $y \in A$, xRy implies $x = y$ (resp., $\neg\exists y \in A$ such that xRy). We denote the set of maximal points of A by $\max A$.

All the systems considered in the paper have the finite model property (fmp). The corresponding completeness theorems are given below:

Theorem 9 ([17]). *GL is complete w.r.t. finite transitive irreflexive Kripke frames.*

Theorem 10 ([1,10]). K4.Grz *is complete w.r.t. finite transitive antisymmetric Kripke frames.*

Theorem 11 ([10]). K4.Grz + (g) *is complete w.r.t. finite transitive antisymmetric frames* $F = (X, R)$ *such that*

$$(\forall x \in X)(xRx \to x \notin \max X).$$

Theorem 12 ([10]). K4.Grz + (wL) *is complete w.r.t. finite transitive antisymmetric frames* $F = (X, R)$ *such that*

$$(\forall x \in X)(xRx \to x \in \max X).$$

We note that all these systems have in fact a stronger tree-like model property. For details see [10].

Duality for K4.Grz

In order to describe the dual category for the category K4.Grz of K4.Grz-algebras and homomorphism we recall that a topological space X is a *Stone space* if X is compact, Hausdorff and zero-dimensional. For a transitive relation R on X and $A \subseteq X$ let $R^{-1}(A) = \{x \in X : y \in A \wedge xRy\}$.

We call (X, R) a *topological Grz-frame* if X is a Stone space and R is a transitive relation on X such that

1. the set $R(x) = \{y \in X : xRy\}$ is closed for every $x \in X$;
2. for every clopen A the set $R^{-1}(A)$ is a clopen;
3. Maximality Principle: for each non-empty clopen A the set $\max A$ is non-empty. For topological Grz-frames X_1, X_2 a map $f : X_1 \to X_2$ is a morphism of the category TGF of topological Grz-frames if f is a continuous function and $fR_1(x) = R_2(f(x))$ for every point $x \in X_1$.

Theorem A (*cf.* [6]). *The category TGF is dually equivalent to the category of* K4.Grz-*algebras and algebraic homomorphisms.*

We note that the category of diagonalizable algebras (alias, GL-algebras) is dually equivalent to the full subcategory of topological K4.Grz-frames, satisfying an addition condition: for every clopen A every maximal point x of A is strongly maximal.

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