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Singular Integral Equations on Piecewise Smooth Curves in Spaces of Smooth Functions

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To Bernd Silbermann on the occasion of his sixtieth birthday

Abstract. We prove the boundedness of the Cauchy singular integral operator in modified weighted Sobolev $\mathbb{KW}_p^m(\Gamma,\rho)$, Hölder-Zygmund $\mathbb{KZ}_\mu^0(\Gamma,\rho)$, Bessel potential $\mathbb{KH}_p^s(\Gamma,\rho)$ and Besov $\mathbb{KB}_{p,q}^s(\Gamma,\rho)$ spaces under the assumption that the smoothness parameters m,μ,s are large. The underlying contour Γ is piecewise smooth with angular points and even with cusps. We obtain Fredholm criteria and an index formula for singular integral equations with piecewise smooth coefficients and complex conjugation in these spaces provided the underlying contour has angular points but no cusps. The Fredholm property and the index turn out to be independent of the integer parts of the smoothness parameters m,μ,s. The results are applied to an oblique derivative problem (the Poincaré problem) in plane domains with angular points and peaks on the boundary.

Introduction

When considering a Cauchy singular integral equation with complex conjugation

$$A\varphi(t) \equiv a(t)\varphi(t) + \frac{b(t)}{\pi i}\int_\Gamma \frac{\varphi(\tau)d\tau}{\tau - t} + \frac{c(t)}{\pi i}\int_\Gamma \frac{\overline{\varphi(\tau)d\tau}}{\overline{\tau} - \overline{t}} = f(t)\,, \quad t \in \Gamma \qquad (0.1)$$

on a piecewise smooth contour Γ (see § 1 below) we are restricted in the choice of the spaces where we can solve equation (0.1). Namely, the operator A in equation (0.1) is not bounded in important spaces of smooth functions: in the usual weighted Sobolev $\mathbb{W}_p^m(\Gamma,\rho)$, Hölder-Zygmund $\mathbb{Z}_\mu(\Gamma,\rho)$, Bessel potential $\mathbb{H}_p^s(\Gamma,\rho)$ and Besov $\mathbb{B}_{p,q}^s(\Gamma,\rho)$ spaces for large values of the smoothness parameters $m = 2,3,\ldots$, $\mu > 1$ and $|s| > 1+1/p$. These spaces cannot be even defined properly (i.e., independently of the choice of a parametrization) if Γ has knots, such as angular points or cusps.

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Even if Γ is sufficiently smooth and the spaces $\mathbb{W}_p^m(\Gamma,\rho)$, $\mathbb{Z}_\mu(\Gamma,\rho)$ etc. can be defined properly, the problem arises again when we take piecewise smooth coefficients $a(t)$, $b(t)$, $c(t)$ with jumps at the knots (for conciseness we relate discontinuity points to the knots of Γ as well).

On the other hand, especially in applications and numerical analysis, it is important to establish additional smoothness properties for the solutions at least outside the knots when the right-hand side f is sufficiently smooth.

We suggest the introduction of weighted spaces $\mathbb{KW}_p^m(\Gamma,\rho)$, $\mathbb{KZ}_\mu(\Gamma,\rho)$, $\mathbb{KH}_p^s(\Gamma,\rho)$, and $\mathbb{KB}_{p,q}^s(\Gamma,\rho)$ with the help of "Fuchs"-derivatives

$$\vartheta(t)\partial_t\varphi(t) := \vartheta(t)\frac{\partial\varphi(t)}{\partial t}, \quad \text{where} \quad \vartheta(t) := \prod_{t_j\in\mathcal{T}_\Gamma}(t-t_j) \tag{0.2}$$

and $\mathcal{T}_\Gamma$ is the collection of knots of Γ, instead of the usual derivatives $\partial_t\varphi(t)$ (see Lemmata 1.2, 1.3, and 2.4). It turns out that the operator A in (0.1) with piecewise smooth coefficients $a,b,c\in\mathbb{PC}^{\mathfrak{m}}(\Gamma,\mathcal{T}_\Gamma)$ (and even with $a,b,c\in\mathbb{KPC}^{\mathfrak{m}}(\Gamma,\mathcal{T}_\Gamma)$; see § 1 for the definitions) is bounded in the modified spaces $\mathbb{KW}_p^m(\Gamma,\rho)$, $\mathbb{KZ}_\mu(\Gamma,\rho)$, $\mathbb{KH}_p^s(\Gamma,\rho)$, and $\mathbb{KB}_{p,q}^s(\Gamma,\rho)$ provided the smoothness parameters m,μ and s are sufficiently large (see Lemmata 1.2, 1.3, 2.4 and Theorem 3.1). Moreover, the operator defined by (0.1) has one and the same kernel and cokernel in the spaces $\mathbb{KW}_p^{\widetilde{m}}(\Gamma,\rho)$, $\mathbb{KZ}_{\widetilde{\mu}}(\Gamma,\rho)$ and $\mathbb{KH}_p^{\widetilde{s}}(\Gamma,\rho)$, $\mathbb{KB}_{p,q}^{\widetilde{s}}(\Gamma,\rho)$ whatever the integer parts of the smoothness parameters $\widetilde{m}=0,\dots,m$, $0<\widetilde{\mu}\le\mu$, and $|\widetilde{s}|\le s$ are (see Theorem 3.2 and Remarks 3.4, 3.5).

The results on the Fredholm properties and also those on the boundedness of the operator A in the usual (non-modified) weighted Bessel potential and Besov spaces $\mathbb{H}_p^s(\Gamma,\rho)$ and $\mathbb{B}_{p,q}^s(\Gamma,\rho)$ for small s, $1/p-1<s<1/p$, when the multiplication by piecewise continuous functions represents a bounded operator (see Theorem 2.3), are new.

Although the space $\mathbb{KW}_p^m(\Gamma,\rho)$ coincides with $\mathbb{KH}_p^m(\Gamma,\rho)$ for any nonnegative integer m, we formulate the results for the modified Sobolev space $\mathbb{KW}_p^m(\Gamma,\rho)$ because these spaces are more common in applications and the proofs are simpler.

It is well known that the Bessel potential spaces are as natural in the theory of pseudodifferential operators as the Sobolev spaces are in the theory of partial differential operators. The norm in $\mathbb{H}_p^s(\mathbb{R}^2)$ is especially simple for even $s=2m, m=0,1,2,\dots$:

$$\|f\,|\,\mathbb{H}_p^{2m}(\mathbb{R}^2)\| = \|(I-\Delta)^m f|\mathbb{L}_p(\mathbb{R}^2)\|, \qquad \Delta=\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}.$$

But in the theory of boundary value problems we cannot confine ourselves to the Bessel potential spaces since the traces of functions $\Phi\in\mathbb{H}_p^s(\Omega^\pm)$ on the boundary belong to the Besov spaces $\mathbb{B}_{p,p}^{s-\frac{1}{p}}(\Gamma)$, provided the boundary Γ is sufficiently smooth and $s>1/p$.

The Besov spaces can be considered as the integral analogue of the Zygmund spaces.