Intuitionistic logic and modality via topology

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Abstract

In the pioneering article and two papers, written jointly with McKinsey, Tarski developed the so-called algebraic and topological frameworks for the Intuitionistic Logic and the Lewis modal system. In this paper, we present an outline of modern (non-Lewis) systems with a topological tinge. We consider topological interpretation of basic systems GL and GRZ of the provability logic in terms of the Cantor derivative and the Hausdorff residue.
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In the pioneering article (appeared in 1938) and two papers [33,34] (written jointly with J.C.C. McKinsey) Tarski developed the so-called algebraic and topological framework for Intuitionistic Logic and Modal systems. These papers influenced a whole group of logicians (we think of Tbilisi Group in particular) who cast their studies in algebraic form and made considerable progress in analysis of various modal systems. Interest in the papers was twofold: not only does it show that topological spaces provide a rich semantical source for intuitionistic logic and modal systems, it also gives insight into the algebraic properties of the collection of open sets of a topological space (Heyting lattice) and of the algebra of the topological closure operation (closure algebra) and the derivative operation (derivative algebra). It is also shown, among other things, that the modal translation of intuitionistic logic proposed by Gödel does indeed rise to an interpretation of the Heyting calculus in the Lewis system S4.

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We shall try to take the risk of presenting a brief survey (that, nevertheless, is likely to be incomplete) of modern (non-Lewis) modal systems, especially those now known as the *Gödel–Löb* system GL and the *Grzegorczyk system* GRZ. What make GL and GRZ of interest is their adequate arithmetical interpretation: when the box $\square$ is interpreted via the formula $\text{Bew}(x)$ expressing provability in a certain standard arithmetical theory. We shall also discuss topological interpretation of the systems and the Intuitionistic logic in term of the *Cantor derivative* and the *Hausdorff residue*. We are happy to contribute a little piece of our own thinking on these matters.

We shall present a diagram with comments, mixing algebraic and modal viewpoints, and will try to emphasize some of the most important modal systems having a topological tinge. We shall not make a point of mentioning Tarski's name at each place in this lecture, where his influence is either directly or indirectly present.

1. Triplets

An arbitrary topological space **X** gives rise to three concrete "algebraic" structures:

$\mathbf{Op}(\mathbf{X}) = $ *The algebra of open sets*,
$(\mathbf{P}(\mathbf{X}), \mathbf{c}) = $ *Closure algebra*,
$(\mathbf{P}(\mathbf{X}), \mathbf{d}) = $ *Derivative algebra*.

Note that the usual intuitively obvious connection between closure and derivative operations is $cA = A \cup dA$.

2. Abstract level

$$(\mathbf{H}, \vee, \wedge, \rightarrow, \perp) \qquad (\mathbf{B}, \vee, \wedge, -, \mathbf{c}) \qquad (\mathbf{B}, \vee, \wedge, -, \mathbf{d})$$
Heyting algebra   Closure algebra   Derivative algebra

Recall that a *Heyting algebra* $(\mathbf{H}, \vee, \wedge, \rightarrow, \perp)$ is a distributive lattice (with smallest element $\perp$), endowed with a binary operation (relative pseudocomplement) $\rightarrow$ such that $x \leqslant a \rightarrow b$ iff $a \wedge x \leqslant b$. An algebra $(\mathbf{B}, \vee, \wedge, -, \mathbf{c})$ is called a *Closure algebra* if $(\mathbf{B}, \vee, \wedge, -)$ is a Boolean algebra and the operator $\mathbf{c}$ satisfies "Kuratowski axioms": $a \leqslant \mathbf{c}a$, $\mathbf{cc}a = \mathbf{c}a$, $\mathbf{c}(a \vee b) = \mathbf{c}a \vee \mathbf{c}b$, $\mathbf{c}\perp = \perp$.

Remark. In *Appendix* I. *Derivative algebra* of the paper [33], McKinsey and Tarski initiated an investigation of the fundamental topological operation of derivation from a purely algebraic (and/or modal) point of view. On p. 182 of [33] the authors say: "Like the topological operation of closure, other topological operations can be treated in an algebraic way. This may be especially interesting in regard to those operations which are not definable in terms of closure... An especially important notion is that of the derivative of a point set A which will be denoted by $\mathbf{d}A$".

Thus, Derivative algebras are Boolean algebras with an unary operation $\mathbf{d}$, which captures algebraic properties of the topological derivation. Recall that $\mathbf{d}A$ is, by definition, the set of all *accumulation* (alias, *limit*) points of a subset A of a topological

space X. A point x is said to be a limit-point of a set A, if every neighborhood of x contains a point of A other than x.

Definition 1. We say, that a Boolean algebra **B** is a *Derivative algebra* with respect to the operation **d**, if (1) $\mathbf{d}\perp = \perp$, (2) $\mathbf{d}(a \vee b) = \mathbf{d}a \vee \mathbf{d}b$, (3*) $\mathbf{d}\mathbf{d}a \leqslant a \vee \mathbf{d}a$.

Remark. It must be pointed out that we weaken the definition of Derivative algebra [33] slightly; namely, we postulate the condition (3*) instead of (3) $\mathbf{d}\mathbf{d}a \leqslant \mathbf{d}a$. We justify this weakening by noting that there are topological spaces, in which condition (3) is not valid (for example, spaces with anti-discrete topology).

3. Three logical systems

HC = *the Heyting Calculus*
S4 = *the Lewis Modal system*
wK4 = *a slightly weakened version of the modal system* $K4$;
namely, **wK4** = **K** + $\mathbf{p} \wedge \square\mathbf{p} \rightarrow \square\square\mathbf{p}$, where the system K (named after Kripke) is the basic normal modal logic whose axioms are all Boolean tautologies and all expressions of the form $\square(p \rightarrow q) \rightarrow (\square p \rightarrow \square q)$ and whose rules are modus ponens and necessitation. The diamond $\diamondsuit$ as usual means the dual $\neg\square\neg$ of $\square$.

Recall that a relational semantics for the system K is based on the notion of a *Kripke frame*, that is, a pair (X,R) where X is nonempty set ("of possible worlds") and R is a binary relation on X ("accessibility relation"). A *valuation* is a function f assigning to each propositional letter p a subset $f(p)$ of X ("the set of worlds in which p is true"). The valuation is then extended to all formulae via the obvious definitions for Boolean connectives, together with $x \in f(\diamondsuit p)$ iff $\exists y \in X$ such that xRy and $y \in f(p)$. A formula p is *valid* in (X,R) iff $f(p) = X$. For detailed exposition of modal logic we refer the reader to the comprehensive textbook [10] or to any other source on modal logic. Relational semantics for the system wK4 is based on the notion of Kripke frame with a weak-transitive accessibility relation. This terminology was inspired by the following "discussion": "This is continuation of the discussion initiated in the papers XXIV 185(1, 2). In spite of disagreements on the way, the polemic ends with all parties agreeing that notion of weak-transitivity of a relation R, characterized by $x \neq y \ \& \ xRy \ \& \ yRz \Rightarrow xRz$ must be distinguished from that of strong transitivity, characterized by $xRy \ \& \ yRz \Rightarrow xRz$" [11].

The reason for our favoring the system wK4 and weak-transitivity as follows.

Proposition 1 (Esakia [12,20]). (a) *Relational completeness of* wK4:
wK4 $\vdash p$ *iff* p *is valid in every weak-transitive Kripke frame*:
(b) *Topological completeness of* wK4 : wK4 $\vdash p$ *iff* p *is valid in every topological space*; *in other word*s, wK4 *is the Logic of topological spaces* (*under reading the diamond-modality* $\diamondsuit$ *as the derivative operation*).

Let X be a topological space. Every topological valuation, i.e., function mapping propositional letters to subsets of X, can be extended to a map assigning subsets of X to all formulae by the inductive definition: $f(p \wedge q) = f(p) \cap f(q)$, $f(p \vee q) = f(p) \cup f(q)$, $f(\neg p) = X - f(p)$ and $f(\Diamond p) = \mathbf{d}f(p)$, where $\mathbf{d}$ is the derivative of the topological space X. A formula p is valid in a space X if $f(p) = X$ for every valuation f. The implications of this proposition are obvious. From the point of view of the axiomatic foundation of topology, Proposition 1(b) shows that the system of postulates for derivative algebras (or for topological spaces in terms of derivative) has a certain completeness property: every "topological" equation, which is identically true in all topological spaces, can be derived from these postulates. (see Sections 8.1 and 8.3).

For the system K4, we need to impose some restriction on topological spaces. Recall that X is said to be a T_d-space if every singleton subset of X is the intersection of an open and a closed subset. This separation axiom, introduced by Aull and Thron [3], proved to play important rôle in the context of lattice-equivalence of topological spaces.

Recall that the system K4 is obtained by adding $\Box p \rightarrow \Box\Box p$ to K as a new axiom schema. It is appropriate to mention here that axioms and necessitation rule of the system K4 are modal simulations of the Hilbert–Bernays derivability conditions; they are called derivability conditions because they are formalizations of the proof predicate Bew(.). In short, K4 axiomatized those properties of Bew(.) that do not depend on the Gödel's diagonal lemma.

Proposition 2 (Esakia [12,20]). *Topological completeness of* K4: K4 $\vdash p$ *iff* p *is valid in every* T_d-*space.*

We now go back a little in time to look at early result of Tarski, who soon began to exert the decisive influence in a semantics of the intuitionistic logic.

First, we recall Tarski's topological interpretation of intuitionistic logic; we assign an open subset $f(p)$ of a topological space X to each propositional letter p. The valuation is extended inductively to all formulae by definition: $f(\bot) = \emptyset$, $f(p \wedge q) = f(p) \cap f(q)$, $f(p \vee q) = f(p) \cup f(q)$, $f(p \rightarrow q) = X - \mathbf{c}(f(p) - f(q))$. A formula p is valid in a topological space X if for every valuation f, $f(p) = X$.

Proposition 3 (Tarski [43]). *Topological completeness of* HC:
(a) HC $\vdash p$ *iff* p *is valid in every topological spaces* (*the intuitionistic connectives interpreted by the operations of the lattice of open sets*);
(b) HC $\vdash p$ *iff* p *is valid in Euclidean space* (*of any number of dimensions*).

Proposition 4 (McKinsey and Tarski [35]). *Topological completeness of* S4:
(a) S4 $\vdash p$ *iff* p *is valid in every topological spaces* (*under reading of the diamond-modality* $\Diamond$ *as closure operator* $\mathbf{c}$),
(b) S4 $\vdash p$ *iff* p *is valid in Euclidean space.*

4. Classical connection

$$\text{HC} \xrightarrow{\text{Tr}} \text{S4} \xrightarrow{\text{Sp}} \text{wK4},$$

where "Tr" is the Gödel translation and "Sp" is the splitting map; namely, Sp is a mapping of the set of the modal formulae into itself, which commutes with Boolean connectives and $\text{Sp}(\Diamond p) = p \vee \Diamond p$, $\text{Sp}(\Box p) = p \wedge \Box p$ (Compare with the topological connection: $\mathbf{c}A = A \cup \mathbf{d}A$). Gödel [23] observed that there is a theorem-preserving translation of Heyting's intuitionistic Calculus HC into the modal system S4. He "presumed" further that the translation is deducibility-invariant, i.e. that a formula is an HC-theorem precisely when its translate is an S4-theorem. This was later verified by McKinsey and Tarski [34], which showed among other things that the modal translation Tr proposed by Gödel does indeed give rise to an interpretation of the Heyting Calculus in the Lewis system S4. Namely, one has

Proposition 5 (Gödel [23], McKinsey and Tarski [34]). *HC* $\vdash p$ *iff* *S4* $\vdash \text{Tr}(p)$.

It is not hard to verify that (see Section 8.3).

Proposition 6 (Esakia [20]). *S4* $\vdash p$ *iff* *wK4* $\vdash \text{Sp}(p)$.

5. Modern connection

5.1. Provability interpretation

$$\text{HC} \xrightarrow{\text{Tr}} \text{Grz} \xrightarrow{\text{Sp}} \text{GL}.$$

In 1967 Grzegorczyk [25] axiomatically defines a modal system Grz (named after him), which is a proper normal extension of the system S4 and proves that HC could be embedded (via the Gödel translation) in the system Grz. Grz is the system that results when the schema $\Box(\Box(p \rightarrow \Box p) \rightarrow p) \rightarrow p$ is added to the modal system S4.

Proposition 7 (Grzegorczyk [25]). *HC* $\vdash p$ *iff* *Grz* $\vdash \text{Tr}(p)$.

The Gödel–Löb Modal system GL (alias, the provability logic) is of interest since it adequately reflects the behavior of formalized Provability Predicate in Peano Arithmetic PA. GL is the result of adding the axiom schema $\Box(\Box p \rightarrow p) \rightarrow \Box p$ to K4. In 1976 Solovay define an arithmetic realization of modal formulae of the system GL (by reading the box-modality $\Box$ as "it is provable in PA" and the diamond-modality $\Diamond$ as "is consistent with PA") and proves its arithmetical completeness. Using more technical terminology, we say that an arithmetic realization $(\_)^*$ of modal formulae is an assignment to each atom p an arithmetic sentence p^* which commutes with non-modal connectives and $(\Box p)^* = \text{Bew}(\ulcorner p^* \urcorner)$ where $\text{Bew}(.)$ is the standard provability predicate for the PA.

Proposition 8 (Solovay [42]). *Arithmetical completeness of* GL:
  GL $\vdash p$ *iff under all arithmetical realizations* $(\_)^*$ *the sentence* p^* *is provable in* PA.

Proposition 9 (Kuznetsov and Muravitsky [28], Goldblatt [24], Boolos [8]). G<small>RZ</small>$\vdash p$ *iff* GL $\vdash$ Sp(p).

We recall that a sentence of PA is *demonstrable* if it is provable and true. Since every provable sentence is true, the distinction between provability and demonstrability is one in "intension" only, but Löb's Theorem shows that Bew(s) $\wedge s$ (the arithmetization of the assertion that s is demonstrable) is equivalent to Bew(s) (the arithmetization of the assertion that s is provable) only if s is actually provable. Let us abbreviate Bew(s) $\wedge s$ as "Dem(s)"; under reading of the box-modality $\square$ as Dem, we have as a by-product of Propositions 8 and 9 the following:

Proposition 10. *Arithmetical completeness of* G<small>RZ</small>:
  G<small>RZ</small>$\vdash p$ *iff* PA $\vdash p^*$.

Thus the formulae of whose provability-true translations are theorems of PA are precisely the theorems of G<small>RZ</small>!
  It should be noted that as a by-product of modern connection we obtain (using the composition Tr, Sp and the Solovay arithmetic interpretation $(\_)^*$) an adequate provability interpretation of HC!
  The following observation was made by Segerberg:

Proposition 11 (Segerberg [39]). *Relational completeness of* GL:
  The theorems of GL *are precisely formulae valid in all Kripke frames in which accessibility relation is transitive and well-founded.*

Proposition 12 (Esakia [13,14]). G<small>RZ</small> *is the largest modal system in which* HC *can be embedded by the Gödel translation* Tr.

Proposition 13 (Blok [6], Esakia [13]).

$$\text{Lat(HC)} \cong \text{Lat(G\textsc{rz})},$$

i.e. the lattice Lat(HC) *of all intermediate logics is isomorphic to the lattice* Lat(G<small>RZ</small>) *of all normal extensions of the system* G<small>RZ</small>.

It is appropriate to mention here system S4.1 first defined in McKinsey [32]. S4.1 is the modal system obtained by adding $\square \diamond p \rightarrow \diamond \square p$ to S4 as a new axiom schema. McKinsey writes [32, p. 83]: "As the intuitive basis for the syntactical definition of possibility, I take the position that to say a sentence is possible means that there exists a true sentence of the same form. Thus, for example, it would be said that the sentence 'Lions are indigenous to Alaska' is possible because of the fact that the sentence 'Lions are indigenous to Africa', has the same form and true".

We say that a reflexive and transitive relation R on X has the McKinsey property if every point has an alternative which has itself as only alternative, i.e.

$$\forall x \exists y (xRy \,\&\, \forall z (yRz \Rightarrow z = y)).$$

Lemmon and Scott [30] (see also [38]) showed that S4.1 was determined by the class of quasi-orders (X, R) in which R has the McKinsey property. What makes S4.1 of interest is that the system S4.1 (just as S4) is a modal "companion" of intuitionistic logic [15], i.e.

$$\text{HC} \vdash p \text{ iff } \text{S4.1} \vdash \text{Tr}(p).$$

As we noted above, the system K4 is evidently sound with respect to arithmetic semantics, but it is not complete. An important example of a formula unprovable in K4 is (G) $\neg\Box\bot \rightarrow \neg\Box\neg\Box\bot$ (a modal version of the famous Gödel's second incompleteness theorem). Denote by K4.G the modal system obtained by adding (G) to K4 as a new axiom. In the interest of historic completeness we also present the following "intermediate" connection:

$$\text{HC} \xrightarrow{\text{Tr}} \text{S4.1} \xrightarrow{\text{Sp}} \text{wK4.G}.$$

Same relevant comments concerning this connection are relegated to Section 8.4.

We conclude this topic with a calculus of sequents for the modal system S4.1 which was discovered by Grigori Mints (private communication):

Definition 2 (Mints). S4.1 is the calculus of sequents that results when the following "enigmatic" rule (MR) is added to the calculus of sequents for S4:

$$\text{(MR)} \qquad \frac{\Gamma^- \rightarrow}{\Box\Gamma \rightarrow},$$

where $\Gamma^- = \{p^- : p \in \Gamma\}$ and p^- is the result of removing from p all occurrences of the modal operator $\Box$. As usual $\Box\Gamma = \{\Box p : p \in \Gamma\}$.

Proposition 14 (Mints). S4.1 *admits cut elimination.*

6. From a topological point of view

Definition 3 (Cantor). A topological space is called a *scattered* space if it has no dense-in-itself non-empty subset.

An "equational" (a lá Kuratowski) characterization of scattered spaces using the derivation operation as a primitive notion is contained in

Proposition 15 (Esakia [16]). *A scattered space is a set X equipped with an operator* **d**, *satisfying the equations*: (1) $\mathbf{d}\perp = \perp$, (2) $\mathbf{d}(A\cup B) = \mathbf{d}A\cup\mathbf{d}B$ *and* (3) $\mathbf{d}A = \mathbf{d}(A - \mathbf{d}A)$ (*the dual form of the Löb formula*).

(See Addendum, Section 8.5)

Corollary 1 (Esakia [16] (see also, Esakia [17] and Bernardi and Aquino [4])). *Topological completeness of* GL:
    GL $\vdash p$ *iff* p *is valid in every scattered space.*

Let $\Gamma(\alpha)$ denote the space of ordinals not exceeding α with its interval topology. It is known that for every ordinal α $\Gamma(\alpha)$ is a scattered space [31].

Proposition 16 (Abashidze [1], Blass [5]). GL $\vdash p$ *iff* p *is valid in* $\Gamma(\omega^\omega)$.

Proposition 17 (Abashidze and Esakia [2]). *The Heyting propositional calculus* HC, *the system* GRZ *and the Provability Logic* GL *admit an adequate topological interpretation in terms of the ordinal space* $\Gamma(\omega^\omega)$.

Note that from the point of view of the axiomatic foundation of topology, Proposition 15 and Corollary 1 show that the axioms for scattered space in terms of derivative have a certain completeness property. Propositions 16 and 17 imply that if a "topological" equation fails in some scattered space (or even in some diagonalizable algebra) we can be sure of finding a counter-example for it in a suitable ordinal space (or even in the ordinal space of the form $\Gamma(\omega^n)$ for suitable n)
    One more bit of notation.

Definition 4 (Hausdorff). If $f(A) = \mathbf{c}A - A$ is the boundary of the complement of a subset A of a topological space X, then the set $f f(A) = \text{Res}(A)$ is called the *residue* of A. Otherwise, $\text{Res}(A) = A\cap\mathbf{c}(\mathbf{c}A - A)$.

We introduce the following.

Definition 5 (Esakia [16] (see also [17])). *A topological space X is called H-reducible if every subset A of X, $A\neq\emptyset$ implies $A - \text{Res}A\neq\emptyset$.*

An "equation" characterization of this class of spaces in terms of the residue can be found in [16]: a space X is H-reducible iff $\mathbf{c}A = \mathbf{c}(A - \text{Res}A)$ for each $A\subseteq X$.
    It is not hard to verify that every scattered space is H-reducible. An example of a scattered space which is not H-reducible has been constructed in [7]; in that paper there has been also shown that scatteredness and H-reducibility coincide on a wide class of spaces, including all of the spectral, first countable, or locally compact Hausdorff spaces (see Corollary 4.7 there).

Proposition 18 (Esakia [16]). *Topological completeness of* GRZ:

GRZ$\vdash p$ *iff p is valid in every H-reducible topological space* (*under reading the diamond* $\Diamond$ *as closure operation* **c**).

Proposition 19 (Gabelaia [22]). *Topological completeness of* S4.1:

S4.1$\vdash p$ *iff p is valid in every topological space X in which the boundary of any subset A of X is nowhere dense*.

Definition 6 (Esakia [19]). A Heyting algebra H is called a *frontal* Heyting algebra if for every element a of H the filter $F_a = \{b \in H : b \to a \leqslant b\}$ is a principal filter.

Example. The lattice Op(X) of all opens of a scattered space is a frontal Heyting algebra [19].

Using a significant result of Kuznetsov [29] and [36] we obtain:

Proposition 20 (Esakia [19]). *Every variety of Heyting algebras is generated by the class of its frontal Heyting algebras*.

In 1976 Segerberg explicitly formulated the logic of elsewhere. Segerberg [40]: says "George Henrik von Wright seems to have been the first to notice the fact, overlooked by everyone else, that 'somewhere else' has an interesting logic of its own, distinct from that of S5-operator 'somewhere' ". Recently this logic (we shall call its KS, after Krister Segerberg) has received a good deal of attention (Goranko, Venema, de Rijke; see, for example, [37]).

In our notation the logic KS can be described as follows: KS $=$ wK4 $+ (p \to \Box \Diamond p)$. The intended kind of Kripke frame (X, R) is one in which R is the "dissident" relation, i.e. xRy iff $x \neq y$. Note that the dissident relation $\neq$ is a simple example of a relation, which is weak-transitive (and even symmetric) but is not transitive!

Segerberg presented a completeness result.

Proposition 21 (Segerberg [41]). *The Logic* KS *may be identified with the set of formulae valid in all frames of this kind*.

In an algebraic context we obtain weak Monadic algebras, i.e. Derivative algebras $(B, \mathbf{d})$, in which the operator $\mathbf{d}$ satisfies the additional equality

$$a \wedge \mathbf{d} - \mathbf{d}a = \bot.$$

It must be pointed out that every Monadic algebra is a weak Monadic algebra satisfying the condition $a \leqslant \mathbf{d}a$. The Monadic algebras were introduced by Halmos [26] in his study of algebraic logic. The equational class of weak Monadic algebras (as well as the class of Monadic algebras) is semisimple and locally finite [20].

The alternative topological semantics for this logic is based on the notion of anti-discrete topology:

Proposition 22 (Esakia [12,20]). *The Logic* KS *may be identified with the set of formulae valid in all spaces with the antidiscrete topology* (*under reading the diamond-modality* $\diamond$ *as a derivative* **d**).

7. A quantifier extension of the Heyting calculus

We now present (and try to justify) an amendment to the standard quantifier extension QHC of the Heyting propositional calculus HC. Namely, our amended calculus Q^+HC is obtained from the usual QHC by postulating the following modified version of the rule of universal generalization:

$$(+) \qquad \frac{\vdash (P(a) \to \forall x P(x)) \to P(a)}{\vdash \forall x P(x)}.$$

This amendment was inspired by the provability interpretation of the intuitionistic predicate logic (via the Gödel modal translation and Solovay's arithmetical completeness Theorem).

Proposition 23 (Esakia [18]). *The Amended Intuitionistic Predicate Logic* Q^+HC *admits a Provability interpretation in Peano arithmetic PA.*

An alternative definition of Q^+HC is expressed by

Proposition 24 (Esakia [18]). *The calculus* Q^+HC *is equivalent to the calculus obtained from the usual QHC by accepting as an additional axiom the formula* (*Casari's schema*):

$$(\text{Cas}) \qquad \forall x[(P(x) \to \forall x P(x)) \to \forall x P(x)] \to \forall x P(x).$$

Recall the remark of Heyting [27, p. 104] in connection with the formula $\neg\neg\forall x P(x) \to \forall x \neg\neg P(x)$: "It is one of the most striking features of intuitionistic logic that the inverse implication does not hold, especially because the formula of the propositional calculus which results if we restrict x to a finite set, is true". And further: "It has been conjectured [S. Kuroda 1951, p.46] that the formula $\forall x \neg\neg P(x) \to \neg\neg\forall x P(x)$ is always true if x ranges over a denumerable infinite species, but no way of proving the conjecture present itself at present".

It is not hard to verify that $\forall x \neg\neg P(x) \to \neg\neg\forall x P(x)$ (and consequently also the biconditional $\forall x \neg\neg P(x) \leftrightarrow \neg\neg\forall x P(x)$) is provable in Q^+HC.

Let us note some semantical features of the amended calculus Q^+HC [18,21]:

(1) Quantifier models of Q^+HC include all Kripke models with well-founded base (X, R) and, hence, all finitary, i.e. the ones with finite base (X, R).
(2) Of the sheaf models, the logic Q^+HC admits the sheaf toposes only over scattered spaces and hence also over ordinal spaces.

8. Addendum

During his visit to Tbilisi, Tarski proposed to us to investigate the derivative algebras and the corresponding modal system and prove its topological completeness. This topic (presented in the Appendix to [33]) opened more questions that it has answered. I found it appropriate to include in the paper results that have been obtained by us in this direction. These results are directly related to the Tarski "topological" problematics and complement the aforementioned Appendix.

It is desirable to exchange some economy for added clarity. We conclude this paper with a few comments mainly concerning with topological and relational semantics for the new modal systems wK4, K4.G and the Gödel–Löb system GL.

8.1. Relational semantics of wK4

Let (X, R) be a Kripke frame, i.e. X is a non-empty set and R is an arbitrary binary relation on X. It can easily be shown that

(a) the frame (X, R) is a wK4-frame (i.e. all axioms of wK4 are valid in (X, R)) iff the relation R is weak-transitive;

(b) the canonical (alias, descriptive) frame of the system wK4 is weak-transitive.

We shall now define (by analogy with Russell) *ancestral* relation R^*.

Definition. $xR^*y \Leftrightarrow xRy$ or ($x \neq y$ and there exist points $x_1, x_2, \ldots, x_n$ of X such that $xRx_1Rx_2 \ldots x_nRy$).

(c) The relation R^* is the weak-transitive closure of R, i.e. R^* is the least weak-transitive relation containing R.

Using the filtration method (mutatis mutandis) it is not hard to verify that the weak-transitive closure of the least filtration of a weak-transitive model X through any set of formulae S is itself a filtration of X through S (cf., [38, Lemma 2]). From these remarks the finite model property of wK4 follows at once: the system wK4 is characterized by the class of all finite weak-transitive frames.

8.2. Topological semantics of wK4

In every topological space X the derived set operation satisfies conditions: $\mathbf{d}(\bot) = \bot$, $\mathbf{d}(A \cup B) = \mathbf{d}A \cup \mathbf{d}B$, which follows directly from the definition of a derivative operator. It is not difficult to verify that $\mathbf{dd}A \subseteq A \cup \mathbf{d}A$ for any $A \subseteq X$. Indeed, suppose $x \in \mathbf{dd}A$. Then for every neighborhood U_x of x there exists $y \neq x$ such that $y \in U_x \cap \mathbf{d}A$. Because of $y \in \mathbf{d}A$, for every neighborhood of y and in particular for U_x we have $U_x \cap \mathbf{d}(A - y) \neq \emptyset$ and all the more $U_x \cap \mathbf{d}A \neq \emptyset$. Since every neighborhood U_x of x contains points of A, it follows that $x \in \mathbf{c}A = A \cup \mathbf{d}A$. Note that we have no reason for the conclusion $x \in \mathbf{d}A$, as $Ux \cap (A - x) = \emptyset$ is possible (for example, if X is a set endowed with the antidiscrete topology). Thus, all axioms of wK4 are valid in every topological space.

It is known that there is a one-to-one correspondence between topologies on a finite set X and quasiordering relations on X. Every topology $(X, \mathbf{c})$ determines a quasiorder

Q by xQy iff $x \in \mathbf{c}(\{x\})$. Conversely, each quasiorder Q defines a topology: for every set $A \subseteq X$ $\mathbf{c}A = Q^{-1}A$, where $Q^{-1}A = \bigcup\{Q^{-1}(x): x \in A\}$ and $Q^{-1}(x) = \{y: yQx\}$.

We need the following more esoteric variant of this correspondence which seems interesting in itself [12,20].

Observation. *There is a one-to-one correspondence between topologies on a finite set X and irreflexive, weak-transitive relations on X.*

We need to modify the correspondence presented above slightly. Namely, each irreflexive, weak-transitive relation R on X defines a topology: the reflexive closure Q of R (i.e. $xQy \Leftrightarrow xRy$ or $x = y$) is a quasiorder on X. Therefore, $(X, \mathbf{c})$ is a topological space where as above $\mathbf{c}A = Q^{-1}A$, besides the operator $\mathbf{d}A = R^{-1}A$ coincide with the derived set of A with respect to this topology. Conversely, with a topological space $(X, \mathbf{c})$ we associate a relation R defined by $xRy \Leftrightarrow x \in \mathbf{d}(y)$, where $\mathbf{d}$ is the derivative operator of the space. It is easy to verify that R is a weak-transitive and irreflexive (as $x \notin \mathbf{d}(x)$) relation. It follows from the above considerations that only irreflexive and weak-transitive Kripke frames (X, R) "give" topologies. However from a modal and/or algebraic point of view we can restrict ourselves to this special "irreflexive" frames. Indeed, the following observation implies that a formula which is valid in every irreflexive weak-transitive finite frame is also valid in every weak-transitive finite frame.

Let (X, R) be a weak-transitive finite frame and $f : Y \to X$ be a map such that:

(a) $|f^{-1}(x)| = 2$ if x is a reflexive point of X, i.e. xRx.
(b) $|f^{-1}(x)| = 1$ otherwise.

We define a relation S on Y the following way: for all $y, z \in Y$

(a′) $ySz \Leftrightarrow y \neq z$ if $y, z \in f^{-1}(x)$ and x is a reflexive point of X,
(b′) $ySz \Leftrightarrow f(y)Rf(z)$ in all other cases.

It is easily seen that (1) (Y, S) is an irreflexive weak-transitive finite frame and (2) the map f satisfies the condition $fS(x) = Rf(x)$ and therefore f is a p-morphism of the frame (Y, S) onto (X, R). From the above considerations we obtain a refined version of the finite approximability of wK4: the modal system wK4 is characterized by all irreflexive weak-transitive finite frames. As a by-product we have topological completeness of wK4 (see, [20]).

8.3. The connection between wK4 and S4

It is easy to see that the splitting of the axioms of S4 is provable in wK4. It should be mentioned that $\text{Sp}(\Box p \to \Box\Box p)$, the splitting of the axiom $\Box p \to \Box\Box p$, is equivalent to the axiom $p \wedge \Box p \to \Box\Box p$ of wK4, i.e. $\text{wK4} \vdash (p \wedge \Box p \to \Box\Box p) \leftrightarrow \text{Sp}(\Box p \to \Box\Box p)$. Thus, $\text{S4} \vdash p$ only if $\text{wK4} \vdash \text{Sp}(p)$. To show the converse implication, we need simple, but useful observation, which already opened the door for a semantical connection between wK4 and S4:

A relation R on a set X is weak-transitive iff the reflexive closure Q of R is a quasiorder on X.

Using this fact and the well-known theorem about completeness of S4 with respect to quasiordered sets we obtain: $S4 \vdash p$ if and only if $wK4 \vdash Sp(p)$. Moreover, the modal system wK4 is the least normal extension of the basic modal system K for which the Sp-translation is true.

8.4. The connection between K4.G and S4.1

Let $(B, \mathbf{C})$ be an arbitrary closure algebra and define $B' = \{a \in B : \mathbf{ICa} \leqslant \mathbf{CIa}\}$, where as usual $\mathbf{Ia} = -\mathbf{C} - a$.

Key observation. *B' is a Boolean subalgebra of B closed under the closure operator $\mathbf{C}$ and therefore the closure algebra $(B', \mathbf{C})$ satisfies the McKinsey condition $\mathbf{ICa} \leqslant \mathbf{CIa}$ for every $a \in B'$. Note that every open element a of B (i.e. $\mathbf{Ia} = a$) belongs to B'.*

It is known that the algebra $H = \{\mathbf{Ia}: a \in B\}$ of all open elements of an arbitrary Closure algebra $(B, \mathbf{C})$ is a Heyting algebra. Moreover it is not hard to show that every Heyting algebra is isomorphic to the algebra of all open elements of a Closure algebra $(B, \mathbf{C})$ which satisfies the McKinsey property: $\mathbf{ICa} \leqslant \mathbf{CIa}$ for all a in B [17]. The remarks made above can be applied to prove the embedding

$$HC \xrightarrow{Tr} S4.1.$$

A comment on the translation: $S4.1 \xrightarrow{Sp} K4.G$.

We must prove that $S4.1 \vdash p$ iff $K4.G \vdash Sp(p)$.

(A) ($\Rightarrow$). We already know that the splitting of all axioms of S4 (see Section 8.3) are derivable in wK4 (and the more, in K4). Therefore, it is only necessary to verify that the splitting $Sp(\square \lozenge p \to \lozenge \square p)$ of the McKinsey axiom $\square \lozenge p \to \lozenge \square p$ is derivable in K4.G. With the algebraic nomenclature at hand the logical terms can be easily translated into algebraic language. Let $(B, \mathbf{d})$ be a derivative algebra such that the operator $\mathbf{d}$ satisfies the following conditions: $\mathbf{dda} \leqslant \mathbf{da}$ (K4-axiom) and $\mathbf{d}\top \leqslant \mathbf{d}(-\mathbf{d}\top)$ (an algebraic version of (G)), where $\top$ is the largest element of the Boolean algebra B. Note that $-\mathbf{d}\top = (a \wedge -\mathbf{d}\top) \vee (-a \wedge -\mathbf{d}\top)$ and hence $\mathbf{d}\top \leqslant \mathbf{d} - \mathbf{d}\top$ iff $-\mathbf{d}\top \vee \mathbf{d} - \mathbf{d}\top = \top$ iff $\mathbf{C}(-\mathbf{d}\top) = \top$ iff (1) $\mathbf{C}((a \wedge -\mathbf{d}\top) \vee (-a \wedge -\mathbf{d}\top)) = \top$. Using the definitions $\mathbf{C}a = a \vee \mathbf{d}a$ and $\mathbf{Ia} = a \wedge -\mathbf{d} - a$ we obtain: (2) $\mathbf{ICa} \leqslant \mathbf{CIa}$ is equivalent to $\mathbf{C}((a \wedge -\mathbf{d} - a) \vee (-a \wedge -\mathbf{da})) = \top$. As $-\mathbf{d}\top \leqslant -\mathbf{da}$ and $-\mathbf{d}\top \leqslant -\mathbf{d} - a$ for any $a \in B$ we have $(a \wedge -\mathbf{d}\top) \vee (-a \wedge -\mathbf{d}\top) \leqslant (a \wedge -\mathbf{da}) \vee (-a \wedge -\mathbf{d} - a)$ and hence (3) $\mathbf{C}((a \wedge -\mathbf{d}\top) \vee (-a \wedge -\mathbf{d}\top)) \leqslant \mathbf{C}((a \wedge -\mathbf{da}) \vee (-a \wedge -\mathbf{d} - a))$. The equation (1) together with (3) implies $\mathbf{C}((a \wedge -\mathbf{da}) \vee (-a \wedge -\mathbf{d} - a)) = \top$, i.e. $\mathbf{ICa} \leqslant \mathbf{CIa}$. Thus we have $S4.1 \vdash p$ only if $K4.G \vdash Sp(p)$.

(B) ($\Leftarrow$). Let (X, R) be a finite transitive frame. It is not hard to verify that the following conditions are equivalent:

(a) the axiom (G) $\neg \square \bot \to \neg \square \neg \square \bot$ is valid in the Kripke frame (X, R),
(b) the relation R has the property:

$$\forall x[\exists x(xRy) \Rightarrow \exists y(xRy \ \& \ \neg \exists z(yRz))].$$

Supposing now that a formula p is not derivable in S4.1. As system S4.1 has the finite frame property (see [38]) there exist a finite quasiorder with the McKinsey property, say (X, Q), such that p is not valid in (X, Q). Denote by Max$X$ the set of all maximal points of (X, Q) (as usually a point $x \in X$ is a maximal point iff $\forall y(xQy \Rightarrow x = y)$). We define a new relation R on X in the following way:

(1) if $x \neq y$ then $xRy \Leftrightarrow xQy$ and (2) if $x = y$ then $xRx \Leftrightarrow xQx$ and $x \notin \text{Max}X$. It is then easily seen that Q is the reflexive closure of R and furthermore R has the property (b). Hence the formula Sp(p) is not valid in (X, R). Thus, we obtain by contraposition K4.G $\vdash$ Sp(p) $\Rightarrow$ S4.1 $\vdash p$ and therefore S4.1 $\vdash p \Leftrightarrow$ K4.G $\vdash$ Sp(p).

8.5. The Löb rule and Cantor's "scatteredness"

It is possible to prove in an algebraic way, avoiding any "point" reasoning the following slightly more general proposition, from which our desired Proposition 15 results immediately. Namely, Proposition 15 could be strengthened to assert

Proposition 15. *For every Derivative algebra* $(B, \mathbf{d})$ *the following conditions are equivalent*:

(A) *The equation* $\mathbf{d}a = \mathbf{d}(a - \mathbf{d}a)$ *(the dual form of the Löb Principle) is identically valid in* $(B, \mathbf{d})$

(B) $(B, \mathbf{d})$ *satisfies the dual form of the Löb Rule, i.e.*
    for all $a \in B$ $a \leqslant \mathbf{d}a$ *only if* $a = \bot$.

Proof. $(A \Rightarrow B)$. Suppose that $a \leqslant \mathbf{d}a$, i.e. $a - \mathbf{d}a = \bot$; then from (A) we obtain $\mathbf{d}a = \mathbf{d}\bot = \bot$. As $a \leqslant \mathbf{d}a$, $a = \bot$. To obtain the implication $(A \Leftarrow B)$ we need the following lemma.

Lemma. *Every derivative algebra* $(B, \mathbf{d})$ *satisfies the following condition*: $\mathbf{d}a - \mathbf{d}(a - \mathbf{d}a) \leqslant \mathbf{d}[\mathbf{d}a - \mathbf{d}(a - \mathbf{d}a)]$ *for every* $a \in B$.

Proof. Using axiom (2) of Definition 1, we obtain:

(a) $\mathbf{d}a = \mathbf{d}(a \wedge \mathbf{d}a) \vee (\mathbf{d}(a - \mathbf{d}a))$. Multiplying this equation throughout by $-\mathbf{d}(a - \mathbf{d}a)$ we have:

(b) $\mathbf{d}a - \mathbf{d}(a - \mathbf{d}a) = \mathbf{d}(a \wedge \mathbf{d}a) - \mathbf{d}(a - \mathbf{d}a)$. Replacing a by $a - \mathbf{d}a$ in the contraposition $-a - \mathbf{d}a \leqslant -\mathbf{dd}a$ of axiom (3^*) of Definition 1, we obtain

(c) $-(a - \mathbf{d}a) - \mathbf{d}(a - \mathbf{d}a) \leqslant -\mathbf{dd}(a - \mathbf{d}a)$. Using $\mathbf{d}a \leqslant -a \vee \mathbf{d}a \leqslant -(a - \mathbf{d}a)$ (c) gives

(d) $\mathbf{d}a - \mathbf{d}(a - \mathbf{d}a) \leqslant -\mathbf{dd}(a - \mathbf{d}a)$; multiplying throughout by $\mathbf{d}(a \wedge \mathbf{d}a)$ we have

(e) $\mathbf{d}(a \wedge \mathbf{d}a) \wedge \mathbf{d}a \wedge -\mathbf{d}(a - \mathbf{d}a) \leqslant \mathbf{d}(a \wedge \mathbf{d}a) - \mathbf{dd}(a - \mathbf{d}a)$. Using $\mathbf{d}(a \wedge \mathbf{d}a) \leqslant \mathbf{d}a$, (e) gives

(f) $\mathbf{d}(a \wedge \mathbf{d}a) - \mathbf{d}(a - \mathbf{d}a) \leqslant \mathbf{d}(a \wedge \mathbf{d}a) - \mathbf{dd}(a - \mathbf{d}a)$; using (b) we obtain

(g) $\mathbf{d}a - \mathbf{d}(a - \mathbf{d}a) \leqslant \mathbf{d}(a \wedge \mathbf{d}a) - \mathbf{dd}(a - \mathbf{d}a)$; using $\mathbf{d}(a \wedge \mathbf{d}a) \leqslant \mathbf{dd}a$ and transitivity of $"\leqslant"$, from the (g) we obtain

(h) $\mathbf{d}a - \mathbf{d}(a - \mathbf{d}a) \leqslant \mathbf{dd}a - \mathbf{dd}(a - \mathbf{d}a)$; finally, using $\mathbf{d}a - \mathbf{d}b \leqslant \mathbf{d}(a - b)$, we have the condition asserted in the lemma:

$\mathbf{d}a - \mathbf{d}(a - \mathbf{d}a) \leqslant \mathbf{d}[\mathbf{d}a - \mathbf{d}(a - \mathbf{d}a)]$.

We now go back to the implication $(A \Leftarrow B)$. Using (B), the lemma gives $\mathbf{d}a - \mathbf{d}(a - \mathbf{d}a) = \perp$, i.e. $\mathbf{d}a \leqslant \mathbf{d}(a - \mathbf{d}a)$. Thus, $\mathbf{d}a = \mathbf{d}(a - \mathbf{d}a)$. □

Corollary 1. *In every scattered space X the equation $\mathbf{d}A = \mathbf{d}(A - \mathbf{d}A)$ is true for every $A \subseteq X$.*

Recall [3] that a topological space X is a T_d-space iff the "pointless" condition $\mathbf{dd}A \leqslant \mathbf{d}A$ is true for every subset A of X. Using a remarkable fact about GL (the proof of which was independently discovered by de Jongh, Kripke and Sambin), namely, GL $\vdash \Box p \rightarrow \Box\Box p$ (see, for example, [9, Theorem 18, p. 11]) we have

Corollary 2. *Every scattered space is a T_d-space.*

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