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Original article

Dirichlet problem for Laplace–Beltrami equation on hypersurfaces—FEM approximation☆

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Abstract

We consider Dirichlet boundary value problem for Laplace–Beltrami Equation On Hypersurface $\mathscr{S}$, when the Laplace–Beltrami operator on the surface is described explicitly in terms of Günter's differential operators. Using the calculus of Günter's tangential differential operators on hypersurfaces we establish Finite Element Method for the considered boundary value problem and obtain approximate solution in explicit form.

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Let $\mathscr{S}$ be a C^2 smooth orientable surface in $\mathbb{R}^3$ with Lipschitz boundary $\partial\mathscr{S}$ given by an immersion

$$\zeta : \omega \to \mathscr{S}, \quad \omega \subset \mathbb{R}^2 \tag{1}$$

where ω is open simple connected domain in $\mathbb{R}^2$ with Lipschitz boundary $\partial\omega$ and let $\theta : \mathscr{S} \to \omega$ be the inverse mapping

$$\zeta \circ \theta = Id : \mathscr{S} \to \mathscr{S}, \quad \theta \circ \zeta = Id : \omega \to \omega.$$

Denote by $\nu(y),\ y \in \mathscr{S}$ the unit normal on $\mathscr{S}$ with the chosen orientation.

Günter's tangential derivatives $\mathscr{D}_j$ on $\mathscr{S}$ are defined by identities

$$\mathscr{D}_j := \partial_j - \nu_j(y)\partial_\nu,\ \ j = 1, 2, 3, \tag{2}$$

where $\partial_\nu = \sum_{k=1}^{3} \nu_k \partial_k$ denotes the normal derivative.

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Tangential derivatives can be applied to the definition of Sobolev spaces $\mathbb{W}^{\ell}_p(\mathscr{S}) = \mathbb{H}^{\ell}(\mathscr{S})$, $\ell \in \mathbb{N}^0$, $1 \le p < \infty$ on an ℓ-smooth surface $\mathscr{S}$ (see [1,2])

$$\mathbb{H}^{\ell}(\mathscr{S}) = \mathbb{W}^{\ell}_p(\mathscr{S}) := \left\{\varphi \in D'(\mathscr{S}) \; : \; \nabla^{\alpha}_{\mathscr{S}}\varphi \in \mathbb{L}_p(\mathscr{S}) \, , \; \forall \, \alpha \in \mathbb{N}^n_0 \, , \; |\alpha| \le \ell \right\}. \tag{3}$$

Equivalently, $\mathbb{W}^{\ell}_p(\mathscr{S})$ is the closure of the space $C^{\ell}(\mathscr{S})$ with respect to the norm

$$\|\varphi \,|\, \mathbb{W}^{\ell}_p(\mathscr{S})\| := \left[\sum_{|\alpha|\le\ell} \|\mathscr{D}_{\alpha}\varphi \,|\, \mathbb{L}_p(\mathscr{S})\|_p\right]^{1/p}.$$

The space $\mathbb{W}^{\ell}_p(\mathscr{S})$ can also be understood in distributional sense: derivative $\mathscr{D}_j\varphi \in \mathbb{L}_2(\mathscr{S})$ means that there exists a function in $\mathbb{L}_2(\mathscr{S})$ denoted by $\mathscr{D}_j\varphi$ such that

$$(\mathscr{D}_j\varphi, \psi) := (\varphi, \mathscr{D}^*_j\psi) := \int_{\mathscr{S}} \varphi(y)\overline{\mathscr{D}^*_j\psi(y)}\, d\sigma \qquad \forall\, \psi \in \mathbb{L}_2(\mathscr{S}).$$

Space $\mathbb{W}^{\ell}_2(\mathscr{S})$ is a Hilbert space with the scalar product

$$(\varphi, \; v)^{(\ell)}_{\mathscr{S}} := \sum_{|\alpha|\le\ell} \int_{\mathscr{S}} \mathscr{D}^{\alpha}_j\varphi(y)\overline{\mathscr{D}^{\alpha}_j \; v(y)}d\sigma. \tag{4}$$

Under the space $\mathbb{W}^{-\ell}_2(\mathscr{S})$ with a negative order $-\ell$, $\ell \in \mathbb{N}$, is understood, as usual, the dual space of distributions to the Sobolev space $\mathbb{W}^{\ell}_2(\mathscr{S})$.

Denote by $\boldsymbol{\Delta}_{\mathscr{S}}$ the Laplace–Beltrami operator on $\mathscr{S}$

$$\boldsymbol{\Delta}_{\mathscr{S}}\,\varphi = \sum_{j=1}^{3} \mathscr{D}^2_j\varphi \qquad \forall\, \varphi \in C^2(\mathscr{S}). \tag{5}$$

Note, that if $\varphi \in C^2_0(\mathscr{S})$, $\quad \psi \in C^1(\mathscr{S})$, then due to Kelvin—Stokes theorem

$$(-\boldsymbol{\Delta}_{\mathscr{S}}\varphi, \psi)_{\mathscr{S}} = \sum_{j=1}^{3}(\mathscr{D}\varphi_j, \mathscr{D}\psi_j)_{\mathscr{S}}. \tag{6}$$

From (6) immediately follows

Theorem 1. *If $\mathscr{S}$ is a C^1 smooth surface in $\mathbb{R}^3$, then Laplace–Beltrami operator*

$$-\boldsymbol{\Delta}_{\mathscr{S}} \; : \; \mathbb{W}^1_2(\mathscr{S}) \to \mathbb{W}^{-1}_2(\mathscr{S}) \tag{7}$$

is positive definite (see [3])

$$(-\boldsymbol{\Delta}_{\mathscr{S}}\varphi, \varphi)_{\mathscr{S}} = \sum_{k=1}^{3}(\mathscr{D}_k\varphi, \mathscr{D}_k\varphi)_{\mathscr{S}} = \|\nabla_{\mathscr{S}}\varphi\big|L_2(\mathscr{S})\|^2 > 0$$
$$\textit{for} \quad \forall\varphi \in \mathbb{W}^1_2(\mathscr{S}), \;\; \varphi \neq 0. \tag{8}$$

We consider the following Dirichlet boundary value problem for the Laplace–Beltrami equation

$$\begin{cases} \Delta_{\mathscr{S}} u(y) = f(y), & y \in \mathscr{S}, \\ u^+(y) = 0, & y \in \partial\mathscr{S}, \end{cases} \tag{9}$$

where $f \in \mathbb{L}_2(\mathscr{S})$.

From (6) follows variational formulation of (9):

Find a vector $\varphi \in \mathbb{H}^1_0(\mathscr{S})$ that

$$\sum_{k=1}^{3}(\mathscr{D}_k\varphi, \mathscr{D}_k\psi)_{\mathscr{S}} = -(f, \psi)_{\mathscr{S}} \qquad \forall\, \psi \in \mathbb{H}^{1/2}(\mathscr{S})^3. \tag{10}$$

Due to Theorem 1 and Poincaré inequality the sesquilinear form

$$a(\varphi, \psi) := \sum_{k=1}^{3}(\mathscr{D}_k\varphi, \mathscr{D}_k\psi)_{\mathscr{S}} \tag{11}$$

is bounded and coercive in $\mathbb{H}^1_0(\mathscr{S})$

$$M_1\left\|\varphi\middle|\mathbb{H}^1(\mathscr{S})\right\|^2 \geq a(\varphi,\varphi) \geq M\left\|\varphi\middle|\mathbb{H}^1(\mathscr{S})\right\|^2, \qquad \forall\, \varphi \in \mathbb{H}^1_0(\mathscr{S}), \tag{12}$$

for some $M > 0,\ M_1 > 0$, therefore problem (10) possesses a unique solution by Lax–Milgram Theorem (see [4]).

Now we describe the discrete counterpart of the problem (cf. [5]).

Let X_h be a family of finite dimensional subspaces approximating $\mathbb{H}^1(\mathscr{S})$, i.e., such that $\bigcup_h X_h$ is dense in $\mathbb{H}^1(\mathscr{S})$.

Consider Eq. (10) in the finite-dimensional space X_h

$$a(\varphi_h, \psi_h) = \widetilde{f}(\psi_h) \quad \forall \psi \in X_h, \tag{13}$$

where $\tilde{f}(\psi_h) = -(f, \psi_h)_{\mathscr{S}}$.

Theorem 2. *Eq. (13) has the unique solution $\varphi_h \in X_h$ for all $h > 0$. This solution converges in $\mathbb{H}^1(\mathscr{S})$ to the solution φ of (10) as $h \to 0$.*

Proof. From the coercivity of sesquilinear form a immediately follows

$$\begin{aligned} c_1\left\|\varphi_h\middle|\mathbb{H}^1(\mathscr{S})\right\|^2 &\leq a(\varphi_h,\varphi_h) = |\tilde{f}(\varphi_h)| \\ &\leq c_2\left\|\varphi_h\middle|\mathbb{H}^1(\mathscr{S})\right\| \ \text{ for all } \ h. \end{aligned} \tag{14}$$

Let φ_h be the unique solution of the homogeneous equation:

$$a(\varphi_h, \psi_h) = 0 \ \text{ for all } \ \psi_h \in X_h. \tag{15}$$

Then (14) implies $\left\|\varphi_h\middle|\mathbb{H}^1(\mathscr{S})\right\| = 0$ and consequently, $\varphi_h = 0$. Therefore Eq. (13) has a unique solution. From (14) it follows also that

$$\left\|\varphi_h\middle|\mathbb{H}^1(\mathscr{S})\right\|^2 \leq \frac{c_2}{c_1}\left\|\varphi_h\middle|\mathbb{H}^1(\mathscr{S})\right\|.$$

Hence sequence $\{\|\varphi_h|\mathbb{H}^1(\mathscr{S})\|\}$ is bounded and we can extract a subsequence $\{\varphi_{h_k}\}$ which converges weakly to some $\varphi \in \mathbb{H}^1(\mathscr{S})$.

Let us take an arbitrary $\psi \in \mathbb{H}^1(\mathscr{S})$ and for each $h > 0$ choose $\psi_h \in X_h$ such, that $\psi_h \to \psi$ in $\mathbb{H}^1(\mathscr{S})$. Then from (13) we have

$$a(\varphi, \psi) = \tilde{f}(\psi), \quad \forall \psi \in \mathbb{H}^1(\mathscr{S}).$$

Hence, φ solves (10). Note, that since (10) is uniquely solvable, each subsequence $\{\varphi_{h_k}\}$ converges weakly to the same solution φ, and consequently the whole sequence $\{\varphi_h\}$ also converges weakly to φ. Now let us prove that it converges in the space $\mathbb{H}^1(\mathscr{S})$.

Indeed, due to (14) we have

$$\begin{aligned} c_1\|\varphi_h - \varphi\|^2 &\leq |a(\varphi_h - \varphi, \varphi_h - \varphi)| \leq |a(\varphi_h, \varphi_h - \varphi) - a(\varphi, \varphi_h - \varphi)| \\ &= c_1|\tilde{f}(\varphi_h) - a(\varphi_h, \varphi) - \tilde{f}(\varphi_h - \varphi)| \to c_1|\tilde{f}(\varphi) - a(\varphi,\varphi)| = 0, \end{aligned}$$

which completes the proof. □

We can choose spaces X_h in different ways. As an example let us describe the discretization method based on the representation of the surface $\mathscr{S}$ as a network of the triangle-shaped elements.

Let $(U_\alpha, \varkappa_\alpha)$ be a parametrization of $\mathscr{S}$. Here $\varkappa_\alpha : U_\alpha \subset \mathbb{R}^2 \to \mathscr{S}$ are injective differentiable mappings (diffeomorphisms) of open sets U_α of $\mathbb{R}^2$ into $\mathscr{S}$ such that $\bigcup_\alpha \varkappa_\alpha(U_\alpha) = \mathscr{S}$.

Let $h > 0$. We call $\mathfrak{S}_h$ a triangulation of $\mathscr{S}$ if $\mathscr{S}$ is represented as $\mathscr{S} = \bigcup_{\mathscr{T}_\gamma \in \mathfrak{S}_h} \mathscr{T}_\alpha$, where the sets $\mathscr{T}_\gamma$ possess the following properties:

1. Each $\mathscr{T}_\gamma$ is a subset of some $\varkappa_{\alpha_\gamma}(U_{\alpha_\gamma})$ and $T_\gamma := \varkappa_{\alpha_\gamma}^{-1}(\mathscr{T}_\gamma) \subset U_{\alpha_\gamma}$ is a triangle.
2. If $T_\gamma = \varkappa_\alpha^{-1}(\mathscr{T}_\gamma)$ and $T_\delta = \varkappa_\alpha^{-1}(\mathscr{T}_\delta)$ are subsets of the same U_α, then their intersection can be only a common vertex or a side.
3. Sides of the triangles $\varkappa_{\alpha_\gamma}^{-1}(\mathscr{T}_\gamma)$ do not exceed h.

Denote by $\mathscr{N}_\mathfrak{S}$ the set of nodes of the triangulation $\mathfrak{S}$, i.e. the set of all points $\varkappa_\alpha(P_\beta) \in \mathscr{S}$, where P_β are vertices of the triangles T_γ. Let $\zeta : \mathscr{N}_\mathfrak{S} \to \mathbb{R}$ be any mapping of $\mathscr{N}_\mathfrak{S}$ into $\mathbb{R}$, then it can be easily proved that there exists function $v_\zeta \in \mathbb{H}^1(\mathscr{S})$ such that:

1. $r_{\mathscr{N}_\mathfrak{S}} v_\zeta = \zeta$.
2. The restriction of $v_\zeta \circ \varkappa_\alpha$ on T_α is an affine function: $v_\zeta \circ \varkappa_\alpha(x_1, x_2) = a_1 x_1 + a_2 x_2 + a_3$.

Denote by X_h the set of all such functions, corresponding to the triangulation $\mathfrak{S}_h$. The set X_h consists of the piecewise-linear functions and therefore $\bigcup_h X_h$ is dense in $\mathbb{H}^{1/2}(\mathscr{S})^3$.

We can replace the triangle-shaped elements in the above-described network by quadrilateral, hexagonal or other type polygonal elements.

In particular, consider a case, when $\omega = U_\alpha$ in the above parametrization is a square part of $\mathbb{R}^2$

$$\omega = \{(x_1, x_2) : \ 0 < x_1 < 1, \ 0 < x_2 < 1\}, \quad \zeta(\omega) = \mathscr{S}.$$

Allocate N^2 nodes $P_{ij} = (i/(N+1), \ j/(N+1)), \ i, j = 1, \dots, N$ on ω.

Let $\alpha_k, \ k = 1, \dots, N$ be piecewise linear functions defined on segment $[0, \ 1]$ as follows:

$$\alpha_k(x) = \begin{cases} 0, & 0 \le x \le \dfrac{k-1}{N+1}, \\ (N+1)\left(x - \dfrac{k-1}{N+1}\right), & \dfrac{k-1}{N+1} < x \le \dfrac{k}{N+1}, \\ (N+1)\left(\dfrac{k+1}{N+1} - x\right), & \dfrac{k}{N+1} < x \le \dfrac{k+1}{N+1}, \\ 0, & \dfrac{k+1}{N+1} < x \le 1, \\ j = k, \dots, N, \end{cases} \tag{16}$$

and denote by $\tilde{\varphi}_{ij}, \ i, j = 1, \dots, N$ functions

$$\tilde{\varphi}_{ij}(x_1, \ x_2) = \alpha_i(x_1)\alpha_j(x_2), \ i, j = 1, \dots, N, \quad (x_1, \ x_2) \in \omega. \tag{17}$$

Evidently, $\tilde{\varphi}_{ij}$ are continuous functions, which take their maximal value $\varphi_{ij}(P_{ij}) = 1$ at point P_{ij} and vanish outside the set

$$\omega_{ij} = \omega \cap \left\{(x_1, x_2) : \ 0 \le \left| x_1 - \frac{i}{N+1} \right| \le 1, \ 0 \le \left| x_2 - \frac{j}{N+1} \right| \le 1 \right\}, \tag{18}$$

consequently, they belong to $\mathbb{H}^1(\omega)$ and are linearly independent (see [6]).

Denote by X_N the linear span of the functions $\varphi_{ij} \circ \vartheta, \quad i, j = 1, \dots, N$. The space X_N is N^2-dimensional space contained into $\mathbb{H}^1(\mathscr{S})$.

Consider Eq. (13) in the space X_N.

$$a(\varphi, \psi) = \tilde{f}(\psi) \quad \forall \psi \in X_N. \tag{19}$$

We sought the solution $\varphi \in X_N$ of Eq. (19) in the form

$$\varphi = \sum_{i,j=1}^{N} C_{ij}\,\varphi_{ij}, \tag{20}$$

where C_{ij} are unknown coefficients. Substituting φ in (19) and replacing ψ successively by $\varphi_{ij},\ i,j = 1,\dots,N$, we get the equivalent system of N^2 linear algebraic equations

$$\sum_{i,j=1}^{N} A_{ijkl}\,C_{ij} = f_{kl}, \quad k,l = 1,\dots,N, \tag{21}$$

where

$$A_{ijkl} = a\Big(\varphi_{ij},\,\varphi_{kl}\Big), \qquad f_{kl} = \tilde{f}(\varphi_{kl}). \tag{22}$$

Matrix $A = A_{(ij)(kl)}$ is the Gram's matrix of the positive semidefinite bilinear form a, therefore it is a nonsingular matrix and Eq. (21) has a unique solution.

$$\varphi = \sum_{i,j,k,l=1}^{N} (A)^{-1}_{(ij)(kl)}\,\varphi_{ij}\,f_{kl}. \tag{23}$$

To calculate explicitly $A_{(ij)(kl)}$ and f_{kl} note, that

$$\begin{aligned}\mathscr{D}_m\varphi_{ij}(y) &= \partial_{y_m}\varphi_{ij}(y) + \nu_m\partial_\nu\varphi_{ij}(y)\\ &= \sum_{p=1}^{2}\partial_p\hat{\varphi}_{ij}(\vartheta(y))\left(\partial_m\vartheta_p(y) + \nu_m\sum_{l=1}^{3}\nu_l\partial_l\vartheta_p(y)\right)\\ &= \sum_{p=1}^{2}\partial_p\hat{\varphi}_{ij}(\vartheta(y))\mathscr{D}_m\vartheta_p(y),\end{aligned} \tag{24}$$

$$\begin{aligned}A_{ijkl} &= a\Big(\varphi_{ij},\,\varphi_{kl}\Big)\\ &= \sum_{p,q=1}^{2}\int_{\mathscr{S}}\big(\partial_p\hat{\varphi}_{ij}(\vartheta(y))\big)\big(\partial_q\hat{\varphi}_{kl}(\vartheta(y))\big)\sum_{m=1}^{3}\mathscr{D}_m\vartheta_p(y)\mathscr{D}_m\vartheta_q(y)d\sigma,\end{aligned} \tag{25}$$

$$f_{kl} = -(f,\,\hat{\varphi}_{kl})_{\mathscr{S}} = -\int_{\mathscr{S}} f(y)\,\hat{\varphi}_{kl}(y)\,d\sigma. \tag{26}$$

Changing variables $y = \zeta(x),\quad x = \vartheta(y)$ on right side of (25) and taking into account, that $\operatorname{supp}\big(\partial_p\hat{\varphi}_{ij}(x)\big) = \omega_{ij}$ we get

$$A_{ijkl} = \sum_{p,q=1}^{2}\int_{\omega_{ij}\cap\omega_{kl}}\big(\partial_p\hat{\varphi}_{ij}(x)\big)\big(\partial_q\hat{\varphi}_{kl}(x)\big)\sum_{m=1}^{3}\mathscr{D}_m\vartheta_p(\zeta(x))\mathscr{D}_m\vartheta_q(\zeta(x))|\sigma'(x)|dx, \tag{27}$$

$$f_{kl} = -\int_{\omega_{kl}} f(\zeta(x))\,\hat{\varphi}_{kl}(\zeta(x))|\sigma'(x)|dx \tag{28}$$

where $|\sigma'(x)|$ is a surface element of $\mathscr{S}$

$$|\sigma'(x)| = |\partial_1\vartheta(x)\times\partial_2\vartheta(x)|.$$

From (16)–(19), (27)–(28) we obtain explicit expressions of A_{ijkl} and f_{kl}, $1 \le i,\ j,\ k,\ l, \le N$

$$A_{ijkl} = 0, \quad \text{if}\ \ i < k-1 \ \text{ or }\ k < i-1 \ \text{ or }\ j < l-1 \ \text{ or }\ l < j-1,$$

$$A_{ijkl} = (N+1)^4 \sum_{p,q=1}^{2} \int_{\frac{i}{N+1}}^{\frac{i+1}{N+1}} \int_{\frac{j}{N+1}}^{\frac{j+1}{N+1}} \Big[\delta_{p1}\Big(x_2 - \frac{j+1}{N+1}\Big) + \delta_{p2}\Big(x_1 - \frac{i+1}{N+1}\Big)\Big]\Big[\delta_{q1}\Big(x_2 - \frac{j}{N+1}\Big)$$

$$+ \delta_{q2}\Big(x_1 - \frac{i}{N+1}\Big)\Big] \sum_{m=1}^{3} \mathscr{D}_m \vartheta_p(\zeta(x)) \mathscr{D}_m \vartheta_q(\zeta(x)) |\sigma'(x)| dx,$$

if $k = i+1, \quad l = j+1,$

$$A_{ijkl} = (N+1)^4 \sum_{p,q=1}^{2} \int_{\frac{i}{N+1}}^{\frac{i+1}{N+1}} \int_{\frac{j-1}{N+1}}^{\frac{j}{N+1}} \Big[\delta_{p1}\Big(\frac{j-1}{N+1} - x_2\Big) + \delta_{p2}\Big(\frac{i+1}{N+1} - x_1\Big)\Big]\Big[\delta_{q1}\Big(x_2 - \frac{j-1}{N+1}\Big)$$

$$+ \delta_{q2}\Big(x_1 - \frac{i}{N+1}\Big)\Big] \sum_{m=1}^{3} \mathscr{D}_m \vartheta_p(\zeta(x)) \mathscr{D}_m \vartheta_q(\zeta(x)) |\sigma'(x)| dx,$$

$$+ (N+1)^4 \sum_{p,q=1}^{2} \int_{\frac{i}{N+1}}^{\frac{i+1}{N+1}} \int_{\frac{j}{N+1}}^{\frac{j+1}{N+1}} \Big[\delta_{p1}\Big(\frac{j+1}{N+1} - x_2\Big) + \delta_{p2}\Big(\frac{i+1}{N+1} - x_1\Big)\Big]\Big[\delta_{q1}\Big(\frac{j+1}{N+1} - x_2\Big)$$

$$+ \delta_{q2}\Big(\frac{i}{N+1} - x_1\Big)\Big] \sum_{m=1}^{3} \mathscr{D}_m \vartheta_p(\zeta(x)) \mathscr{D}_m \vartheta_q(\zeta(x)) |\sigma'(x)| dx,$$

if $k = i+1, \quad l = j,$

$$A_{ijkl} = (N+1)^4 \sum_{p,q=1}^{2} \int_{\frac{i}{N+1}}^{\frac{i+1}{N+1}} \int_{\frac{j-1}{N+1}}^{\frac{j}{N+1}} \Big[\delta_{p1}\Big(\frac{j-1}{N+1} - x_2\Big) + \delta_{p2}\Big(\frac{i+1}{N+1} - x_1\Big)\Big]\Big[\delta_{q1}\Big(\frac{j}{N+1} - x_2\Big)$$

$$+ \delta_{q2}\Big(x_1 - \frac{i}{N+1}\Big)\Big] \sum_{m=1}^{3} \mathscr{D}_m \vartheta_p(\zeta(x)) \mathscr{D}_m \vartheta_q(\zeta(x)) |\sigma'(x)| dx,$$

if $k = i+1, \quad l = j-1,$

$$A_{ijkl} = (N+1)^4 \sum_{p,q=1}^{2} \int_{\frac{i-1}{N+1}}^{\frac{i}{N+1}} \int_{\frac{j}{N+1}}^{\frac{j+1}{N+1}} \Big[\delta_{p1}\Big(\frac{j+1}{N+1} - x_2\Big) + \delta_{p2}\Big(\frac{i-1}{N+1} - x_1\Big)\Big]\Big[\delta_{q1}\Big(\frac{j}{N+1} - x_2\Big)$$

$$+ \delta_{q2}\Big(\frac{i}{N+1} - x_1\Big)\Big] \sum_{m=1}^{3} \mathscr{D}_m \vartheta_p(\zeta(x)) \mathscr{D}_m \vartheta_q(\zeta(x)) |\sigma'(x)| dx,$$

if $k = i-1, \quad l = j+1,$

$$A_{ijkl} = (N+1)^4 \sum_{p,q=1}^{2} \int_{\frac{i-1}{N+1}}^{\frac{i}{N+1}} \int_{\frac{j-1}{N+1}}^{\frac{j}{N+1}} \Big[\delta_{p1}\Big(x_2 - \frac{j-1}{N+1}\Big) + \delta_{p2}\Big(x_1 - \frac{i-1}{N+1}\Big)\Big]\Big[\delta_{q1}\Big(x_2 - \frac{j-1}{N+1}\Big)$$

$$+ \delta_{q2}\Big(x_1 - \frac{i-1}{N+1}\Big)\Big] \sum_{m=1}^{3} \mathscr{D}_m \vartheta_p(\zeta(x)) \mathscr{D}_m \vartheta_q(\zeta(x)) |\sigma'(x)| dx,$$

$$+ (N+1)^4 \sum_{p,q=1}^{2} \int_{\frac{i-1}{N+1}}^{\frac{i}{N+1}} \int_{\frac{j}{N+1}}^{\frac{j+1}{N+1}} \Big[\delta_{p1}\Big(\frac{j+1}{N+1} - x_2\Big) + \delta_{p2}\Big(x_1 - \frac{i-1}{N+1}\Big)\Big]\Big[\delta_{q1}\Big(\frac{j+1}{N+1} - x_2\Big)$$

$$+ \delta_{q2}\Big(\frac{i-1}{N+1} - x_1\Big)\Big] \sum_{m=1}^{3} \mathscr{D}_m \vartheta_p(\zeta(x)) \mathscr{D}_m \vartheta_q(\zeta(x)) |\sigma'(x)| dx,$$

$$+ (N+1)^4 \sum_{p,q=1}^{2} \int_{\frac{i}{N+1}}^{\frac{i+1}{N+1}} \int_{\frac{j-1}{N+1}}^{\frac{j}{N+1}} \Big[\delta_{p1}\Big(x_2 - \frac{j-1}{N+1}\Big) + \delta_{p2}\Big(\frac{i+1}{N+1} - x_1\Big)\Big]\Big[\delta_{q1}\Big(\frac{j-1}{N+1} - x_2\Big)$$

$$+\delta_{q2}\Big(\frac{i+1}{N+1}-x_1\Big)\Big]\sum_{m=1}^{3}\mathscr{D}_m\vartheta_p(\zeta(x))\mathscr{D}_m\vartheta_q(\zeta(x))|\sigma'(x)|dx,$$

$$+(N+1)^4\sum_{p,q=1}^{2}\int_{\frac{i}{N+1}}^{\frac{i+1}{N+1}}\int_{\frac{j}{N+1}}^{\frac{j+1}{N+1}}\Big[\delta_{p1}\Big(\frac{j+1}{N+1}-x_2\Big)+\delta_{p2}\Big(\frac{i+1}{N+1}-x_1\Big)\Big]\Big[\delta_{q1}\Big(\frac{j+1}{N+1}-x_2\Big)$$

$$+\delta_{q2}\Big(\frac{i+1}{N+1}-x_1\Big)\Big]\sum_{m=1}^{3}\mathscr{D}_m\vartheta_p(\zeta(x))\mathscr{D}_m\vartheta_q(\zeta(x))|\sigma'(x)|dx,$$

if $k=i,\ l=j,$

$$A_{ijkl}=(N+1)^4\sum_{p,q=1}^{2}\int_{\frac{i-1}{N+1}}^{\frac{i}{N+1}}\int_{\frac{j-1}{N+1}}^{\frac{j}{N+1}}\Big[\delta_{p1}\Big(x_2-\frac{j-1}{N+1}\Big)+\delta_{p2}\Big(x_1-\frac{i-1}{N+1}\Big)\Big]\Big[\delta_{q1}\Big(\frac{j}{N+1}-x_2\Big)$$

$$+\delta_{q2}\Big(\frac{i-1}{N+1}-x_1\Big)\Big]\sum_{m=1}^{3}\mathscr{D}_m\vartheta_p(\zeta(x))\mathscr{D}_m\vartheta_q(\zeta(x))|\sigma'(x)|dx,$$

$$+(N+1)^4\sum_{p,q=1}^{2}\int_{\frac{i}{N+1}}^{\frac{i+1}{N+1}}\int_{\frac{j-1}{N+1}}^{\frac{j}{N+1}}\Big[\delta_{p1}\Big(\frac{j-1}{N+1}-x_2\Big)+\delta_{p2}\Big(\frac{i+1}{N+1}-x_1\Big)\Big]\Big[\delta_{q1}\Big(x_2-\frac{j}{N+1}\Big)$$

$$+\delta_{q2}\Big(x_1-\frac{i+1}{N+1}\Big)\Big]\sum_{m=1}^{3}\mathscr{D}_m\vartheta_p(\zeta(x))\mathscr{D}_m\vartheta_q(\zeta(x))|\sigma'(x)|dx,$$

if $k=i,\ l=j-1,$

$$A_{ijkl}=(N+1)^4\sum_{p,q=1}^{2}\int_{\frac{i-1}{N+1}}^{\frac{i}{N+1}}\int_{\frac{j-1}{N+1}}^{\frac{j}{N+1}}\Big[\delta_{p1}\Big(x_2-\frac{j-1}{N+1}\Big)+\delta_{p2}\Big(x_1-\frac{i-1}{N+1}\Big)\Big]\Big[\delta_{q1}\Big(\frac{j-1}{N+1}-x_2\Big)$$

$$+\delta_{q2}\Big(\frac{i}{N+1}-x_1\Big)\Big]\sum_{m=1}^{3}\mathscr{D}_m\vartheta_p(\zeta(x))\mathscr{D}_m\vartheta_q(\zeta(x))|\sigma'(x)|dx,$$

$$+(N+1)^4\sum_{p,q=1}^{2}\int_{\frac{i-1}{N+1}}^{\frac{i}{N+1}}\int_{\frac{j}{N+1}}^{\frac{j+1}{N+1}}\Big[\delta_{p1}\Big(x_2-\frac{j-1}{N+1}\Big)+\delta_{p2}\Big(x_1-\frac{i-1}{N+1}\Big)\Big]\Big[\delta_{q1}\Big(x_2-\frac{j+1}{N+1}\Big)$$

$$+\delta_{q2}\Big(x_1-\frac{i}{N+1}\Big)\Big]\sum_{m=1}^{3}\mathscr{D}_m\vartheta_p(\zeta(x))\mathscr{D}_m\vartheta_q(\zeta(x))|\sigma'(x)|dx,$$

if $k=i-1,\ l=j,$

$$A_{ijkl}=(N+1)^4\sum_{p,q=1}^{2}\int_{\frac{i-1}{N+1}}^{\frac{i}{N+1}}\int_{\frac{j-1}{N+1}}^{\frac{j}{N+1}}\Big[\delta_{p1}\Big(x_2-\frac{j-1}{N+1}\Big)+\delta_{p2}\Big(x_1-\frac{i-1}{N+1}\Big)\Big]\Big[\delta_{q1}\Big(x_2-\frac{j}{N+1}\Big)$$

$$+\delta_{q2}\Big(x_1-\frac{i}{N+1}\Big)\Big]\sum_{m=1}^{3}\mathscr{D}_m\vartheta_p(\zeta(x))\mathscr{D}_m\vartheta_q(\zeta(x))|\sigma'(x)|dx,$$

if $k=i-1,\ l=j-1,$

$$f_{kl}=-(N+1)^2\int_{\frac{k-1}{N+1}}^{\frac{k}{N+1}}\int_{\frac{l-1}{N+1}}^{\frac{l}{N+1}}\Big(x_1-\frac{k-1}{N+1}\Big)\Big(x_2-\frac{l-1}{N+1}\Big)f(\zeta(x))|\sigma'(x)|dx$$

$$-(N+1)^2\int_{\frac{k}{N+1}}^{\frac{k+1}{N+1}}\int_{\frac{l-1}{N+1}}^{\frac{l}{N+1}}\Big(\frac{k+1}{N+1}-x_1\Big)\Big(x_2-\frac{l-1}{N+1}\Big)f(\zeta(x))|\sigma'(x)|dx$$

$$-(N+1)^2\int_{\frac{k-1}{N+1}}^{\frac{k}{N+1}}\int_{\frac{l}{N+1}}^{\frac{l+1}{N+1}}\Big(x_1-\frac{k-1}{N+1}\Big)\Big(\frac{l+1}{N+1}-x_2\Big)f(\zeta(x))|\sigma'(x)|dx$$
$$-(N+1)^2\int_{\frac{k}{N+1}}^{\frac{k+1}{N+1}}\int_{\frac{l}{N+1}}^{\frac{l+1}{N+1}}\Big(\frac{k+1}{N+1}-x_1\Big)\Big(\frac{l+1}{N+1}-x_2\Big)f(\zeta(x))|\sigma'(x)|dx.$$

As an application of the aforementioned boundary value problem we can consider stationary state of heat conduction by an isotropic media, governed by the Laplace equations and constrained by classical Dirichlet–Neumann mixed boundary conditions for the Laplace equation in the layer domain $\Omega^\varepsilon := \mathscr{S}\times(-\varepsilon,\varepsilon)$ of thickness 2ε, where $\mathscr{S}\subset\mathscr{C}$ is a smooth subsurface of a closed hypersurface $\mathscr{C}$ with smooth nonempty boundary $\partial\mathscr{S}$

$$\Delta_{\Omega^\varepsilon}\varphi(y,t)=f(y,t),\quad (y,t)\in\mathscr{S}\times(-\varepsilon,\varepsilon), \tag{29}$$
$$\varphi^+(y,t)=g(y,t),\quad (y,t)\in\partial\mathscr{S}\times(-\varepsilon,\varepsilon), \tag{30}$$
$$\pm(\partial_t\varphi)^+(y,\pm\varepsilon)=q^\pm(y),\quad y\in\mathscr{S}, \tag{31}$$

where

$$\Delta_{\Omega^\varepsilon}\varphi=\sum_{j=1}^{4}\mathscr{D}_j^2\varphi+\mathscr{H}_{\mathscr{S}}^0\partial_\nu\varphi=\Delta_{\mathscr{S}}\varphi+\partial_t^2\varphi+\mathscr{H}_{\mathscr{S}}^0\partial_\nu\varphi$$

and $\mathscr{H}_{\mathscr{S}}^0$ is a Weingarten matrix

$$\mathscr{H}_{\mathscr{S}}^0(\mathscr{X}):=\big[\mathscr{D}_j\nu_k(\mathscr{X})\big]_{n\times n},\qquad \mathscr{X}\in\mathscr{S}. \tag{32}$$

It can be proved that when the thickness 2ε of the layer domain Ω^ε with the mid-surface $\mathscr{C}$, tends to zero, this boundary-value problem "converges" in the sense of Γ convergence to the following Dirichlet boundary value problem for Laplace–Beltrami equation on $\mathscr{S}$

$$\begin{array}{ll}\Delta_{\mathscr{S}}\varphi(y)=f_0(y) & y\in\mathscr{S},\\ \varphi^+(\tau)=g(\tau,0), & \tau\in\partial\mathscr{S},\end{array} \tag{33}$$

where

$$f_0(y):=f(y,0)-(\partial_t^2G)(y,0)-\frac{1}{2}[q^+(y)+q^-(y)]\frac{1}{2}\big[(\partial_tG)(y,0)-(\partial_tG)(y,0)\big] \tag{34}$$

and $G(x,t)$ is a continuation of the boundary data $g(y,t)$, $(y,t)\in\partial\mathscr{S}\times(-\varepsilon,\varepsilon)$, from the boundary into the domain $(x,t)\in\Omega^\varepsilon$.

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Further reading

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