J. Math. Anal. Appl. 443 (2016) 707–731

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Mellin convolution operators in Bessel potential spaces ☆

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ARTICLE INFO

Article history:
Received 31 March 2015
Available online 27 May 2016
Submitted by W.L. Wendland

Keywords:
Fourier convolutions
Mellin convolutions
Bessel potential spaces
Operator algebras
Symbol
Fixed singularities

ABSTRACT

Mellin convolution equations acting in Bessel potential spaces are considered. The study is based upon two results. The first one concerns the interaction of Mellin convolutions and Bessel potential operators (BPOs). In contrast to the Fourier convolutions, BPOs and Mellin convolutions do not commute and we derive an explicit formula for the corresponding commutator in the case of Mellin convolutions with meromorphic symbols. These results are used in the lifting of the Mellin convolution operators acting on Bessel potential spaces up to operators on Lebesgue spaces. The operators arising belong to an algebra generated by Mellin and Fourier convolutions acting on $\mathbb{L}_p$-spaces. Fredholm conditions and index formulae for such operators have been obtained earlier by one of the authors and are employed here. The results of the present work have numerous applications in boundary value problems for partial differential equations, in particular, for equations in domains with angular points.

0. Introduction

Boundary value problems for elliptic equations in domains with angular points play an important role in applications and have a rich and exciting history. A prominent representative of this family is the Helmholtz equation. In the classical $\mathbb{W}^1$-setting, the existence and uniqueness of the solution of coercive systems with various types of boundary conditions and various elliptic and even non-linear partial differential operators are easily obtainable by using the celebrated Lax–Milgram Theorem (see, e.g., [8,30] and the recent paper [21] where Laplace–Beltrami equations are considered on smooth surface with Lipschitz boundary). Similar problems arise in new applications in physics, mechanics and engineering. Thus recent publications on nano-photonics [1,25] deal with physical and engineering problems described by BVPs for the

☆ This work was carried out when the second author visited the Universiti Brunei Darussalam (UBD). The support of UBD provided via Grant UBD/GSR/S&T/19 is highly appreciated. The work is also supported by the Georgian National Science Foundation, Contracts No. 13/14 and No. 31/39.

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http://dx.doi.org/10.1016/j.jmaa.2016.05.043

Helmholts equation in $2D$ domains with angular points. They are investigated with the help of a modified Lax–Milgram Lemma for so-called T-coercive operators. Similar problems occur for the Lamé systems in elasticity, Cauchy–Riemann systems, Carleman–Vekua systems in generalized analytic function theory etc.

Despite an impressive number of publications and ever growing interest to such problems, the results available to date are not complete. In particular, serious difficulties arise if information on the solvability in non-classical setting in the Sobolev spaces $\mathbb{W}^1_p$, $1 < p < \infty$ is required, and one wants to study the solvability of equivalent boundary integral equations in the trace spaces $\mathbb{W}^{1-1/p}_p$ on the boundary. Integral equations arising in this case often have fixed singularities in the kernel and are of Mellin convolution type. For example, [6] describes how model BVPs in corners emerge from the localization of BVP for the Helmholtz equation in domains with Lipschitz boundary. Consequently, an attempt to study the corresponding Mellin convolution operators in Bessel potential spaces has been undertaken in [19]. However, the main Theorem 2.7 and Theorem 4.1 (based on Theorem 2.7) are incorrect. The aim of the present work is to provide correct formulations and proofs of Theorem 2.7 and 4.1 from [19]. We also hope that the results of the present paper will be helpful in further studies of boundary value problems for various elliptic equations in Lipschitz domains.

Consider the following BVP with the mixed Dirichlet–Neumann boundary conditions

$$\begin{cases} \Delta u(x) + k^2 u(x) = 0, & x \in \Omega_\alpha, \\ u^+(t) = g(t), & t \in \mathbb{R}^+, \\ (\partial_{\boldsymbol{\nu}} u)^+(t) = h(t), & t \in \mathbb{R}_\alpha \end{cases} \tag{1}$$

in the corner Ω_α of magnitude α,

$$\partial\Omega_\alpha = \mathbb{R}^+ \cup \mathbb{R}_\alpha, \quad \mathbb{R}^+ = (0, \infty),$$
$$\mathbb{R}_\alpha := \{te^{i\alpha} = (t\cos\alpha, t\sin\alpha) \ : \ t \in \mathbb{R}^+\}$$

with a complex wave number $\operatorname{Im} k \neq 0$. In [20] the BVP (1) is reduced to the following equivalent system of boundary integral equations on $\mathbb{R}^+$:

$$\begin{cases} \varphi + \dfrac{1}{2}\left[\boldsymbol{K}^1_{e^{i\alpha}} + \boldsymbol{K}^1_{e^{-i\alpha}}\right]\psi = G_1, \\ \psi - \dfrac{1}{2}\left[\boldsymbol{K}^1_{e^{i\alpha}} + \boldsymbol{K}^1_{e^{-i\alpha}}\right]\varphi = H_1. \end{cases} \tag{2}$$

Here

$$\boldsymbol{K}^1_{e^{\pm i\alpha}}\psi(t) := \frac{1}{\pi}\int_0^\infty \frac{\psi(\tau)d\tau}{t - e^{\pm i\alpha}\tau}, \qquad 0 < |\alpha| < \pi, \tag{3}$$

are Mellin convolution operators with homogeneous kernels of order -1 (see e.g. [16,17] and Section 1 below), also called integral equations with fixed singularities in the kernel. Similar integral operators arise in the theory of singular integral equations with complex conjugation if the contour of integration possesses corner points. A complete theory of such equations was worked out by R. Duduchava and T. Latsabidze, whereas various approximation methods have been investigated in [13]. For a more detailed survey of this theory, applications in elasticity, and numerical methods for the corresponding equations we refer the reader to [16,17,32] and [11,12]. Note that a similar approach has been employed by M. Costabel and E. Stephan [9,10] in order to study boundary integral equations on curves with corner points.

All of the above mentioned investigations consider integral equations in the Lebesgue space $\mathbb{L}_p$ with and without exponential weights. However, the available results are insufficient to fully investigate the BVP (1). On the other hand, if we are looking for solutions of BVP (1) in the classical (finite energy) formulation

$$\begin{gathered} g \in \mathbb{H}^{1/2}(\mathbb{R}^+), \quad h \in \mathbb{H}^{-1/2}(\mathbb{R}_\alpha), \quad u \in \mathbb{H}^1(\Omega_\alpha) = \mathbb{W}^1(\Omega_\alpha), \\ u(x) = \mathcal{O}(1) \quad \text{as} \quad |x| \to \infty, \end{gathered} \tag{4}$$

we have to investigate the corresponding equivalent system of boundary integral equation (2) in the Bessel potential space $\widetilde{\mathbb{H}}^{-1/2}(\mathbb{R}^+)$. There exist several different approaches to the study of BVPs (1) in the classical setting, which in some cases go beyond the Hilbert space setting. We refer to the recent paper [7] for one of approaches and the survey of relevant investigations.

If we consider the BVP (1) in the non-classical formulation

$$\begin{gathered} g \in \mathbb{W}_p^{1-1/p}(\mathbb{R}^+), \quad h \in \mathbb{W}_p^{-1/p}(\mathbb{R}_\alpha), \quad u \in \mathbb{H}_p^1(\Omega_\alpha) = \mathbb{W}_p^1(\Omega_\alpha), \\ u(x) = \mathcal{O}(1) \quad \text{as} \quad |x| \to \infty, \quad 1 < p < \infty, \end{gathered} \tag{5}$$

then the corresponding equivalent system of boundary integral equation (2) has to be studied in the Besov (Sobolev–Slobodeckij) space $\widetilde{\mathbb{W}}_p^{-1/p}(\mathbb{R}^+)$. The non-classical formulation is very helpful to explore the maximal smoothness of a solution to the BVP. This plays an important role in approximation methods and other applications.

While considering equation (2) in the Besov (Sobolev–Slobodeckij) space $\widetilde{\mathbb{W}}_p^s(\mathbb{R}^+)$ one encounters the three major tasks.

- In general, Mellin convolution operators are not bounded in neither Besov nor Bessel potential spaces. Therefore, in order to study equations (2) in the spaces of interest, one has to find a subclass of multipliers with the boundedness property.
- If boundedness criteria for the operators associated with equation (2) are available, one can lift this equation from the Besov or the Bessel potential space to a Lebesgue space.
- The lifted equations should be studied in the Lebesgue space.

Here we offer a complete solution to all the problems mentioned above.

The boundedness of general Mellin pseudodifferential operators was addressed, for example, in [14] and solved by introducing the additional constraints on the space. We choose a different approach and introduce constraints on the kernel of the operator, prompted by application to boundary value problems for elliptic equations. A suitable class of Mellin convolution operators bounded in the Bessel potential spaces is introduced in [19]. These are Mellin convolutions with *admissible meromorphic kernels* (see (20) below).

Having proved the boundedness result, one can study convolution equations in Bessel potential spaces. In particular, by lifting an equation with Mellin convolution operator $\mathfrak{M}_a^0$ with the help of Bessel potential operators $\mathbf{\Lambda}_+^s$ and $\mathbf{\Lambda}_-^{s-r}$, one obtains an equation in $\mathbb{L}_p$-space with the operator $\mathbf{\Lambda}_-^{s-r}\mathfrak{M}_a^0\mathbf{\Lambda}_+^{-s}$. However, the resulting operator $\mathbf{\Lambda}_-^{s-r}\mathfrak{M}_a^0\mathbf{\Lambda}_+^{-s}$ is neither a Mellin nor a Fourier convolution and in order to describe its properties, one first has to study the commutators of Bessel potential operators and Fourier convolutions with discontinuous symbols. As was already mentioned, this problem has been considered in [19], but not all of the results of that work are correct. Therefore, in Section 1 the commutator problem is discussed once again, Theorem 3.3 and Corollary 3.4 below provide correct formulae for the commutators in question.

The lifted operator $\mathbf{\Lambda}_-^{s-r}\mathfrak{M}_a\mathbf{\Lambda}_+^{-s}$ belongs to the Banach algebra generated by Mellin and Fourier convolution operators with discontinuous symbols. Such algebras have been studied before in [18] and the results obtained are systematized and updated in the recent paper [19]. In Section 2, these results are applied to the lifted equation, hereby establishing properties of the initial Mellin convolution equation in the Bessel potential space.

The results of the present paper are applied to BVP (1) in [20] (more correctly [20] is based on corrected results of the paper [19]). These results can also be employed to other elliptic equations on planar Lipschitz domains and also on 2D Lipschitz surfaces. The relevant investigations of R. Duduchava, M. Tsaava and T. Tsutsunava will appear soon.

The advantage of the present approach in comparison with other methods, especially in comparison with the approach based on the Lax–Milgram Theorem and its generalization for so-called T-coercive operators (see [1]), is that it provides better tools to analyze the solvability of the equations involved (provides the solvability criteria) and the asymptotic behavior of their solutions. Moreover, it can also be of use in studying the Schrödinger operator on combinatorial and quantum graphs. Such a problem has attracted a lot of attention recently, since the operator mentioned has a wide range of applications in nano-structures [28,29] and possesses interesting properties. Another area where the results of the present paper can be useful, is the study of Mellin pseudodifferential operators on graphs. This problem has been considered in [31] but in the periodic case only. Moreover, some of the results obtained below play an important role in the theory of approximation methods for Mellin operators in Bessel potential spaces.

The present paper is organized as follows. In the first two sections we define Mellin convolution operators and recall some of their properties. In the second section we also consider Fourier convolution operators in the Bessel potential spaces and discuss the lifting of these operators from the Bessel potential spaces to Lebesgue spaces, mostly according the papers [16,22]. For Mellin convolutions such a lifting operation has not been studied before, and in Section 3 the interaction between Bessel potential operators and the Mellin convolution $\mathbf{K}^1_c$ with the kernel $(t-c\tau)^{-1}$ is considered. In particular, we derive formulae for commutators of Bessel potential operators and Mellin convolutions, and these results are crucial for our further considerations.

Section 4 recalls results from [18,19] concerning the Banach algebra generated by Fourier and Mellin convolution operators in Lebesgue spaces with weight. These results, together with Theorem 3.3 and Corollary 3.4, are used in Section 5 in order to describe the lifting of Mellin convolution operators from the Bessel potential spaces up to operators in Lebesgue spaces. It turns out that the objects arising belong to a Banach algebra generated by Mellin and Fourier convolutions in $\mathbb{L}_p$-space on the semi-axis. The main result here is represented by Theorem 5.1 and Theorem 5.2, where the interaction between Bessel potential operators and the Mellin convolution resulting from the lifting of a model operator $\mathbf{K}^1_c$ is described. Theorem 5.3 deals with the lifting of the operator $\mathbf{K}^2_c$. In conclusion of Section 5, we present explicit formulae for the symbols of Mellin convolution operators with meromorphic kernels, which allow us to find Fredholm criteria and an index formula for the operators under consideration (see Theorem 5.4 and Corollary 5.5).

1. Mellin convolution operators

Equations (2) are a particular case of the Mellin convolution equation

$$\mathfrak{M}^0_a\varphi(t) := c_0\varphi(t) + \frac{c_1}{\pi i}\int_0^\infty \frac{\varphi(\tau)\,dt}{\tau - t} + \int_0^\infty \mathcal{K}\left(\frac{t}{\tau}\right)\varphi(\tau)\frac{d\tau}{\tau} = f(t) \tag{6}$$

where $c_0, c_1 \in \mathbb{C}$. If the kernel $\mathcal{K}$ satisfies the condition

$$\int_0^\infty t^\beta |\mathcal{K}(t)|\frac{dt}{t} < \infty, \quad 0 < \beta < 1,$$

then both equation (6) and analogous equations on the unit interval $I := (0,1)$ considered, respectively, on Lebesgue spaces $\mathbb{L}_p(\mathbb{R}^+)$ and $\mathbb{L}_p(I)$, are fully studied in [16].

Let a be an essentially bounded measurable $N \times N$ matrix function $a \in \mathbb{L}_\infty(\mathbb{R})$, and let $\mathcal{M}_\beta$ and $\mathcal{M}_\beta^{-1}$ denote, respectively, the Mellin transform and its inverse, i.e.

$$\mathcal{M}_\beta\psi(\xi) := \int_0^\infty t^{\beta-i\xi}\psi(t)\,\frac{dt}{t}, \quad \xi \in \mathbb{R},$$

$$\mathcal{M}_\beta^{-1}\varphi(t) := \frac{1}{2\pi}\int_{-\infty}^{\infty} t^{i\xi-\beta}\varphi(\xi)\,d\xi, \quad t \in \mathbb{R}^+.$$

On the Schwartz space $\mathbb{S}(\mathbb{R}^+)$ of the rapidly decaying functions on $\mathbb{R}^+$, consider the equation

$$\mathfrak{M}_a^0\varphi(t) = f(t), \tag{7}$$

where $\mathfrak{M}_a^0$ is the Mellin convolution operator,

$$\begin{aligned}\mathfrak{M}_a^0\varphi(t) &:= \mathcal{M}_\beta^{-1} a \mathcal{M}_\beta\varphi(t)\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} a(\xi)\int_0^\infty \left(\frac{t}{\tau}\right)^{i\xi-\beta}\varphi(\tau)\,\frac{d\tau}{\tau}\,d\xi, \quad \varphi \in \mathbb{S}(\mathbb{R}^+).\end{aligned} \tag{8}$$

Note that equation (6) has the form (7) with the function a defined by

$$a(\xi) := c_0 + c_1\coth[\,\pi(i\beta+\xi)] + (\mathcal{M}_\beta\mathcal{K})(\xi).$$

Equations of the form (6), (7) and similar equations on finite intervals often arise in various areas of mathematics and mechanics (see [16,26]).

The function $a(\xi)$ in (8) is usually referred to as the symbol of the Mellin operator $\mathfrak{M}_a^0$. Further, if the corresponding Mellin convolution operator $\mathfrak{M}_a^0$ is bounded on the weighted Lebesgue space $\mathbb{L}_p(\mathbb{R}^+, t^\gamma)$ endowed with the norm

$$\big\|\varphi \mid \mathbb{L}_p(\mathbb{R}^+, t^\gamma)\big\| := \left[\int_0^\infty t^\gamma|\varphi(t)|^p\,dt\right]^{1/p},$$

then the symbol $a(\xi)$ is called a Mellin $\mathbb{L}_{p,\gamma}$-multiplier.

The two most important examples of Mellin convolution operators are

$$S_{\mathbb{R}^+}\varphi(t) := \frac{1}{\pi i}\int_0^\infty \frac{\varphi(\tau)\,d\tau}{\tau - t}, \qquad \boldsymbol{K}_c^m\varphi(t) := \frac{1}{\pi}\int_0^\infty \frac{\tau^{m-1}\varphi(\tau)\,d\tau}{(t-c\,\tau)^m},$$

where $\operatorname{Im} c \neq 0$ and $m \in \mathbb{N}$ (see (3), (6)). The operator $S_{\mathbb{R}^+}$ is the celebrated Cauchy singular integral operator. The Mellin symbols of these operators are (cf. [19, § 2])

$$\sigma(S_{\mathbb{R}^+})(\xi) := -i\cot[\pi(\beta - i\xi)], \quad \xi \in \mathbb{R},$$

$$\sigma(\boldsymbol{K}_c^m)(\xi) := \binom{\beta - i\xi - 1}{m-1}\frac{e^{\mp\pi(\beta-i\xi)i}}{\sin[\pi(\beta-i\xi)]}\,c^{\beta-i\xi-m}, \quad 0 < \pm\arg c < \pi,$$

where

$$\binom{\theta-1}{m-1} := \frac{(\theta-1)\cdots(\theta-m+1)}{(m-1)!}, \qquad \binom{\theta-1}{0} := 1.$$

In particular,

$$\mathcal{M}_\beta \mathcal{K}^1_{-c}(\xi) = \frac{c^{\beta - i\xi - 1}}{\sin[\pi(\beta - i\xi)]}, \qquad -\pi < \arg c < \pi, \tag{9}$$

$$\mathcal{M}_\beta \mathcal{K}^1_{-1}(\xi) = \frac{1}{\sin[\pi(\beta - i\xi)]}, \qquad \arg c = \pi. \tag{10}$$

The study of the equation (7) does not require much effort. The Mellin transform $\mathcal{M}_\beta$ converts (7) into the equation

$$a(\xi)\mathcal{M}_\beta \varphi(\xi) = \mathcal{M}_\beta f(\xi). \tag{11}$$

If $\inf |\det a(\xi)| > 0$ and the matrix-function a^{-1} is a Mellin $\mathbb{L}_{p,\gamma}$-multiplier, then equation (11) has the unique solution $\varphi = \mathcal{M}^0_{a^{-1}} f$.

The solvability of analogues of equation (8) on the unit interval $I = (0,1)$ in a weighted Lebesgue space $\mathbb{L}_p([0,1], t^\gamma)$ is also well understood. Namely if

$$1 < p < \infty, \qquad -1 < \gamma < p-1, \qquad \beta := \frac{1+\gamma}{p}, \qquad 0 < \beta < 1, \tag{12}$$

then one can use the isomorphisms

$$\begin{aligned} Z_\beta &: \mathbb{L}_p([0,1], t^\gamma) \to \mathbb{L}_p(\mathbb{R}^+), \qquad Z_\beta \varphi(\xi) := e^{-\beta\xi}\varphi(e^{-\xi}), \quad \xi \in \mathbb{R}^+, \\ Z_\beta^{-1} &: \mathbb{L}_p(\mathbb{R}^+) \to \mathbb{L}_p([0,1], t^\gamma), \quad Z_\beta^{-1}\psi(t) := t^{-\beta}\psi(-\ln t), \quad t \in (0,1], \end{aligned} \tag{13}$$

which transform the corresponding equation on the unit interval I into an equivalent Wiener–Hopf equation, i.e. into the equation

$$c_0 \psi(x) + \int_0^\infty \mathcal{K}_1(x-y)\varphi(y)dy = f_0(t). \tag{14}$$

The Fourier transform of the kernel $\mathcal{K}_1$ is called the symbol of the corresponding Fourier convolution operator and is used to describe Fredholm properties, index and solvability of the equation (14). In passing note that Fourier convolution equations with discontinuous symbols are well studied [2,3,16,34].

2. Fourier convolution operators in the Bessel potential spaces: definition and lifting

Let N be a positive integer and let $\mathfrak{A}$ be a Banach algebra. We will write $\mathfrak{A}$ for both scalar and $N \times N$ matrix algebras with entries from $\mathfrak{A}$ if no confusion arises. Similarly, the same notation $\mathfrak{A}$ is used for the set of N-dimensional vectors with entries from $\mathfrak{A}$. It will be usually clear from the context what kind of space or algebra is considered.

Along with Mellin convolutions $\mathfrak{M}^0_a$, let us consider the Fourier convolution operators

$$W^0_a \varphi := \mathcal{F}^{-1} a \mathcal{F}\varphi, \quad \varphi \in \mathbb{S}(\mathbb{R}),$$

where $a \in \mathbb{L}_{\infty,loc}(\mathbb{R})$ is a locally bounded $N \times N$ matrix function, called the symbol of W^0_a and $\mathcal{F}$ and $\mathcal{F}^{-1}$ are, respectively, the direct and inverse Fourier transforms, i.e.

$$\mathcal{F}\varphi(\xi) := \int_{-\infty}^{\infty} e^{i\xi x}\varphi(x)\,dx, \quad \mathcal{F}^{-1}\psi(x) := \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\xi x}\psi(\xi)\,d\xi, \quad x \in \mathbb{R}.$$

Let $1 < p < \infty$. An $N \times N$ matrix symbol $a(\xi)$, $\xi \in \mathbb{R}$ is called $\mathbb{L}_p$-multiplier if the corresponding convolution operator $W_a^0 \ : \ \mathbb{L}_p(\mathbb{R}) \longrightarrow \mathbb{L}_p(\mathbb{R})$ is bounded. The set of all $\mathbb{L}_p$-multipliers is denoted by $\mathfrak{M}_p(\mathbb{R})$. It is known (see, e.g. [16]), that $\mathfrak{M}_p(\mathbb{R})$ is a Banach subalgebra of $\mathbb{L}_\infty(\mathbb{R})$ which contains the algebra $\mathbf{V}_1(\mathbb{R})$ of all functions with finite variation. For $p = 2$ we have the exact equality $\mathfrak{M}_2(\mathbb{R}) = \mathbb{L}_\infty(\mathbb{R})$.

The operator

$$W_a := r_{\mathbb{R}^+} W_a^0 \Big| \mathbb{L}_p(\mathbb{R}^+) \ : \ \mathbb{L}_p(\mathbb{R}^+) \longrightarrow \mathbb{L}_p(\mathbb{R}^+),$$

where $r_{\mathbb{R}^+} : \mathbb{L}_p(\mathbb{R}) \to \mathbb{L}_p(\mathbb{R}^+)$ denotes the restriction operator, is called the convolution on the semi-axis $\mathbb{R}^+$ or the Wiener–Hopf operator. It is worth noting that unlike the operators W_a^0 and $\mathfrak{M}_a^0$, which possess the property

$$W_a^0 W_b^0 = W_{ab}^0, \quad \mathfrak{M}_a^0 \mathfrak{M}_b^0 = \mathfrak{M}_{ab}^0 \quad \text{for all} \quad a, b \in \mathfrak{M}_p(\mathbb{R}), \tag{15}$$

the product of Wiener–Hopf operators cannot be computed by the simple rule (15). In fact for the operators W_a and W_b, a similar relation

$$W_a W_b = W_{ab} \tag{16}$$

is valid if and only if either $a(\xi)$ has an analytic extension into the lower half plane or $b(\xi)$ has an analytic extension into the upper half plane [16].

If the conditions (12) hold, the isometrical isomorphisms (13) are extended to the following isomorphisms of Lebesgue spaces:

$$\mathbf{Z}_\beta : \mathbb{L}_p(\mathbb{R}^+, t^\gamma) \to \mathbb{L}_p(\mathbb{R}), \qquad \mathbf{Z}_\beta \varphi(\xi) := e^{-\beta\xi} \varphi(e^{-\xi}), \quad \xi \in \mathbb{R},$$
$$\mathbf{Z}_\beta^{-1} : \mathbb{L}_p(\mathbb{R}) \to \mathbb{L}_p(\mathbb{R}^+, t^\gamma), \quad \mathbf{Z}_\beta^{-1} \psi(t) := t^{-\beta} \psi(-\ln t), \quad t \in \mathbb{R}^+,$$

and provide the following connection between the Fourier and Mellin transformations and the corresponding convolution operators:

$$\mathcal{M}_\beta = \mathcal{F} \mathbf{Z}_\beta, \qquad \mathcal{M}_\beta^{-1} = \mathbf{Z}_\beta^{-1} \mathcal{F}^{-1},$$
$$\mathfrak{M}_a^0 = \mathcal{M}_\beta^{-1} a \mathcal{M}_\beta = \mathbf{Z}_\beta^{-1} \mathcal{F}^{-1} a \mathcal{F} \mathbf{Z}_\beta = \mathbf{Z}_\beta^{-1} W_a^0 \mathbf{Z}_\beta.$$

These identities also justify the following assertion.

Proposition 2.1 *([16]). Let $1 < p < \infty$ and $-1 < \gamma < p - 1$. The class of Mellin $\mathbb{L}_{p,\gamma}$-multipliers does not depend on the parameter γ and coincides with the Banach algebra $\mathfrak{M}_p(\mathbb{R})$ of Fourier $\mathbb{L}_p$-multipliers.*

Corollary 2.2 *([16]). A Mellin convolution operator of the form (8) is bounded in the setting $\mathfrak{M}_a^0 \ : \ \mathbb{L}_p(\mathbb{R}^+, t^\gamma) \to \mathbb{L}_p(\mathbb{R}^+, t^\gamma)$ if and only if $a \in \mathfrak{M}_p(\mathbb{R})$.*

For $s \in \mathbb{R}$ and $1 < p < \infty$, the Bessel potential space, known also as a fractional Sobolev space, is a subspace of the Schwartz space $\mathbb{S}'(\mathbb{R})$ of the distributions having the finite norm

$$\|\varphi | \mathbb{H}_p^s(\mathbb{R})\| := \left[\int_{-\infty}^{\infty} \left| \mathcal{F}^{-1} (1 + |\xi|^2)^{s/2} (\mathcal{F}\varphi)(t) \right|^p dt \right]^{1/p} < \infty.$$

For the integer parameters $s = m \in \mathbb{N}$, space $\mathbb{H}^s_p(\mathbb{R})$ coincides with the Sobolev space $\mathbb{W}^m_p(\mathbb{R})$ endowed with an equivalent norm

$$\|\varphi|\mathbb{W}^m_p(\mathbb{R})\| := \left[\sum_{k=0}^{m}\int_{-\infty}^{\infty}\left|\frac{d^k\varphi(t)}{dt^k}\right|^p dt\right]^{1/p}.$$

If $s < 0$, one gets the space of distributions. Moreover, $\mathbb{H}^{-s}_{p'}(\mathbb{R})$ is the dual to the space $\mathbb{H}^s_p(\mathbb{R})$, provided that $p' := \dfrac{p}{p-1}$, $1 < p < \infty$. Note that $\mathbb{H}^s_2(\mathbb{R})$ is a Hilbert space with the inner product

$$\langle\varphi,\psi\rangle_s = \int_{\mathbb{R}}(\mathcal{F}\varphi)(\xi)\overline{(\mathcal{F}\psi)(\xi)}(1+\xi^2)^s d\xi\ , \quad \varphi,\psi \in \mathbb{H}^s(\mathbb{R}).$$

By r_Σ we denote the operator restricting functions or distributions defined on $\mathbb{R}$ to the subset $\Sigma \subset \mathbb{R}$. Thus $\mathbb{H}^s_p(\mathbb{R}^+) = r_{\mathbb{R}^+}(\mathbb{H}^s_p(\mathbb{R}))$, and the norm in $\mathbb{H}^s_p(\mathbb{R}^+)$ is defined by

$$\|f|\mathbb{H}^s_p(\mathbb{R}^+)\| = \inf_{\ell}\|\ell f|\mathbb{H}^s_p(\mathbb{R})\|,$$

where ℓf stands for any extension of f to a distribution in $\mathbb{H}^s_p(\mathbb{R})$.

Further, we denote by $\widetilde{\mathbb{H}}^s_p(\mathbb{R}^+)$ the (closed) subspace of $\mathbb{H}^s_p(\mathbb{R})$ which consists of all distributions supported in the closure of $\mathbb{R}^+$.

Note that $\widetilde{\mathbb{H}}^s_p(\mathbb{R}^+)$ is always continuously embedded in $\mathbb{H}^s_p(\mathbb{R}^+)$ and for $s \in (1/p-1, 1/p)$ these two spaces coincide. Moreover, $\mathbb{H}^s_p(\mathbb{R}^+)$ may be viewed as the quotient-space $\mathbb{H}^s_p(\mathbb{R}^+) := \mathbb{H}^s_p(\mathbb{R})/\widetilde{\mathbb{H}}^s_p(\mathbb{R}^-)$, $\mathbb{R}^- := (-\infty, 0)$.

If the Fourier convolution operator (FCO) on the semi-axis $\mathbb{R}^+$ with the symbol $a \in \mathbb{L}_{\infty,loc}(\mathbb{R})$ is bounded in the space setting

$$W_a\ :\ \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) \longrightarrow \mathbb{H}^{s-r}_p(\mathbb{R}^+),$$

we say that W_a has order r and a is an $\mathbb{L}_p$ multiplier of order r. The set of all $\mathbb{L}_p$ multipliers of order r is denoted by $\mathfrak{M}^r_p(\mathbb{R})$. Let us mention another description of the space $\mathfrak{M}^r_p(\mathbb{R})$, viz. $a \in \mathfrak{M}^r_p(\mathbb{R})$ if and only if $\lambda^{-r}a \in \mathfrak{M}_p(\mathbb{R}) = \mathfrak{M}^0_p(\mathbb{R})$, where $\lambda^r(\xi) := (1+|\xi|^2)^{r/2}$.

Note that, FCOs are particular cases of pseudodifferential operators (ΨDOs).

Theorem 2.3. *Let $1 < p < \infty$. Then*

1. *For any $r, s \in \mathbb{R}$ and for any $\gamma \in \mathbb{C}$, $\operatorname{Im}\gamma > 0$, the pseudodifferential operators $\mathbf{\Lambda}^r_\gamma := \mathbf{\Lambda}^r_{+\gamma}$ and $\mathbf{\Lambda}^r_{-\gamma}$ defined by*

$$\begin{aligned}\mathbf{\Lambda}^r_\gamma = W_{\lambda^r_\gamma}\ &:\ \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) \to \widetilde{\mathbb{H}}^{s-r}_p(\mathbb{R}^+),\\ \mathbf{\Lambda}^r_{-\gamma} = W_{\lambda^r_{-\gamma}}\ &:\ \mathbb{H}^s_p(\mathbb{R}^+) \to \mathbb{H}^{s-r}_p(\mathbb{R}^+),\end{aligned} \tag{17}$$

   where $\lambda^r_{\pm\gamma}(\xi) := (\xi \pm \gamma)^r$, $\xi \in \mathbb{R}^+$, are isomorphisms between the corresponding spaces.

2. *For any operator $\mathbf{A}\ :\ \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) \to \mathbb{H}^{s-r}_p(\mathbb{R}^+)$ of order r, the following diagram is commutative:*

$$\begin{array}{ccc}\widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) & \xrightarrow{\mathbf{A}} & \mathbb{H}^{s-r}_p(\mathbb{R}^+)\\ \uparrow \mathbf{\Lambda}^{-s}_\gamma & & \downarrow \mathbf{\Lambda}^{s-r}_{-\gamma}\\ \mathbb{L}_p(\mathbb{R}^+) & \xrightarrow{\mathbf{\Lambda}^{s-r}_{-\gamma}\mathbf{A}\mathbf{\Lambda}^{-s}_\gamma} & \mathbb{L}_p(\mathbb{R}^+).\end{array} \tag{18}$$

Thus the diagram (18) provides an equivalent lifting of the operator $\mathbf{A}$ *of order* r *up to the operator* $\mathbf{\Lambda}^{s-r}_{-\gamma}\mathbf{A}\mathbf{\Lambda}^{-s}_{\gamma} : \mathbb{L}_p(\mathbb{R}^+) \longrightarrow \mathbb{L}_p(\mathbb{R}^+)$ *of order* 0.

3. *Let* $\mu, \nu \in \mathbb{R}$. *If* a *is an* $\mathbb{L}_p$*-multiplier of order* r, *then for any complex numbers* γ_1, γ_2 *such that* $\operatorname{Im}\gamma_j > 0$, $j = 1, 2$, *the operator* $\mathbf{\Lambda}^{\mu}_{-\gamma_1} W_a \mathbf{\Lambda}^{\nu}_{\gamma_2}$ *is a Fourier convolution* $W_{a_{\mu,\nu}}$ *of order* $r + \mu + \nu$,

$$W_{a_{\mu,\nu}} \ : \ \widetilde{\mathbb{H}}^{s+\nu}_p(\mathbb{R}^+) \longrightarrow \mathbb{H}^{s-r-\mu}_p(\mathbb{R}^+), \tag{19}$$

with the symbol

$$a_{\mu,\nu}(\xi) := (\xi - \gamma_1)^{\mu} a(\xi)(\xi + \gamma_2)^{\nu}.$$

In particular, the lifting of the operator W_a *up to the operator* $\mathbf{\Lambda}^{s-r}_{-\gamma} W_a \mathbf{\Lambda}^{-s}_{\gamma}$ *acting in the space* $\mathbb{L}_p(\mathbb{R}^+)$ *is FCO of order zero with the symbol*

$$a_{s-r,-s}(\xi) = \lambda^{s-r}_{-\gamma}(\xi) a(\xi) \lambda^{-s}_{\gamma}(\xi) = \left(\frac{\xi - \gamma}{\xi + \gamma}\right)^{s-r} \frac{a(\xi)}{(\xi + \gamma)^r}\,.$$

4. *The Hilbert transform* $\mathbf{K}^1_1 = iS_{\mathbb{R}^+} = W_{-i\,\mathrm{sign}}$ *is a Fourier convolution operator and*

$$\mathbf{\Lambda}^{s}_{-\gamma_1} \boldsymbol{K}^1_1 \mathbf{\Lambda}^{-s}_{\gamma_2} = W_{-i\, g^s_{-\gamma_1,\gamma_2}\,\mathrm{sign}},$$

where

$$g^s_{-\gamma_1,\gamma_2}(\xi) := \left(\frac{\xi - \gamma_1}{\xi + \gamma_2}\right)^s.$$

Proof. For the proof of items 1–3 we refer the reader to [16, Lemma 5.1] and [22]. The item 4 is a consequence of 2–3 (see [16,19]). ◻

Note that the operator equality in (19) is in fact a consequence of the relation (16).

3. Mellin convolution operators in the Bessel potential spaces – lifting

In contrast to the Fourier convolution operators the lifted Mellin convolution operator is not a Mellin convolution anymore. Moreover, there are Mellin convolution operators $\mathfrak{M}^0_{a_\beta}$ with symbols $a_\beta \in \mathfrak{M}_p(\mathbb{R})$ which are unbounded in the Bessel potential spaces. Thus in order to study the Mellin convolutions in the space of Bessel potentials, one has to address the boundedness problem first. To this end, a class of integral operators with admissible kernels was introduced in [19]. For the sake of simplicity, here we consider a lighter version of such kernels.

Definition 3.1. *The function* $\mathcal{K}$ *is called an* admissible meromorphic kernel *if it can be represented in the form*

$$\mathcal{K}(t) := \sum_{j=0}^{\ell} \frac{d_j}{t - c_j} + \sum_{j=\ell+1}^{N} \frac{d_j}{(t - c_j)^{m_j}}, \tag{20}$$

where $d_j, c_j \in \mathbb{C}$, $j = 0, 1, \ldots, N$, $m_{\ell+1}, \ldots, m_N \in \{2, 3, \ldots\}$, *and* $0 < \alpha_k := |\arg c_k| \leqslant \pi$ *for* $k = \ell + 1, \ldots, N$.

Note that the kernel $\mathcal{K}(t)$ has poles at the points $c_0, c_1, \ldots, c_N \in \mathbb{C}$.

Recall that boundary integral operators for BVPs in planar domains with corners have admissible kernels (see (2) and [16,17,19,20]).

Theorem 3.2 *([19, Theorem 2.5 and Corollary 2.6]). If $\mathcal{K}$ is an admissible kernel, then the corresponding Mellin convolution operator with the kernel $\mathcal{K}$*

$$\begin{gathered}\mathbf{K}\varphi(t) := \int_0^\infty \mathcal{K}\left(\frac{t}{\tau}\right)\varphi(\tau)\frac{d\tau}{\tau},\\ \mathbf{K} \,:\, \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) \longrightarrow \mathbb{H}^s_p(\mathbb{R}^+),\end{gathered} \tag{21}$$

is bounded for all $1 < p < \infty$ and $s \in \mathbb{R}$.

The next result is crucial for what follows. Note that a similar assertion appears in [19], but the proof there contains fatal errors.

Theorem 3.3. *Let $s \in \mathbb{R}$, $c, \gamma \in \mathbb{C}$, $0 < \arg c < 2\pi$, $0 < \arg\gamma < \pi$ and $-\pi < \arg(c\,\gamma) < 0$. Then*

$$\mathbf{\Lambda}^s_{-\gamma}\mathbf{K}^1_c = c^{-s}\mathbf{K}^1_c\mathbf{\Lambda}^s_{-c\,\gamma}, \tag{22}$$

where $c^{-s} = |c|^{-s}e^{-s\arg(c)\,i}$.

Proof. Taking into account the mapping properties of Bessel potential operators (17) and the mapping properties of a Mellin convolution operator with an admissible kernel (21), one observes that both operators

$$\begin{aligned}\mathbf{\Lambda}^s_{-\gamma}\mathbf{K}^1_c \;\;&: \widetilde{\mathbb{H}}^r_p(\mathbb{R}^+) \longrightarrow \mathbb{H}^{r-s}_p(\mathbb{R}^+),\\ \mathbf{K}^1_c\mathbf{\Lambda}^s_{-c\,\gamma} \;&: \widetilde{\mathbb{H}}^r_p(\mathbb{R}^+) \longrightarrow \mathbb{H}^{r-s}_p(\mathbb{R}^+)\end{aligned} \tag{23}$$

are correctly defined and bounded for all $s \in \mathbb{R}$, $1 < p < \infty$, since $0 < \arg\gamma < \pi$ and $0 < -\arg(c\,\gamma) < \pi$.

On the other hand, let us note that the reverse superpositions $\mathbf{K}^1_c\mathbf{\Lambda}^s_{-\gamma}$ and $\mathbf{\Lambda}^s_{-c\,\gamma}\mathbf{K}^1_c$ are correctly defined only for $1/p - 1 < s < 1/p$ and $s = 1, 2, \ldots$.

For a smooth function $\varphi \in C^\infty_0(\mathbb{R}^+)$ with compact support and for $k = 1, 2, \ldots$ we can use integration by parts and obtain

$$\begin{aligned}\frac{d^k}{dt^k}\boldsymbol{K}^1_c\varphi(t) &= \frac{1}{\pi}\int_0^\infty \frac{d^k}{dt^k}\frac{1}{t - c\,\tau}\varphi(\tau)\,d\tau = \frac{(-c)^{-k}}{\pi}\int_0^\infty \frac{d^k}{d\tau^k}\frac{1}{t - c\,\tau}\varphi(\tau)\,d\tau =\\ &= \frac{c^{-k}}{\pi}\int_0^\infty \frac{1}{t - c\,\tau}\frac{d^k\varphi(\tau)}{d\tau^k}\,d\tau = c^{-k}\Big(\boldsymbol{K}^1_c\frac{d^k}{dt^k}\varphi\Big)(t).\end{aligned} \tag{24}$$

Let us consider the case where s is a positive integer, i.e. $s = m = 1, 2, \ldots$. The Bessel potentials $\mathbf{\Lambda}^m_{\pm} = W_{\lambda^m_{\pm\gamma}}$ are the Fourier convolutions of order m and they represent ordinary differential operators of the order m, namely,

$$\mathbf{\Lambda}^m_{\pm\gamma} = W_{\lambda^m_{\pm\gamma}} = \Big(i\,\frac{d}{dt} \pm \gamma\Big)^m = \sum_{k=0}^m \binom{m}{k} i^k(\pm\gamma)^{m-k}\frac{d^k}{dt^k}\,. \tag{25}$$

Since $C^\infty_0(\mathbb{R}^+)$ is a dense subset of $\widetilde{\mathbb{H}}^s_p(\mathbb{R}^+)$, by taking into account formulae (24) and (25) we get the following:

$$\mathbf{\Lambda}^m_{-\gamma}\mathbf{K}^1_c\varphi = \Big(i\,\frac{d}{dt} - \gamma\Big)^m\mathbf{K}^1_c\varphi = \sum_{k=0}^m \binom{m}{k} i^k(-\gamma)^{m-k}\frac{d^k}{dt^k}\mathbf{K}^1_c\varphi$$

$$
\begin{aligned}
&= \sum_{k=0}^{m} \binom{m}{k} i^k (-\gamma)^{m-k} c^{-k} \Big(\mathbf{K}_c^1 \frac{d^k}{dt^k} \varphi \Big)(t) = \\
&= c^{-m} \mathbf{K}_c^1 \Big(\sum_{k=0}^{m} \binom{m}{k} i^k (-c\gamma)^{m-k} \frac{d^k}{dt^k} \varphi \Big)(t) = \\
&= c^{-m} \mathbf{K}_c^1 \mathbf{\Lambda}_{-c\gamma}^m \varphi, \qquad \varphi \in \widetilde{\mathbb{H}}_p^r(\mathbb{R}^+).
\end{aligned}
$$

Thus for $s = m = 1, 2, \ldots$, formula (22) is proved.

By the relations (17) the mappings

$$
\begin{aligned}
&\mathbf{\Lambda}_{-c\gamma}^r : \widetilde{\mathbb{H}}_p^s(\mathbb{R}^+) \longrightarrow \widetilde{\mathbb{H}}_p^{s-r}(\mathbb{R}^+), \\
&\mathbf{\Lambda}_{-\gamma}^r : \mathbb{H}_p^s(\mathbb{R}^+) \longrightarrow \mathbb{H}_p^{s-r}(\mathbb{R}^+),
\end{aligned}
$$

are isomorphisms of the corresponding spaces for arbitrary $r \in \mathbb{R}$ since we have assumed that $\mathrm{Im}(-\gamma) < 0$ and $\mathrm{Im}\,(-c\gamma) > 0$.

If $s = -m$, $m \in \mathbb{N}$ is a negative integer, then we apply the inverse operators $\mathbf{\Lambda}_{-\gamma}^{-m}$ and $\mathbf{\Lambda}_{-c\gamma}^{-m}$ to the already proven operator equality

$$
\mathbf{\Lambda}_{-\gamma}^m \mathbf{K}_c^1 = c^{-m} \mathbf{K}_c^1 \mathbf{\Lambda}_{-c\gamma}^m, \quad m = 1, 2, \ldots,
$$

from the left-hand side and from the right-hand side, respectively. This leads to the relation:

$$
\mathbf{K}_c^1 \mathbf{\Lambda}_{-c\gamma}^{-m} = c^{-m} \mathbf{\Lambda}_{-\gamma}^{-m} \mathbf{K}_c^1 \quad \text{or} \quad \mathbf{\Lambda}_{-\gamma}^{-m} \mathbf{K}_c^1 = c^m \mathbf{K}_c^1 \mathbf{\Lambda}_{-c\gamma}^{-m}.
$$

The equality (22) is now proved also for a negative $s = -1, -2, \ldots$.

In order to derive formula (22) for non-integer values of s, we can confine ourselves to the case $-2 < s < -1$. Indeed, any non-integer value $s \in \mathbb{R}$ can be represented in the form $s = s_0 + m$, where $-2 < s_0 < -1$ and m is an integer. Therefore, if for $s = s_0 + m$ the operators in (23) are correctly defined and bounded, and if the relations in question are valid for $-2 < s_0 < -1$, then we can write

$$
\begin{aligned}
\mathbf{\Lambda}_{-\gamma}^s \mathbf{K}_c^1 = \mathbf{\Lambda}_{-\gamma}^{s_0+m} \mathbf{K}_c^1 &= c^{-m} \mathbf{\Lambda}_{-\gamma}^{s_0} \mathbf{K}_c^1 \mathbf{\Lambda}_{-c\gamma}^m = c^{-s_0-m} \mathbf{K}_c^1 \mathbf{\Lambda}_{-c\gamma}^{s_0} \mathbf{\Lambda}_{-c\gamma}^m \\
&= c^{-s_0-m} \mathbf{K}_c^1 \mathbf{\Lambda}_{-c\gamma}^{s_0+m} = c^{-s} \mathbf{K}_c^1 \mathbf{\Lambda}_{-c\gamma}^s.
\end{aligned}
$$

Thus let us assume that $-2 < s < -1$ and consider the expression

$$
\mathbf{\Lambda}_{-\gamma}^s \mathbf{K}_c^1 \varphi(t) = \frac{1}{2\pi^2} r_+ \int_{-\infty}^{\infty} e^{-i\xi t} (\xi - \gamma)^s \int_0^\infty e^{i\xi y} \int_0^\infty \frac{\varphi(\tau)}{y - c\tau} \, d\tau \, dy \, d\xi, \tag{26}
$$

where r_+ is the restriction to $\mathbb{R}^+$. It is clear that the integral in the right-hand-side of (26) exists. Indeed, if $\varphi \in \mathbb{L}_2$, then $\mathbf{K}_c^1 \varphi \in \mathbb{L}_2 \cap C^\infty$ and $\mathbf{\Lambda}_{-\gamma}^s \mathbf{K}_c^1 \varphi \in \mathbb{H}^{-s} \cap C^\infty \subset \mathbb{L}_2 \cap C^\infty$.

Now consider the function $e^{-izt}(z - \gamma)^s e^{izy}$, $z \in \mathbb{C}$. Since $\mathrm{Im}\gamma \neq 0$, $s < -1$, then for sufficiently small $\varepsilon > 0$ this function is analytic in the strip between the lines $\mathbb{R}$ and $\mathbb{R} + i\varepsilon$ and vanishes at the infinity for all finite $t \in \mathbb{R}$ and for all $y > 0$. Therefore, the integration over the real line $\mathbb{R}$ in the first integral of (26) can be replaced by the integration over the line $\mathbb{R} + i\varepsilon$, i.e.

$$
\mathbf{\Lambda}_{-\gamma}^s \mathbf{K}_c^1 \varphi(t) = \frac{1}{2\pi^2} r_+ \int_{-\infty}^{\infty} e^{-i\xi t + \varepsilon t} (\xi + i\varepsilon - \gamma)^s \int_0^\infty e^{i\xi y - \varepsilon y} \int_0^\infty \frac{\varphi(\tau)}{y - c\tau} \, d\tau \, dy \, d\xi. \tag{27}
$$

Let us use the density of the set $C_0^\infty(\mathbb{R}^+)$ in $\widetilde{\mathbb{H}}_p^s(\mathbb{R}^+)$. Thus for all finite $t \in \mathbb{R}$ and for all functions $\varphi \in C_0^\infty(\mathbb{R})$ with compact supports the integrand in the corresponding triple integral for (27) is absolutely integrable. Therefore, for such functions one can use Fubini–Tonelli theorem and change the order of integration in (27). Thereafter, one returns to the integration over the real line $\mathbb{R}$ and obtains

$$\mathbf{\Lambda}^s_{-\gamma}\mathbf{K}^1_c\varphi(t) \;= \frac{1}{2\pi^2}\, r_+ \int_0^\infty \varphi(\tau) \int_0^\infty \frac{1}{y - c\tau} \int_{-\infty}^\infty e^{i\xi(y-t)}(\xi-\gamma)^s d\xi\, dy\, d\tau. \tag{28}$$

In order to study the expression in the right-hand side of (28), one can use a well known formula

$$\int_{-\infty}^\infty (\beta+ix)^{-\nu} e^{-ipx}\, dx = \begin{cases} 0 & \text{for} \quad p>0,\\ \dfrac{2\pi(-p)^{\nu-1}e^{\beta\, p}}{\Gamma(\nu)} & \text{for} \;\; p<0, \end{cases}$$
$$\operatorname{Re}\beta > 0, \qquad \operatorname{Re}\nu > 0$$

[24, Formula 3.382.6]. It can be rewritten in a more convenient form – viz.,

$$\int_{-\infty}^\infty e^{i\mu\,\xi}(\xi-\gamma)^s\, d\xi = \begin{cases} 0 & \text{if} \;\; \mu<0,\\ \dfrac{2\pi\,\mu^{-s-1}e^{-\frac{\pi}{2}si+\mu\,\gamma i}}{\Gamma(-s)} & \text{if} \;\; \mu>0, \end{cases} \tag{29}$$
$$\operatorname{Im}\gamma > 0, \qquad \operatorname{Re}s < 0.$$

Applying (29) to the last integral in (28), one obtains

$$\begin{aligned} \mathbf{\Lambda}^s_{-\gamma}\mathbf{K}^1_c\varphi(t) &= \frac{e^{-\frac{\pi}{2}si}}{\pi\Gamma(-s)} r_+ \int_0^\infty \varphi(\tau)\, d\tau \int_t^\infty \frac{e^{i(y-t)\gamma}dy}{(y-t)^{1+s}(y-c\tau)} \\ &= \frac{e^{-\frac{\pi}{2}si}}{\pi\Gamma(-s)} r_+ \int_0^\infty \varphi(\tau)\, d\tau \int_0^\infty \frac{y^{-s-1}e^{i\gamma\, y}dy}{y+t-c\tau}, \end{aligned} \tag{30}$$

where the integrals exist since $0 < -s-1 < 1$ and $0 < \arg\gamma < \pi$ (i.e., $\operatorname{Im}\gamma > 0$).

Let us recall the formula

$$\int_0^\infty \frac{x^{\nu-1}e^{-\mu\, x}\, dx}{x+\beta} = \beta^{\nu-1}e^{\beta\,\mu}\Gamma(\nu)\Gamma(1-\nu,\beta\mu), \tag{31}$$
$$\operatorname{Re}\nu > 0, \quad \operatorname{Re}\mu > 0, \quad |\arg\,\beta| < \pi$$

(cf. [24, formula 3.383.10]). Due to the conditions $0 < \arg c < 2\pi$, $t > 0$, $\tau > 0$ we have $|\arg(t - c\tau)| < \pi$ and, therefore, we can apply (31) to the equality (30). Then (30) acquires the following final form:

$$\mathbf{\Lambda}^s_{-\gamma}\mathbf{K}^1_c\varphi(t) = \frac{e^{-\frac{\pi}{2}si}}{\pi} r_+ \int_0^\infty \frac{e^{-i\gamma(t-c\tau)}\Gamma(1+s,-i\gamma(t-c\tau))\varphi(\tau)\, d\tau}{(t-c\tau)^{1+s}}. \tag{32}$$

Consider now the reverse composition $\mathbf{K}^1_c\mathbf{\Lambda}^s_{-c\,\gamma}\varphi(t)$. Changing the order of integration in the corresponding expression (see (28) for a similar motivation), one obtains

$$\mathbf{K}^1_c\mathbf{\Lambda}^s_{-c\,\gamma}\varphi(t) := \frac{1}{2\pi^2}\, r_+\int_0^\infty \frac{1}{t-c\,y}\int_{-\infty}^\infty e^{-i\xi\,y}(\xi - c\,\gamma)^s\int_0^\infty e^{i\xi\,\tau}\varphi(\tau)d\tau\, d\xi\, dy$$
$$= \frac{1}{2\pi^2} r_+\int_0^\infty \varphi(\tau)\int_0^\infty \frac{1}{t-c\,y}\int_{-\infty}^\infty e^{i\xi(\tau-y)}(\xi - c\,\gamma)^s d\xi\, dy\, d\tau. \tag{33}$$

In order to compute the expression in the right-hand side of (33), let us recall formula 3.382.7 from [24]:

$$\int_{-\infty}^\infty (\beta - ix)^{-\nu}e^{-ipx}\,dx = \begin{cases} 0 & \text{for} \quad p<0, \\ \dfrac{2\pi\, p^{\nu-1}e^{-\beta\, p}}{\Gamma(\nu)} & \text{for} \quad p>0, \end{cases}$$
$$\mathrm{Re}\,\nu > 0, \quad \mathrm{Re}\beta > 0$$

and rewrite it in a form more suitable for our consideration – viz.,

$$\int_{-\infty}^\infty e^{i\mu\,\xi}(\xi+\omega)^s\,d\xi = \begin{cases} 0 & \mu>0,\ \mathrm{Im}\,\omega > 0, \\ \dfrac{2\pi\,(-\mu)^{-s-1}e^{\frac{\pi}{2}si - \mu\,\omega i}}{\Gamma(-s)} & \mu<0,\ \mathrm{Im}\,\omega>0, \end{cases} \tag{34}$$
$$\mathrm{Re}\,s < 0, \quad \mu\in\mathbb{R}, \quad \omega,\, s\in\mathbb{C}.$$

Using (34), we represent (33) in the form

$$\mathbf{K}^1_c\mathbf{\Lambda}^s_{-c\,\gamma}\varphi(t) = \frac{e^{\frac{\pi}{2}si}}{\pi\Gamma(-s)}\, r_+\int_0^\infty \varphi(\tau)\,d\tau\int_\tau^\infty \frac{e^{-ic\,\gamma(y-\tau)}\,dy}{(y-\tau)^{s+1}(t-c\,y)}$$
$$= -\frac{e^{\frac{\pi}{2}si}}{\pi c\Gamma(-s)} r_+\int_0^\infty \varphi(\tau)\,d\tau\int_0^\infty \frac{y^{-s-1}e^{-ic\gamma\,y}\,dy}{y+\tau-c^{-1}t}, \tag{35}$$

where the integrals exist since $-s-1 > -1$ and $-\pi < \arg(c\,\gamma) < 0$ (i.e., $\mathrm{Im}(c\,\gamma) < 0$).

Due to the conditions $0 < \arg\, c < 2\pi$, $t>0$, $\tau > 0$ we have $|\arg(\tau - c^{-1}t)| < \pi$. Therefore, we can apply formulae (31) to (35) and get the following representation:

$$\mathbf{K}^1_c\mathbf{\Lambda}^s_{-c\,\gamma}\varphi(t) = -\frac{c^{-1}e^{\frac{\pi}{2}si}}{\pi} r_+\int_0^\infty \frac{e^{-ic\gamma(c^{-1}t-\tau)}\Gamma(1+s, -ic\gamma(c^{-1}t-\tau))\varphi(\tau)\,d\tau}{(\tau - c^{-1}t)^{1+s}}$$
$$= \frac{c^s e^{-\frac{\pi}{2}si}}{\pi} r_+\int_0^\infty \frac{e^{-i\gamma(t-c\,\tau)}\Gamma(1+s, -i\gamma(t-c\,\tau))\varphi(\tau)\,d\tau}{(t-c\,\tau)^{1+s}}. \tag{36}$$

If we multiply (36) by c^{-s} we get precisely the expression in (32) and, therefore, $\mathbf{\Lambda}^s_{-\gamma}\mathbf{K}^1_c\varphi(t) = c^{-s}\mathbf{K}^1_c\mathbf{\Lambda}^s_{-c\,\gamma}\varphi(t)$, which proves the claimed equality (22) for $-2 < s < -1$ and accomplishes the proof. □

Corollary 3.4. *Let $0 < \arg\, c < 2\pi$ and $0 < \arg\gamma < \pi$. Then for arbitrary $\gamma_0 \in \mathbb{C}$ such that $0 < \arg\,\gamma_0 < \pi$ and $-\pi < \arg(c\,\gamma_0) < 0$, one has*

$$\mathbf{\Lambda}^s_{-\gamma}\mathbf{K}^1_c = c^{-s}W_{g_{-\gamma,-\gamma_0}}\mathbf{K}^1_c\mathbf{\Lambda}^s_{-c\,\gamma_0}, \tag{37}$$

where

$$g^s_{-\gamma,-\gamma_0}(\xi) := \left(\frac{\xi-\gamma}{\xi-\gamma_0}\right)^s. \tag{38}$$

If, in addition, $1 < p < \infty$ and $1/p - 1 < r < 1/p$ then equality (37) can be supplemented as follows:

$$\mathbf{\Lambda}^s_{-\gamma}\mathbf{K}^1_c = c^{-s}\left[\mathbf{K}^1_c W_{g^s_{-\gamma,-\gamma_0}} + \mathbf{T}\right]\mathbf{\Lambda}^s_{-c\,\gamma_0}, \tag{39}$$

where $\mathbf{T} : \widetilde{\mathbb{H}}^r_p(\mathbb{R}^+) \to \mathbb{H}^r_p(\mathbb{R}^+)$ is a compact operator, and if c is a real negative number, then $c^{-s} := |c|^{-s}e^{-\pi si}$.

Proof. It follows from equalities (16) and (22) that

$$\mathbf{\Lambda}^s_{-\gamma}\mathbf{K}^1_c = \mathbf{\Lambda}^s_{-\gamma}\mathbf{\Lambda}^{-s}_{-\gamma_0}\mathbf{\Lambda}^s_{-\gamma_0}\mathbf{K}^1_c = c^{-s}W_{g_{-\gamma,-\gamma_0}}\mathbf{K}^1_c\mathbf{\Lambda}^s_{-c\,\gamma_0}$$

and (37) is proved. If $1 < p < \infty$ and $1/p - 1 < r < 1/p$, then the commutator

$$\mathbf{T} := W_{g^s_{-\gamma,-\gamma_0}}\mathbf{K}^1_c - \mathbf{K}^1_c W_{g^s_{-\gamma,-\gamma_0}} : \widetilde{\mathbb{H}}^r_p(\mathbb{R}^+) \to \mathbb{H}^r_p(\mathbb{R}^+)$$

of Mellin and Fourier convolution operators is correctly defined and bounded. It is compact for $r = 0$ and all $1 < p < \infty$ (see [15,18]). Due to Krasnoselsky's interpolation theorem (see [27] and also [35, Sections 1.10.1 and 1.17.4]), the operator $\mathbf{T}$ is compact in all $\mathbb{L}_r$-spaces for $1/p - 1 < r < 1/p$. Therefore, the equality (37) can be rewritten as

$$\mathbf{\Lambda}^s_{-\gamma}\mathbf{K}^1_c = c^{-s}\left[\mathbf{K}^1_c W_{g^s_{-\gamma,-\gamma_0}} + \mathbf{T}\right]\mathbf{\Lambda}^s_{-c\,\gamma_0},$$

and we are done. □

Remark 1. The assumption $1/p-1 < r < 1/p$ in (39) cannot be relaxed. Indeed, the operator $W_{g^s_{-\gamma,-\gamma_0}}\mathbf{K}^1_c = \mathbf{\Lambda}^s_{-\gamma}\mathbf{\Lambda}^{-s}_{-\gamma_0}\mathbf{K}^1_c : \widetilde{\mathbb{H}}^r_p(\mathbb{R}^+) \to \mathbb{H}^r_p(\mathbb{R}^+)$ is bounded for all $r \in \mathbb{R}$ (see (23)). But the operator $\mathbf{K}^1_c W_{g^s_{-\gamma,-\gamma_0}} : \widetilde{\mathbb{H}}^r_p(\mathbb{R}^+) \to \mathbb{H}^r_p(\mathbb{R}^+)$ is bounded only for $1/p - 1 < r < 1/p$ because the function $g^s_{-\gamma,-\gamma_0}(\xi)$ has an analytic extension into the lower half-plane but not into the upper one.

4. The algebra generated by Mellin and Fourier convolution operators

In the present section we recall some results about the Banach algebra generated by Fourier and Mellin convolution operators in the Lebesgue space with weight from [18], revised in [19]. The exposition follows [19, Section 2]. For more general algebras we refer the reader to [18] and to [2–5,15,34].

Let us consider the Banach algebra $\mathfrak{A}_p(\mathbb{R}^+)$ generated by Mellin convolution and Fourier convolution operators in the Lebesgue space $\mathbb{L}_p(\mathbb{R}^+)$. In particular, this algebra contains the operators

$$\mathbf{A} := \sum_{j=1}^m \mathfrak{M}^0_{a_j} W_{b_j}, \tag{40}$$

and their compositions. Here $\mathfrak{M}^0_{a_j}$ are Mellin convolution operators with continuous $N \times N$ matrix symbols $a_j \in C\mathfrak{M}_p(\dot{\mathbb{R}})$ including the infinity $a_j(-\infty) = a_j(+\infty)$, W_{b_j} are Fourier convolution operators with $N \times N$ matrix symbols $b_j \in C\mathfrak{M}_p(\overline{\mathbb{R}}\setminus\{0\}) := C\mathfrak{M}_p(\overline{\mathbb{R}}^- \cup \overline{\mathbb{R}}^+)$ which might have jump discontinuities at 0 and at the infinity. The algebra of $N \times N$ matrix $\mathbb{L}_p$-multipliers $C\mathfrak{M}_p(\overline{\mathbb{R}} \setminus \{0\})$ consists of those piecewise-continuous

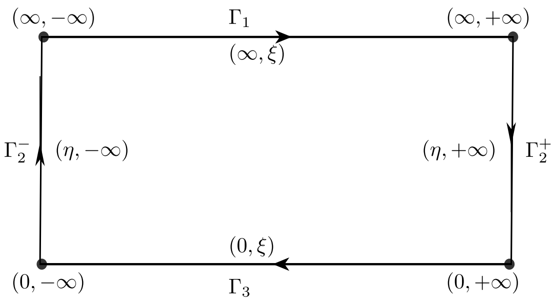

Fig. 1. The domain $\mathfrak{R}$ of definition of the symbol $\mathcal{A}_p^s(\omega)$.

$N \times N$ matrix multipliers $b \in \mathfrak{M}_p(\mathbb{R}) \cap PC(\overline{\mathbb{R}})$ which are continuous on the semi-axes $\mathbb{R}^-$ and $\mathbb{R}^+$ but might have finite jump discontinuities at 0 and at the infinity.

Note that the algebra $\mathfrak{A}_p(\mathbb{R}^+)$ is actually a subalgebra of the Banach algebra $\mathfrak{F}_p(\mathbb{R}^+)$ generated by the Fourier convolution operators W_a with piecewise-constant symbols $a(\xi)$ in the space $\mathbb{L}_p(\mathbb{R}^+)$. Let $\mathfrak{S}(\mathbb{L}_p(\mathbb{R}^+))$ denote the ideal of all compact operators in $\mathbb{L}_p(\mathbb{R}^+)$. Since in the scalar case $N = 1$ the quotient algebra $\mathfrak{F}_p(\mathbb{R}^+)/\mathfrak{S}(\mathbb{L}_p(\mathbb{R}^+))$ is commutative, the following proposition is true.

Proposition 4.1 *([18] and [19, Corollary 3.10]). If $N = 1$, then the quotient algebra $\mathfrak{A}_p(\mathbb{R}^+)/\mathfrak{S}(\mathbb{L}_p(\mathbb{R}^+))$ is commutative.*

To describe the symbol of the operator $\boldsymbol{A}$ in (40), consider the infinite clockwise oriented "rectangle" $\mathfrak{R} := \Gamma_1 \cup \Gamma_2^- \cup \Gamma_2^+ \cup \Gamma_3$, where (cf. Fig. 1)

$$\Gamma_1 := \{\infty\} \times \overline{\mathbb{R}}, \quad \Gamma_2^{\pm} := \overline{\mathbb{R}}^+ \times \{\pm\infty\}, \quad \Gamma_3 := \{0\} \times \overline{\mathbb{R}}.$$

Let for a piecewise continuous function $g \in PC(\overline{\mathbb{R}})$ introduce the notation

$$\begin{aligned} g_p(\infty,\xi) &:= \frac{g(+\infty)+g(-\infty)}{2} + \frac{g(+\infty)-g(-\infty)}{2i} \cot\left[\pi\Big(\frac{1}{p} - i\xi\Big)\right], \\ g_p(t,\xi) &:= \frac{g(t+0)+g(t-0)}{2} + \frac{g(t+0)-g(t-0)}{2i} \cot\left[\pi\Big(\frac{1}{p} - i\xi\Big)\right], \end{aligned} \tag{41}$$

where $t, \xi \in \mathbb{R}$. Then the symbol $\mathcal{A}_p(\omega)$ of the operator $\boldsymbol{A}$ in (40) is a function on the set $\mathfrak{R}$, viz.

$$\mathcal{A}_p(\omega) := \begin{cases} \displaystyle\sum_{j=1}^m a_j(\xi)(b_j)_p(\infty,\xi), & \omega = (\infty,\xi) \in \overline{\Gamma}_1, \\ \displaystyle\sum_{j=1}^m a_j(\infty) b_j(\eta), & \omega = (\eta,+\infty) \in \Gamma_2^+, \\ \displaystyle\sum_{j=1}^m a_j(\infty) b_j(-\eta), & \omega = (\eta,-\infty) \in \Gamma_2^-, \\ \displaystyle\sum_{j=1}^m a_j(\xi)(b_j)_p(0,\xi), & \omega = (0,\xi) \in \overline{\Gamma}_3. \end{cases} \tag{42}$$

Arc condition ([23,36]): The function $g_p(\infty,\xi)$ connects the point $g(-\infty)$ with $g(+\infty)$. More precisely, it fills up the discontinuity of the function g at ∞ with an oriented arc of the circle such that from every point of the arc the oriented interval $[g(-\infty), g(+\infty)]$ is seen under the angle π/p. Moreover, the oriented arc lies on the left of the oriented interval if $1/2 < 1/p < 1$ (i.e., if $1 < p < 2$) and the oriented arc is on the right

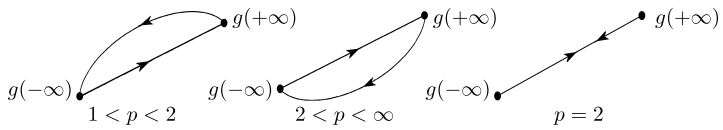

Fig. 2. Arc condition.

of the oriented interval if $0 < 1/p < 1/2$ (i.e., if $2 < p < \infty$). For $p = 2$ the oriented arc coincides with the oriented interval (see Fig. 2).

A similar geometric interpretation is valid for the function $g_p(t, \xi)$, which connects the points $g(t-0)$ and $g(t+0)$ when g has a jump at $t \in \overline{\mathbb{R}}$.

The image of the function $\det \mathcal{A}_p(\omega)$, $\omega \in \mathfrak{R}$ is a closed curve in the complex plane. This follows from the continuity of the symbol at the angular points of the rectangle $\mathfrak{R}$ where the one-sided limits coincide. Thus

$$\mathcal{A}_p(\pm\infty, \infty) = \sum_{j=1}^{m} a_j(\infty) b_j(\pm\infty),$$

$$\mathcal{A}_p(\pm\infty, 0) = \sum_{j=1}^{m} a_j(\infty) b_j(0 \pm 0).$$

Hence, if the symbol of the corresponding operator is elliptic, i.e. if

$$\inf_{\omega \in \mathfrak{R}} \left| \det \mathcal{A}_p(\omega) \right| > 0,$$

the increment of the argument $(1/2\pi) \arg \mathcal{A}_p(\omega)$ when ω ranges through $\mathfrak{R}$ in the direction of orientation, is an integer. It is called the winding number or the index of the curve $\Gamma := \{z \in \mathbb{C} : z = \det \mathcal{A}_p(\omega),\ \omega \in \mathfrak{R}\}$ and is denoted by $\operatorname{ind} \det \mathcal{A}_p$.

Theorem 4.2 *([19, Theorem 3.13]). Let $1 < p < \infty$ and let $\mathbf{A}$ be defined by (40). The operator $\mathbf{A} : \mathbb{L}_p(\mathbb{R}^+) \longrightarrow \mathbb{L}_p(\mathbb{R}^+)$ is Fredholm if and only if its symbol $\mathcal{A}_p(\omega)$ is elliptic. If $\mathbf{A}$ is Fredholm, then the index of this operator is*

$$\operatorname{Ind} \mathbf{A} = -\operatorname{ind} \det \mathcal{A}_p.$$

If $\mathcal{A}_p(\omega)$ is the symbol of an operator $\mathbf{A}$ in (40), then the set $\mathcal{R}(\mathcal{A}_p) := \{\mathcal{A}_p(\omega) \in \mathbb{C} : \omega \in \mathfrak{R}\}$ coincides with the essential spectrum of $\mathbf{A}$. Recall that the essential spectrum $\sigma_{ess}(\mathbf{A})$ of a bounded operator $\mathbf{A}$ is the set of all $\lambda \in \mathbb{C}$ such that the operator $\mathbf{A} - \lambda I$ is not Fredholm in $\mathbb{L}_p(\mathbb{R}^+)$ or, equivalently, the coset $[\mathbf{A} - \lambda I]$ is not invertible in the quotient algebra $\mathfrak{A}_p(\mathbb{R}^+)/\mathfrak{S}(\mathbb{L}_p(\mathbb{R}^+))$. Then, due to Banach's theorem, the essential norm $|||\mathbf{A}|||$ of the operator $\mathbf{A}$ can be estimated as follows

$$\sup_{\omega \in \omega} |\mathcal{A}_p(\omega)| \leqslant |||\mathbf{A}||| := \inf_{\mathbf{T} \in \mathfrak{S}(\mathbb{L}_p(\mathbb{R}^+))} \left\| (\mathbf{A} + \mathbf{T}) \mid \mathcal{L}(\mathbb{L}_p(\mathbb{R}^+)) \right\|. \tag{43}$$

The inequality (43) enables one to extend the symbol map (42)

$$[\mathbf{A}] \longrightarrow \mathcal{A}_p(\omega), \quad [\mathbf{A}] \in \mathfrak{A}_p(\mathbb{R}^+)/\mathfrak{S}(\mathbb{L}_p(\mathbb{R}^+))$$

continuously onto the whole Banach algebra $\mathfrak{A}_p(\mathbb{R}^+)$. Now, applying Theorem 4.2 and standard methods, cf. [18, Theorem 3.2], one can derive the following result.

Corollary 4.3 *([19, Corollary 3.15]). Let $1 < p < \infty$ and $\mathbf{A} \in \mathfrak{A}_p(\mathbb{R}^+)$. The operator $\mathbf{A} : \mathbb{L}_p(\mathbb{R}^+) \longrightarrow \mathbb{L}_p(\mathbb{R}^+)$ is Fredholm if and only if its symbol $\mathcal{A}_p(\omega)$ is elliptic. If $\mathbf{A}$ is Fredholm, then*

$$\operatorname{Ind}\mathbf{A} = -\operatorname{ind}\mathcal{A}_p.$$

5. Fredholm properties of Mellin convolution operators in the Bessel potential spaces

As it was already mentioned, the primary aim of the present paper is to study Fredholm properties and the invertibility of Mellin convolution operators $\mathfrak{M}^0_a$ acting in Bessel potential spaces, namely,

$$\mathfrak{M}^0_a : \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) \longrightarrow \mathbb{H}^s_p(\mathbb{R}^+).$$

The symbols of these operators are $N \times N$ matrix functions $a \in C\mathfrak{M}^0_p(\overline{\mathbb{R}})$, continuous on the real axis $\mathbb{R}$ with the only possible jump at infinity.

Theorem 5.1. *Let $s \in \mathbb{R}$ and $1 < p < \infty$.*

1. *If the conditions of Theorem 3.3 hold, the Mellin convolution operator*

$$\mathbf{K}^1_c \ : \ \widetilde{\mathbb{H}}^r_p(\mathbb{R}^+) \to \mathbb{H}^r_p(\mathbb{R}^+), \tag{44}$$

is lifted to the equivalent operator

$$\mathbf{\Lambda}^s_{-\gamma}\mathbf{K}^1_c\mathbf{\Lambda}^{-s}_{\gamma} = c^{-s}\mathbf{K}^1_c W_{g^s_{-c\gamma,\gamma}} : \mathbb{L}_p(\mathbb{R}^+) \to \mathbb{L}_p(\mathbb{R}^+),$$

where $c^{-s} = |c|^{-s}e^{-is\arg c}$ and the function $g^s_{-c\gamma,\gamma}$ is defined in (38).

2. *If the conditions of Corollary 3.4 hold, the Mellin convolution operator between Bessel potential spaces (44) is lifted to the equivalent operator*

$$\begin{aligned}\mathbf{\Lambda}^s_{-\gamma}\mathbf{K}^1_c\mathbf{\Lambda}^{-s}_{\gamma} &= c^{-s}W_{g^s_{-\gamma,-\gamma_0}}\mathbf{K}^1_c W_{g^s_{-c\gamma_0,\gamma}} \\ &= c^{-s}\mathbf{K}^1_c W_{g^s_{-\gamma,-\gamma_0}g^s_{-c\gamma_0,\gamma}} + \mathbf{T} : \mathbb{L}_p(\mathbb{R}^+) \to \mathbb{L}_p(\mathbb{R}^+),\end{aligned}$$

where $\mathbf{T} \ : \ \mathbb{L}_p(\mathbb{R}^+) \to \mathbb{L}_p(\mathbb{R}^+)$ is a compact operator.

Proof. By Theorem 2.3, using the lifting procedure, one obtains the following equivalent operator:

$$\mathbf{\Lambda}^s_{-\gamma}\mathbf{K}^1_c\mathbf{\Lambda}^{-s}_{\gamma} \ : \ \mathbb{L}_p(\mathbb{R}^+) \to \mathbb{L}_p(\mathbb{R}^+).$$

In order to proceed, we need two formulae

$$\mathbf{\Lambda}^s_{-c\gamma}\mathbf{\Lambda}^{-s}_{\gamma} = W_{g^s_{-c\gamma,\gamma}}, \qquad W_{g^s_{-\gamma,-\gamma_0}}W_{g^s_{-c\gamma_0,\gamma}} = W_{g^s_{-\gamma,-\gamma_0}g^s_{-c\gamma_0,\gamma}}. \tag{45}$$

The first relation holds because, by the conditions of Theorem 3.3, $0 < \arg\gamma < \pi$ and the second one holds because $g^s_{-\gamma,-\gamma_0}(\xi)$ has a smooth, uniformly bounded analytic extension in the lower complex half plane.

If the conditions of Theorem 3.3 are fulfilled, we apply the equalities (22), (45) and get the following:

$$\mathbf{\Lambda}^s_{-\gamma}\mathbf{K}^1_c\mathbf{\Lambda}^{-s}_{\gamma} = c^{-s}\mathbf{K}^1_c\mathbf{\Lambda}^s_{-c\gamma}\mathbf{\Lambda}^{-s}_{\gamma} = c^{-s}\mathbf{K}^1_c W_{g^s_{-c\gamma,\gamma}}.$$

If the conditions of Corollary 3.4 hold, we successively apply formulae (37), (39), both formulae (45) and get the following:

$$\mathbf{\Lambda}^s_{-\gamma}\mathbf{K}^1_c\mathbf{\Lambda}^{-s}_{\gamma} = c^{-s}W_{g^s_{-\gamma,-\gamma_0}}\mathbf{K}^1_c\mathbf{\Lambda}^s_{-c\gamma}\mathbf{\Lambda}^{-s}_{\gamma}$$
$$= c^{-s}W_{g^s_{-\gamma,-\gamma_0}}\mathbf{K}^1_c W_{g^s_{-c\gamma_0,\gamma}} = c^{-s}\mathbf{K}^1_c W_{g^s_{-\gamma,-\gamma_0}} W_{g^s_{-c\gamma_0,\gamma}} + \mathbf{T}.$$

The obtained equality accomplishes the proof. □

Remark 2. The operator $\mathbf{K}^1_1$ is the Hilbert transform $\mathbf{K}^1_1 = iS_{\mathbb{R}^+} = W_{-\mathrm{i}\,\mathrm{sign}}$ and does not satisfy the condition $\arg c \neq 0$ of Theorem 5.1. As already emphasized in Theorem 2.3, this case is essentially different. Considered as acting between the Bessel potential spaces (44), $\mathbf{K}^1_1$ is lifted to the equivalent Fourier convolution operator

$$\mathbf{\Lambda}^s_{-\gamma}\mathbf{K}^1_1\mathbf{\Lambda}^{-s}_{\gamma} = W_{-ig^s_{-\gamma,\gamma}\mathrm{sign}} \,:\, \mathbb{L}_p(\mathbb{R}^+) \to \mathbb{L}_p(\mathbb{R}^+),$$

cf. Theorem 2.3.

Theorem 5.2. *Let $c_j, d_j \in \mathbb{C}$, $0 < \arg c_j < 2\pi$ for $j = 1, \dots, n$, $0 < \arg\gamma < \pi$, $-\pi < \arg(c_j\gamma) < 0$ for $j = 1, \dots, m$ and $0 < \arg(c_j\gamma) < \pi$ for $j = m+1, \dots, n$. The Mellin convolution operator $\mathbf{A}$,*

$$\mathbf{A} = \sum_{j=1}^{n} d_j\mathbf{K}^1_{c_j} \,:\, \widetilde{\mathbb{H}}^r_p(\mathbb{R}^+) \to \mathbb{H}^r_p(\mathbb{R}^+),$$

is lifted to the equivalent operator

$$\mathbf{\Lambda}^s_{-\gamma}\mathbf{A}\mathbf{\Lambda}^{-s}_{\gamma} = \sum_{j=0}^{m} d_j c_j^{-s}\mathbf{K}^1_{c_j} W_{g^s_{-c_j\gamma,-\gamma}} + \sum_{j=m+1}^{n} d_j c_j^{-s} W_{g^s_{-\gamma,-\gamma_j}}\mathbf{K}^1_{c_j} W_{g^s_{-c_j\gamma_j,\gamma}} \tag{46}$$
$$= \sum_{j=0}^{m} d_j c_j^{-s}\mathbf{K}^1_{c_j} W_{g^s_{-c_j\gamma,\gamma}} + \sum_{j=m+1}^{n} d_j c_j^{-s}\mathbf{K}^1_{c_j} W_{g^s_{-\gamma,-\gamma_j} g^s_{-c_j\gamma_j,\gamma}} + \mathbf{T}$$

in the $\mathbb{L}_p(\mathbb{R}^+)$ space, where $c^{-s} = |c|^{-s}e^{-is\arg c}$ and γ_j are such that $0 < \arg\gamma_j < \pi$, $-\pi < \arg(c_j\,\gamma_j) < 0$ for $j = m+1, \dots, n$ and $\mathbf{T} \,:\, \mathbb{L}_p(\mathbb{R}^+) \to \mathbb{L}_p(\mathbb{R}^+)$ is a compact operator.

Proof. The proof is a direct consequence of Theorem 5.1. □

Theorem 5.3. *Let $s \in \mathbb{R}$ and $1 < p < \infty$. If the conditions of Theorem 3.3 hold, then the Mellin convolution operator*

$$\mathbf{K}^2_c \,:\, \widetilde{\mathbb{H}}^r_p(\mathbb{R}^+) \to \mathbb{H}^r_p(\mathbb{R}^+), \tag{47}$$

is lifted to the equivalent operator in $\mathbb{L}_p(\mathbb{R}^+)$ space

$$\mathbf{\Lambda}^s_{-\gamma}\mathbf{K}^2_c\mathbf{\Lambda}^{-s}_{\gamma} = c^{-s}\left[\mathbf{K}^2_c - sc^{-1}\mathbf{K}^1_c\right] W_{g^s_{-c\gamma,\gamma}} + s\,\gamma\, c^{-s}\mathbf{K}^1_c W_{(\xi+\gamma)^{-1}g^{s-1}_{-c\gamma,\gamma}}\,, \tag{48}$$

where $c^{-s} = |c|^{-s}e^{-is\arg c}$, the function $g^s_{-c\gamma,\gamma}$ is defined in (38).

If the conditions of Corollary 3.4 hold, the Mellin convolution operator $\mathbf{K}^2_c$ between Bessel potential spaces (47) is lifted to the equivalent operator in the space $\mathbb{L}_p(\mathbb{R}^+)$

$$\mathbf{\Lambda}^s_{-\gamma}\mathbf{K}^2_c\mathbf{\Lambda}^{-s}_{\gamma} = \tag{49}$$
$$= c^{-s}W_{g^s_{-\gamma,-\gamma_0}}\left[\mathbf{K}^2_c - sc^{-1}\mathbf{K}^1_c\right] W_{g^s_{-c\gamma_0,\gamma}} + s\,\gamma\, c^{-s}W_{g^s_{-\gamma,-\gamma_0}}\mathbf{K}^1_c W_{(\xi+\gamma)^{-1}g^{s-1}_{-c\gamma_0,\gamma}}$$

$$= c^{-s}\left[\mathbf{K}^2_c - sc^{-1}\mathbf{K}^1_c\right] W_{g^s_{-\gamma,-\gamma_0}g^s_{-c\,\gamma_0,\gamma}} + s\,\gamma\, c^{-s}\mathbf{K}^1_c W_{(\xi-c\gamma_0)^{-1}g^s_{-c\gamma_0,-\gamma_0}g^s_{-\gamma,\gamma}} + \mathbf{T},$$

where the operator $\mathbf{T} : \mathbb{L}_p(\mathbb{R}^+) \to \mathbb{L}_p(\mathbb{R}^+)$ *is compact.*

Proof. If the conditions of Theorem 3.3 are satisfied, then $\operatorname{Im}\gamma > 0$ and $\operatorname{Im} c\,\gamma < 0$. Hence

$$\frac{1}{(t-c)^2} = \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon i}\left[\frac{1}{t-c-\varepsilon i} - \frac{1}{t-c+\varepsilon i}\right]$$

and we have

$$\begin{aligned}
\mathbf{\Lambda}^s_{-\gamma}\mathbf{K}^2_c\mathbf{\Lambda}^{-s}_\gamma &= \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon i}\mathbf{\Lambda}^s_{-\gamma}\left[\mathbf{K}^1_{c+\varepsilon i} - \mathbf{K}^1_{c-\varepsilon i}\right]\mathbf{\Lambda}^{-s}_\gamma \\
&= \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon i}\left[(c+\varepsilon i)^{-s}\mathbf{K}^1_{c+\varepsilon i}\mathbf{\Lambda}^s_{-(c+\varepsilon i)\gamma} - (c-\varepsilon i)^{-s}\mathbf{K}^1_{c-\varepsilon i}\mathbf{\Lambda}^s_{-(c-\varepsilon i)\gamma}\right]\mathbf{\Lambda}^{-s}_\gamma \\
&= \lim_{\varepsilon\to 0}\left\{\frac{(c+\varepsilon i)^{-s} - (c-\varepsilon i)^{-s}}{2\varepsilon i}\mathbf{K}^1_{c+\varepsilon i}\mathbf{\Lambda}^s_{-(c+\varepsilon i)\gamma}\right. \\
&\quad + (c-\varepsilon i)^{-s}\frac{1}{2\varepsilon i}\left[\mathbf{K}^1_{c+\varepsilon i} - \mathbf{K}^1_{c-\varepsilon i}\right]\mathbf{\Lambda}^s_{-(c+\varepsilon i)\gamma} \\
&\quad \left. + (c-\varepsilon i)^{-s}\mathbf{K}^1_{c-\varepsilon i}\frac{1}{2\varepsilon i}\left[\mathbf{\Lambda}^s_{-(c+\varepsilon i)\gamma} - \mathbf{\Lambda}^s_{-(c-\varepsilon i)\gamma}\right]\right\}\mathbf{\Lambda}^{-s}_\gamma \\
&= -s\,c^{-s-1}\mathbf{K}^1_c\mathbf{\Lambda}^s_{-c\,\gamma}\mathbf{\Lambda}^{-s}_\gamma + c^{-s}\mathbf{K}^2_c\mathbf{\Lambda}^s_{-c\,\gamma}\mathbf{\Lambda}^{-s}_\gamma \\
&\quad + c^{-s}\mathbf{K}^1_c\lim_{\varepsilon\to 0}\mathcal{F}^{-1}\frac{(\xi - c\,\gamma - \varepsilon\gamma i)^s - (\xi - c\,\gamma + \varepsilon\gamma i)^s}{2\varepsilon i}\mathcal{F}\mathbf{\Lambda}^{-s}_\gamma \\
&= c^{-s}\left[\mathbf{K}^2_c - sc^{-1}\mathbf{K}^1_c\right]W_{g^s_{-c\gamma,\gamma}} + s\,\gamma\,c^{-s}\mathbf{K}^1_c\mathbf{\Lambda}^{s-1}_{-c\,\gamma}\mathbf{\Lambda}^{-s}_\gamma \\
&= c^{-s}\left[\mathbf{K}^2_c - sc^{-1}\mathbf{K}^1_c\right]W_{g^s_{-c\gamma,\gamma}} + s\,\gamma\,c^{-s}\mathbf{K}^1_c W_{(\xi+\gamma)^{-1}g^{s-1}_{-c\gamma,\gamma}}.
\end{aligned}$$

Thus formula (48) is proved.

The formula (49) can be derived from (48) similarly Theorem 5.1. □

Remark 3. The operators $\mathbf{K}^n_c$, $n = 3, 4, \ldots$, can be treated analogously to the approach of Corollary 5.3. Indeed, let us represent the operator $\mathbf{K}^n_c$ in the form

$$\mathbf{K}^n_c\varphi = \lim_{\varepsilon\to 0}\mathbf{K}_{c_{1,\varepsilon},\ldots,c_{n,\varepsilon}}\varphi, \qquad \forall\,\varphi\in\widetilde{\mathbb{H}}^r_p(\mathbb{R}^+),$$

where

$$\begin{aligned}
\mathbf{K}_{c_{1,\varepsilon},\ldots,c_{n,\varepsilon}}\varphi(t) &:= \int_0^\infty \mathcal{K}_{c_{1,\varepsilon},\ldots,c_{n,\varepsilon}}\left(\frac{t}{\tau}\right)\varphi(\tau)\frac{d\tau}{\tau} = \sum_{j=1}^n d_j(\varepsilon)\mathbf{K}^1_{c_{j,\varepsilon}}\varphi(t), \\
\mathcal{K}_{c_{1,\varepsilon},\ldots,c_{m,\varepsilon}}(t) &:= \frac{1}{(t-c_{1,\varepsilon})\cdots(t-c_{n,\varepsilon})} = \sum_{j=1}^n\frac{d_j(\varepsilon)}{t-c_{j,\varepsilon}}, \qquad (50)\\
c_{j,\varepsilon} &= c(1+\varepsilon e^{i\omega_j}),\ \omega_j\in(-\pi,\pi),\ \arg c_{j,\varepsilon},\ \arg(c_{j,\varepsilon}\,\gamma)\neq 0,\ j = 1,\ldots,m.
\end{aligned}$$

Since $n \in \{3, 4, \dots\}$ the argument $\arg c$ does not vanish. Hence, the points $\omega_1, \dots, \omega_n \in (-\pi, \pi]$ are pairwise different, i.e., $\omega_j \neq \omega_k$ for $j \neq k$. By equating the numerators in the formula (50) we find the coefficients $d_1(\varepsilon), \dots, d_{n-1}(\varepsilon)$.

Note that the operators $\mathbf{K}^3_c, \mathbf{K}^4_c, \dots$ appear rather rarely in applications. Therefore, in this work exact formulae are given in the case of the operators $\mathbf{K}^1_c$ and $\mathbf{K}^2_c$ only.

Let $a_0, \dots, a_n, b_1, \dots, b_n \in C\mathfrak{M}_p(\overline{\mathbb{R}} \setminus \{0\})$, $d_0, c_1, \dots, c_n \in \mathbb{C}$ and consider the model operator $\mathbf{A} : \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) \to \mathbb{H}^s_p(\mathbb{R}^+)$,

$$\mathbf{A} := d_0 I + W_{a_0} + \sum_{j=1}^{n} W_{a_j} \mathbf{K}^1_{c_j} W_{b_j}, \tag{51}$$

comprising the identity I, Fourier $W_{a_0}, \dots, W_{a_n}$, $W_{b_1}, \dots, W_{b_n}$ and Mellin $\mathbf{K}^1_{c_1}, \dots, \mathbf{K}^1_{c_n}$ convolution operators. In order to ensure proper mapping properties of the operator $\mathbf{A} : \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) \to \mathbb{H}^s_p(\mathbb{R}^+)$, we additionally assume that if $s \leq 1/p - 1$ or if $s \geq 1/p$, then the functions $a_1(\xi), \dots, a_n(\xi)$ and $b_1(\xi), \dots, b_n(\xi)$ have bounded analytic extensions in the lower $\operatorname{Im} \xi < 0$ and the upper $\operatorname{Im} \xi > 0$ half planes, correspondingly.

If $1/p - 1 < s < 1/p$, then the spaces $\widetilde{\mathbb{H}}^s_p(\mathbb{R}^+)$ and $\mathbb{H}^s_p(\mathbb{R}^+)$ coincide (can be identified) and the analytic extendability assumptions are superfluous.

Now we can describe the symbol $\mathcal{A}^s_p$ of the model operator $\mathbf{A}$. According to the formulae (42) and (41) one has

$$\mathcal{A}^s_p(\omega) := d_0 \mathcal{I}^s_p(\omega) + \mathcal{W}^s_{a_0,p}(\omega) + \sum_{j=1}^{n} \mathcal{W}^0_{a_j,p}(\omega) \mathcal{K}^{1,s}_{c_j,p}(\omega) \mathcal{W}^0_{b_j,p}(\omega), \tag{52}$$

where the symbols $\mathcal{I}^s_p(\omega)$, $\mathcal{W}^0_{a,p}(\omega)$, $\mathcal{W}^s_{a,p}(\omega)$ and $\mathcal{K}^{1,s}_{c_j,p}(\omega)$ have the form

$$\mathcal{I}^s_p(\omega) := \begin{cases} g^s_{-\gamma,\gamma,p}(\infty, \xi), & \omega = (\infty, \xi) \in \overline{\Gamma}_1, \\ \left(\dfrac{\eta - \gamma}{\eta + \gamma}\right)^{\mp s}, & \omega = (\eta, \pm\infty) \in \Gamma^\pm_2, \\ e^{\pi s i}, & \omega = (0, \xi) \in \overline{\Gamma}_3, \end{cases} \tag{53a}$$

$$\mathcal{W}^0_{a,p}(\omega) := \begin{cases} a_p(\infty, \xi), & \omega = (\infty, \xi) \in \overline{\Gamma}_1, \\ a(\pm\eta), & \omega = (\eta, \pm\infty) \in \Gamma^\pm_2, \\ a_p(0, \xi), & \omega = (0, \xi) \in \overline{\Gamma}_3, \end{cases} \tag{53b}$$

$$\mathcal{W}^s_{a,p}(\omega) := \begin{cases} a^s_p(\infty, \xi), & \omega = (\infty, \xi) \in \overline{\Gamma}_1, \\ a(\pm\eta)\left(\dfrac{\eta - \gamma}{\eta + \gamma}\right)^{\mp s}, & \omega = (\eta, \pm\infty) \in \Gamma^\pm_2, \\ e^{\pi s i} a_p(0, \xi), & \omega = (0, \xi) \in \overline{\Gamma}_3, \end{cases} \tag{53c}$$

$$\mathcal{K}^{1,s}_{c,p}(\omega) := \begin{cases} \dfrac{e^{-i\pi\left(\frac{1}{p} - i\xi - 1\right)} c^{\frac{1}{p} - i\xi - s - 1}}{\sin[\pi(\frac{1}{p} - i\xi)]}, & \omega = (\infty, \xi) \in \overline{\Gamma}_1, \\ 0, & \omega \in \Gamma^\pm_2, \\ \dfrac{e^{-i\pi\left(\frac{1}{p} - i\xi - 1\right)} c^{\frac{1}{p} - i\xi - s - 1}}{\sin[\pi(\frac{1}{p} - i\xi)]}, & \omega = (0, \xi) \in \overline{\Gamma}_3, \end{cases} \tag{53d}$$

$$a_p^s(\infty,\xi) := \frac{e^{2\pi si}a(\infty)+a(-\infty)}{2} + \frac{e^{2\pi si}a(\infty)-a(-\infty)}{2i}\cot\left[\pi\Big(\frac{1}{p}-i\xi\Big)\right],$$

$$a_p(x,\xi) := \frac{a(x+0)+a(x-0)}{2} + \frac{a(x+0)-a(x-0)}{2i}\cot\left[\pi\Big(\frac{1}{p}-i\xi\Big)\right], \qquad x=0,\infty,$$

$$g^s_{-\gamma,\gamma,p}(\infty,\xi) := \frac{e^{2\pi si}+1}{2} + \frac{e^{2\pi si}-1}{2i}\cot\left[\pi\Big(\frac{1}{p}-i\xi\Big)\right]$$
$$= e^{\pi si}\frac{\sin\left[\pi\Big(\frac{1}{p}+s-i\xi\Big)\right]}{\sin\left[\pi\Big(\frac{1}{p}-i\xi\Big)\right]}, \qquad \xi\in\mathbb{R}, \quad \eta\in\mathbb{R}^+,$$

where $a(\infty\pm 0) := a(\pm\infty)$, $0<|\arg(c\gamma)|<\pi$,, $-\pi<\arg(c\gamma_0)<0$, $0<\arg c<\pi$, $0<\arg\gamma,\arg\gamma_0<\pi$ and $c^\delta = |c|^\delta e^{i\delta\arg c}$.

In the case where $a(-\infty)=1$ and $a(+\infty)=e^{2\pi\alpha i}$ the symbol $a_p^s(\infty,\xi)$ takes the form

$$a_p^s(\infty,\xi) = e^{\pi(s+\alpha)i}\frac{\sin\left[\pi\left(\frac{1}{p}+s+\alpha-i\xi\right)\right]}{\sin\left[\pi\Big(\frac{1}{p}-i\xi\Big)\right]}. \tag{53e}$$

Note that, the Mellin convolution operator $\mathbf{K}^1_{-1}$,

$$\mathbf{K}^1_{-1}\varphi(t) := \frac{1}{\pi}\int_0^\infty \frac{\varphi(\tau)\,d\tau}{t+\tau} = \mathfrak{M}^0_{\mathcal{M}_{\frac{1}{p}}\mathcal{K}^1_{-1}}, \quad \mathcal{M}_{\frac{1}{p}}\mathcal{K}^1_{-1}(\xi) = \frac{1}{\sin\left[\pi\left(\frac{1}{p}-i\xi\right)\right]},$$

which often appears in applications, has a rather simple symbol if considered in the Bessel potential space $\mathbb{H}^s_p(\mathbb{R}^+)$. Thus using formula (53d) with $c=-1$, one obtains

$$\mathcal{K}^{1,s}_{-1,p}(\omega) := \begin{cases} \dfrac{e^{\pi si}}{\sin[\pi(\beta-i\xi)]}, & \omega\in\overline{\Gamma}_1\cup\overline{\Gamma}_3,\\ 0, & \omega\in\Gamma_2^\pm. \end{cases}$$

Theorem 5.4. *Let $1<p<\infty$, $s\in\mathbb{R}$. The operator*

$$\mathbf{A} : \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) \longrightarrow \mathbb{H}^s_p(\mathbb{R}^+) \tag{54}$$

defined in (51) is Fredholm if and only if its symbol $\mathcal{A}^s_p(\omega)$ described by the relations (52), (53a)–(53e), is elliptic. If $\mathbf{A}$ is Fredholm, then

$$\mathrm{Ind}\mathbf{A} = -\mathrm{ind}\det\mathcal{A}^s_p.$$

The operator (54) is locally invertible at $0\in\mathbb{R}^+$ if and only if its symbol $\mathcal{A}^s_p(\omega)$, defined in (52), (53a)–(53e), is elliptic on Γ_1, i.e.

$$\inf_{\omega\in\Gamma_1}\left|\det\mathcal{A}^s_p(\omega)\right| = \inf_{\xi\in\mathbb{R}}\left|\det\mathcal{A}^s_p(\xi,\infty)\right| > 0.$$

Proof. Let $d_0, c_1,\dots,c_n\in\mathbb{C}$, $-\pi\leqslant\arg c_j<\pi$ $\arg c_j\neq 0$, for $j=1,\dots,n$. Lifting $\mathbf{A}$ up to an operator on the space $\mathbb{L}_p(\mathbb{R}^+)$ we get

$$\mathbf{\Lambda}^s_{-\gamma}\mathbf{A}\mathbf{\Lambda}^{-s}_{\gamma} = d_0\mathbf{\Lambda}^s_{-\gamma}\mathbf{\Lambda}^{-s}_{\gamma} + \mathbf{\Lambda}^s_{-\gamma}W_{a_0}\mathbf{\Lambda}^{-s}_{\gamma} + \sum_{j=1}^{n} W_{a_j}\mathbf{\Lambda}^s_{-\gamma}\mathbf{K}^1_{c_j}\mathbf{\Lambda}^{-s}_{\gamma}W_{b_j}, \tag{55}$$

where γ is such that $0 < \arg\gamma < \pi$, $-\pi < \arg(c_j\gamma) < 0$ for all $j = m+1, \ldots, n$.

In (55) we used special properties of convolution operators, namely,

$$\mathbf{\Lambda}^s_{-\gamma}W_{a_j} = W_{a_j}\mathbf{\Lambda}^s_{-\gamma}, \quad W_{b_j}\mathbf{\Lambda}^s_{\gamma} = \mathbf{\Lambda}^s_{\gamma}W_{b_j}, \quad \mathbf{\Lambda}^{\mp s}_{\pm\gamma} = W_{\lambda^{\mp s}_{\pm\gamma}},$$

which follows from the analytic extendability of functions $\lambda^s_{-\gamma}, a_1(\xi), \ldots, a_n(\xi)$ and $\lambda^{-s}_{\gamma}, b_1(\xi), \ldots, b_n(\xi)$ into the lower $\operatorname{Im}\xi < 0$ and upper $\operatorname{Im}\xi > 0$ half planes, respectively.

The model operators I, W_a and $\mathbf{K}^1_c$ lifted to the space $\mathbb{L}_p(\mathbb{R}^+)$ have the form

$$\mathbf{\Lambda}^s_{-\gamma}I\mathbf{\Lambda}^{-s}_{\gamma} = W_{g^s_{-\gamma,\gamma}}, \qquad \mathbf{\Lambda}^s_{-\gamma}W_a\mathbf{\Lambda}^{-s}_{\gamma} = W_{ag^s_{-\gamma,\gamma}},$$
$$\mathbf{\Lambda}^s_{-\gamma}\mathbf{K}^1_c\mathbf{\Lambda}^{-s}_{\gamma} = \begin{cases} c^{-s}\mathbf{K}^1_c W_{g^s_{-c\,\gamma,\gamma}} & \text{if} \quad -\pi < \arg(c\,\gamma) < 0, \\ c^{-s}\mathbf{K}^1_c W_{g^s_{-\,\gamma,-\gamma_0}g^s_{-c\,\gamma_0,\gamma}} + \mathbf{T}, & \text{if} \quad \begin{matrix} 0 < \arg(c\,\gamma) < \pi \\ -\pi < \arg(c\,\gamma_0) < 0, \end{matrix} \end{cases} \tag{56}$$

where $\mathbf{T}$ is a compact operator. Here, as above, $0 < \arg c < 2\pi$, $0 < \arg\gamma < \pi$, $0 < \arg\gamma_0 < \pi$ and either $-\pi < \arg(c\,\gamma) < 0$ or, if $-\pi < \arg(c\,\gamma) < 0$, then $-\pi < \arg(c\,\gamma_0) < 0$.

Therefore, the operator $\mathbf{\Lambda}^s_{-\gamma}\mathbf{A}\mathbf{\Lambda}^{-s}_{\gamma}$ in (55) can be rewritten as follows

$$\mathbf{\Lambda}^s_{-\gamma}\mathbf{A}\mathbf{\Lambda}^{-s}_{\gamma} = d_0W_{g^s_{-\gamma,\gamma}} + W_{a_0g^s_{-\gamma,\gamma}} + \sum_{j=1}^{m} c_j^{-s}W_{a_j}\mathbf{K}^1_{c_j}W_{g^s_{-c_j\gamma,-\gamma}}W_{b_j}$$
$$+ \sum_{j=m+1}^{n} c_j^{-s}W_{a_j}\mathbf{K}^1_{c_j}W_{g^s_{-\gamma,-\gamma_j}g^s_{-c_j\gamma_j,\gamma}}W_{b_j} + \mathbf{T} \;:\; \mathbb{L}_p(\mathbb{R}^+) \longrightarrow \mathbb{L}_p(\mathbb{R}^+), \tag{57}$$

where $\mathbf{T}$ is a compact operator and we ignore it when writing the symbol of $\mathbf{A}$.

Now we define the symbol of the initial operator $\mathbf{A} : \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) \to \mathbb{H}^s_p(\mathbb{R}^+)$ of (51) as the symbol of the corresponding lifted operator $\mathbf{\Lambda}^s_{-\gamma}\mathbf{A}\mathbf{\Lambda}^{-s}_{\gamma} : \mathbb{L}_p(\mathbb{R}^+) \to \mathbb{L}_p(\mathbb{R}^+)$ of (57).

To write the symbol of the lifted operator in the Lebesgue space $\mathbb{L}_p(\mathbb{R}^+)$ let us first find the limits of involved functions (symbols). The function $g^s_{-\gamma,\gamma} \in C(\mathbb{R})$ is continuous on $\mathbb{R}$, but has different limits at the infinity, viz.,

$$g^s_{-\gamma,\gamma}(-\infty) = 1, \quad g^s_{-\gamma,\gamma}(+\infty) = e^{2\pi si}, \qquad g^s_{-\gamma,\gamma}(0) = e^{\pi si}, \tag{58}$$

while the functions $g^s_{-\gamma,-\gamma_0}$, $g^s_{-c\gamma,\gamma}, g^s_{-c\gamma_0,\gamma} \in C(\mathbb{R})$ are continuous on $\mathbb{R}$ including the infinity. Thus

$$g^s_{-c\,\gamma,\gamma}(\pm\infty) = g^s_{-\gamma,-\gamma_0}(\pm\infty) = g^s_{-c\,\gamma_0,\gamma}(\pm\infty) = 1,$$
$$g^s_{-\gamma,-\gamma_0}(0)g^s_{-c\,\gamma_0,\gamma}(0) = \left(\frac{-\gamma}{-\gamma_0}\right)^s\left(\frac{-c\gamma_0}{\gamma}\right)^s = (-c)^s, \tag{59}$$
$$g^s_{-c\,\gamma,\gamma}(0) = (-c)^s \quad \text{if} \quad -0 < \arg c < 2\pi, \quad \arg c \neq 0.$$

In the Lebesgue space $\mathbb{L}_p(\mathbb{R}^+)$, the symbols of the first two operators in (57), are written according the formulae (42)–(41) by taking into account the equalities (58) and (59). The symbols of these operators have, respectively, the form (53a) and (53c).

For the operators $W_{a_1}, \ldots, W_{a_n}$ and $W_{b_1}, \ldots, W_{b_n}$ we can use the formulae (42)–(41) and write their symbols in the form (53b).

The lifted Mellin convolution operators

$$\mathbf{\Lambda}^s_\gamma \mathbf{K}^1_{c_j} \mathbf{\Lambda}^{-s}_\gamma \ : \ \mathbb{L}_p(\mathbb{R}^+) \longrightarrow, \mathbb{L}_p(\mathbb{R}^+)$$

comprise both Fourier convolution operators $W_{g^s_{-c_j\,\gamma_0,\gamma}}$ and $W_{g^s_{-\,\gamma,-\gamma_0} g^s_{-c_j\,\gamma_0,\gamma}}$ and Mellin convolution operator $\mathbf{K}^1_{c_j} = \mathfrak{M}^0_{\mathcal{K}^1_{c_j,p}(\xi)}$, with the symbol $\mathcal{K}^1_{c_j,p}(\xi) := \mathcal{M}_{1/p}\mathcal{K}^1_{c_j}(\xi)$ defined in (9) and (10). The symbol of the operators $\mathbf{\Lambda}^s_\gamma \mathbf{K}^1_{c_j} \mathbf{\Lambda}^{-s}_\gamma$ from (56) in the Lebesgue space $\mathbb{L}_p(\mathbb{R}^+)$ is found according formulae (42)–(41), has the form (53d) and is declared the symbol of $\mathbf{K}^1_{c_j} \ : \ \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) \to \mathbb{H}^s_p(\mathbb{R}^+)$. The symbols of Fourier convolution factors $W_{g^s_{-c_j\,\gamma_0,\gamma}}$ and $W_{g^s_{-\,\gamma,-\gamma_0} g^s_{-c_j\,\gamma_0,\gamma}}$, which participate in the symbol of $\mathbf{K}^1_{c_j} = \mathfrak{M}^0_{\mathcal{K}^1_{c_j,p}}$ are written again according formulae (42)–(41) by taking into account the equalities (58) and (59). Now Theorem 4.2 applies and that gives the result formulated in Theorem 5.4.

Concerning the concluding assertion of the theorem: $\mathbf{A}$ is, after lifting to $\mathbb{L}_p$-space, locally equivalent at 0 to the Mellin convolution operator $\mathfrak{M}^0_{\mathcal{A}^s_p(\infty,\xi)}$, which is locally invertible if and only if is globally invertible and this is the case iff $\inf\limits_{\xi\in\mathbb{R}} |\mathcal{A}^s_p(\infty,\xi)| > 0$. □

In the proof of the foregoing Theorem 5.4 a local principle is used. The definition of the local invertibility and a short introduction to a local principle can be found in [23,33].

The next results are concerned with the operators acting in the Sobolev–Slobodeckij (Besov) spaces. For the definition of the corresponding spaces $\mathbb{W}^s_p(\Omega) = \mathbb{B}^s_{p,p}(\Omega)$, $\widetilde{\mathbb{W}}^s_p(\Omega) = \widetilde{\mathbb{B}}^s_{p,p}(\Omega)$ for an arbitrary domain $\Omega \subset \mathbb{R}^n$, including the semi-axis $\mathbb{R}^+$, we refer the reader to the monograph [35].

Corollary 5.5. *Let $1 < p < \infty$, $s \in \mathbb{R}$. If the operator $\mathbf{A} : \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) \longrightarrow \mathbb{H}^s_p(\mathbb{R}^+)$, defined in (51), is Fredholm (invertible) for all $s \in (s_0, s_1)$ and $p \in (p_0, p_1)$, where $-\infty < s_0 < s_1 < \infty$, $1 < p_o < p_1 < \infty$, then the operator*

$$\mathbf{A} : \widetilde{\mathbb{W}}^s_p(\mathbb{R}^+) \longrightarrow \mathbb{W}^s_p(\mathbb{R}^+), \quad s \in (s_0, s_1), \quad p \in (p_0, p_1) \tag{60}$$

is Fredholm (invertible) in the Sobolev–Slobodeckij (Besov) spaces $\mathbb{W}^s_p = \mathbb{B}^s_{p,p}$, and

$$\operatorname{Ind}\mathbf{A} = -\operatorname{ind}\det \mathcal{A}^s_p. \tag{61}$$

Proof. Recall that the Sobolev–Slobodeckij (Besov) spaces $\mathbb{W}^s_p = \mathbb{B}^s_{p,p}$ emerge as the result of interpolation with the real interpolation method between Bessel potential spaces

$$\begin{aligned} \left(\mathbb{H}^{s_0}_{p_0}(\Omega), \mathbb{H}^{s_1}_{p_1}(\Omega)\right)_{\theta,p} &= \mathbb{W}^s_p(\Omega), \ \ s := s_0(1-\theta) + s_1\theta, \\ \left(\widetilde{\mathbb{H}}^{s_0}_{p_0}(\Omega), \widetilde{\mathbb{H}}^{s_1}_{p_1}(\Omega)\right)_{\theta,p} &= \widetilde{\mathbb{W}}^s_p(\Omega), \ \ p := \frac{1}{p_0}(1-\theta) + \frac{1}{p_1}\theta, \ \ 0 < \theta < 1. \end{aligned} \tag{62}$$

If $\mathbf{A} : \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) \longrightarrow \mathbb{H}^s_p(\mathbb{R}^+)$ is Fredholm (invertible) for all $s \in (s_0, s_1)$ and $p \in (p_0, p_1)$, it has a regularizer $\mathbf{R}$ (the inverse $\mathbf{A}^{-1} = \mathbf{R}$, respectively), which is bounded in the setting

$$\mathbf{R} : \mathbb{W}^s_p(\mathbb{R}^+) \longrightarrow \widetilde{\mathbb{W}}^s_p(\mathbb{R}^+)$$

due to the interpolation (62) and

$$\mathbf{RA} = I + \mathbf{T}_1, \quad \mathbf{AR} = I + \mathbf{T}_2,$$

where $\mathbf{T}_1$ and $\mathbf{T}_2$ are compact in $\widetilde{\mathbb{H}}^s_p(\mathbb{R}^+)$ and in $\mathbb{H}^s_p(\mathbb{R}^+)$, or $\mathbf{T}_1 = \mathbf{T}_2 = 0$ if $\mathbf{A}$ is invertible.

Due to the Krasnoselsky interpolation theorem (see [35]), $\mathbf{T}_1$ and $\mathbf{T}_2$ are compact in $\widetilde{\mathbb{W}}^s_p(\mathbb{R}^+)$ and in $\mathbb{W}^s_p(\mathbb{R}^+)$, respectively for all $s \in (s_0, s_1)$ and $p \in (p_0, p_1)$ and, therefore, $\mathbf{A}$ in (60) is Fredholm (is invertible, respectively).

The index formulae (61) follows from the embedding properties of the Sobolev–Slobodeckij and Bessel potential spaces by standard well-known arguments. □

Acknowledgments

The authors are greatly indebted to an anonymous referee whose careful reading of the manuscript, constructive criticism and helpful suggestions improved the manuscript and prevented the authors from embarrassing mistakes.

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