The Review of Symbolic Logic
Volume 8, Number 1, March 2015

MODAL LOGICS OF METRIC SPACES

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Abstract. It is a classic result (McKinsey & Tarski, 1944; Rasiowa & Sikorski, 1963) that if we interpret modal diamond as topological closure, then the modal logic of any dense-in-itself metric space is the well-known modal system $\mathsf{S4}$. In this paper, as a natural follow-up, we study the modal logic of an arbitrary metric space. Our main result establishes that modal logics arising from metric spaces form the following chain which is order-isomorphic (with respect to the $\supset$ relation) to the ordinal $\omega + 3$:

$$\mathsf{S4.Grz}_1 \supset \mathsf{S4.Grz}_2 \supset \mathsf{S4.Grz}_3 \supset \cdots \mathsf{S4.Grz} \supset \mathsf{S4.1} \supset \mathsf{S4}.$$

It follows that the modal logic of an arbitrary metric space is finitely axiomatizable, has the finite model property, and hence is decidable.

§1. Introduction. In topological semantics of modal logic, modal diamond is interpreted as topological closure (and modal box as topological interior). It is well known that under such interpretation the modal system $\mathsf{S4}$ defines and is complete for the class of all topological spaces. It is a celebrated result of McKinsey & Tarski (1944) that $\mathsf{S4}$ is in fact the modal logic of any dense-in-itself separable metric space. Rasiowa & Sikorski (1963) showed that this result can be strengthened by dropping the separability assumption. To give credit to all four authors, we refer to this strengthened result as the MTRS-theorem.

Our main goal is to also drop the dense-in-itself assumption, and study the modal logic of an arbitrary metric space. As follows from Bezhanishvili & Harding (2012), there are infinitely many modal logics that arise this way. We recall that a Stone space is a zero-dimensional compact Hausdorff space. By Stone duality (Stone, 1936), Stone spaces are exactly the ultrafilter spaces of Boolean algebras, and a Boolean algebra is countable iff its dual Stone space is metrizable. Let X be a metrizable Stone space. As was shown in Bezhanishvili & Harding (2012), if X is not scattered, then the modal logic of X is $\mathsf{S4.1}$ or $\mathsf{S4}$ (depending on whether or not the set of isolated points is dense in X); and if X is scattered, then the modal logic of X is $\mathsf{S4.Grz}$ or $\mathsf{S4.Grz}_n$, $n \geq 1$ (depending on the Cantor-Bendixson rank of X). So Bezhanishvili & Harding (2012) axiomatizes the modal logic of an arbitrary metric Stone space.

Received: May 21, 2014.

doi:10.1017/S1755020314000446

Since metric spaces play an important role in topology, one should ask what happens if being Stone is dropped from the assumptions? That is, what modal logics can be realized by arbitrary metric spaces? Our main result shows that, surprisingly enough, no new modal logics arise even in this general case. Namely, we prove that modal logics arising from metric spaces form the following chain:

$$\mathsf{S4.Grz}_1 \supset \mathsf{S4.Grz}_2 \supset \mathsf{S4.Grz}_3 \supset \cdots \mathsf{S4.Grz} \supset \mathsf{S4.1} \supset \mathsf{S4}.$$

We briefly summarize our key techniques used in arriving at this general result. One of the standard tools we employ is building validity preserving maps from an arbitrary metric space onto suitable finite (counter)models. The difficulty lies in showing that such maps exist in the general setting. We utilize the Cantor-Bendixson decomposition paired with a powerful result of Telgarsky (1968) that each scattered metric space is strongly zero-dimensional.

In a little more detail, let X be a metric space. Using the Cantor-Bendixson theorem, we decompose X into scattered and dense-in-itself parts S and D. If D is empty, then $X = S$ is scattered, in which case we show that every finite tree of depth not exceeding the Cantor-Bendixson rank of X is an interior image of an open subspace of X. Thus, if the Cantor-Bendixson rank of X is infinite, then the modal logic of X is $\mathsf{S4.Grz}$; and if the Cantor-Bendixson rank of X is n, then the modal logic of X is $\mathsf{S4.Grz}_n$ (see Theorem 3.7).

On the other hand, if D is nonempty, then by the MTRS-theorem, the modal logic of D is $\mathsf{S4}$. But the modal logic of X varies depending on whether or not S is dense in X. If S is not dense in X, then as we show in Theorem 3.1, the modal logic of X is $\mathsf{S4}$. If S is dense in X, then the MTRS-theorem gives that there is an interior map from D onto every suitable finite model. Given such a mapping, thanks to Telgarsky's theorem, we show that S can be divided into clopen subsets, which are sufficiently well-behaved to allow us to extend the mapping to the whole X. This yields that the modal logic of X is $\mathsf{S4.1}$ (see Theorem 3.4).

As a corollary to our main result, we obtain that the modal logic of an arbitrary metric space is finitely axiomatizable, has the finite model property, and hence is decidable. We also axiomatize superintuitionistic logics arising from metric spaces.

§2. Background.

In this section we briefly recall the basic facts from modal logic and topology that will be used. We use Chagrov & Zakharyaschev (1997) and Blackburn *et al.* (2001) as basic references in modal logic and Engelking (1989) as a basic reference in topology.

2.1. Modal logic.

The modal logic $\mathsf{S4}$ is the least set of formulas containing classical tautologies, the axioms

$$\begin{aligned}&\Box(p \to q) \to (\Box p \to \Box q),\\ &\Box p \to p,\\ &\Box p \to \Box\Box p,\end{aligned}$$

and closed under Modus Ponens $\frac{\varphi,\ \varphi\to\psi}{\psi}$, substitution $\frac{\varphi(p_1,\dots,p_n)}{\varphi(\psi_1,\dots,\psi_n)}$, and necessitation $\frac{\varphi}{\Box\varphi}$. As usual, we use $\Diamond\varphi$ as an abbreviation for $\neg\Box\neg\varphi$.

For $n \geq 1$, define

$$\begin{aligned}\mathsf{bd}_1 &= \Diamond\Box p_1 \to p_1,\\ \mathsf{bd}_{n+1} &= \Diamond(\Box p_{n+1} \wedge \neg\mathsf{bd}_n) \to p_{n+1},\end{aligned}$$

and set

$$\begin{aligned} \mathsf{S4.1} &= \mathsf{S4} + \Box\Diamond p \to \Diamond\Box p, \\ \mathsf{S4.Grz} &= \mathsf{S4} + \Box(\Box(p \to \Box p) \to p) \to p, \\ \mathsf{S4.Grz}_n &= \mathsf{S4.Grz} + \mathsf{bd}_n. \end{aligned}$$

It is well known that these logics form the following chain:

$$\mathsf{S4.Grz}_1 \supset \mathsf{S4.Grz}_2 \supset \mathsf{S4.Grz}_3 \supset \cdots \mathsf{S4.Grz} \supset \mathsf{S4.1} \supset \mathsf{S4}.$$

This will become apparent after describing relational semantics of these logics.

2.2. Relational semantics. A *frame* $\mathfrak{F}$ is a pair (W, R) consisting of a nonempty set W and a binary relation $R \subseteq W \times W$. For $w \in W$, let $R(w) = \{v \in W : wRv\}$ and $R^{-1}(w) = \{v \in W : vRw\}$. We will only work with $\mathsf{S4}$-*frames*; that is, frames where the binary relation R is reflexive and transitive. Thus, $\mathsf{S4}$-frames are quasi-ordered sets (qosets), while antisymmetric $\mathsf{S4}$-frames are partially ordered sets (posets).

Recall that a qoset $\mathfrak{F}$ is *rooted* provided there is $r \in W$, called a *root* of $\mathfrak{F}$, such that $R(r) = W$. A *chain* A in a poset $\mathfrak{F}$ is a subset of W satisfying wRv or vRw for each $w, v \in A$. A *tree* is a rooted poset in which $R^{-1}(w)$ is a chain for each $w \in W$. The *height* or *depth* of a tree is $n \geq 1$ provided there is a chain with n elements and no chain has more than n elements. Call $v \in W$ a *child* of $w \in W$ provided v *covers* w; that is, wRv, $w \neq v$, and for each $u \in W$, from $wRuRv$ it follows that $w = u$ or $u = v$.

We recall that a *cluster* C in $\mathfrak{F}$ is a subset of W which is maximal with respect to set inclusion satisfying the property that wRv and vRw for each $v, w \in C$. The cluster generated by $w \in W$ is $C(w) = R(w) \cap R^{-1}(w)$. The set of all clusters of a qoset forms a partition of W. The relation R induces a partial ordering of the clusters of $\mathfrak{F}$ and the associated poset is known as the *skeleton* of $\mathfrak{F}$.

Call $\mathfrak{F}$ a *quasi-tree* or *qtree* provided the skeleton of $\mathfrak{F}$ is a tree. The *root cluster* of a qtree is the root of its skeleton. In a qtree, a cluster C is a *child cluster* of a cluster C' whenever C is a child of C' in its skeleton. The *height* or *depth* of a qtree is the height of its skeleton. Call $w \in W$ *quasi-maximal* (*maximal*) in $\mathfrak{F}$ provided wRv implies vRw ($w = v$) for all $v \in W$. We denote the quasi-maximal and maximal nodes of $\mathfrak{F}$ by $\mathrm{qmax}(\mathfrak{F})$ and $\max(\mathfrak{F})$, respectively. Note that $\max(\mathfrak{F}) \subseteq \mathrm{qmax}(\mathfrak{F})$ and the containment can be proper.

The modal language has a natural interpretation in a frame $\mathfrak{F}$. Namely formulas are interpreted as subsets, the classical connectives as Booleans, and for modal operators, we set:

$$\begin{aligned} w \vDash \Box\varphi \quad &\text{iff} \quad (\forall v \in W)(wRv \Rightarrow v \vDash \varphi), \\ w \vDash \Diamond\varphi \quad &\text{iff} \quad (\exists v \in W)(wRv \;\&\; v \vDash \varphi). \end{aligned}$$

If $w \models \varphi$, then we say that φ is *true* at $w \in W$. If $w \models \varphi$ for all $w \in W$, then we say that φ is *true* in $\mathfrak{F}$ under a given valuation. We call φ *valid* in $\mathfrak{F}$ if φ is true in $\mathfrak{F}$ under all valuations. It is well known that all formulas valid in a frame $\mathfrak{F}$ form a modal logic, called the modal logic of $\mathfrak{F}$ and denoted $L(\mathfrak{F})$. The modal logic of a class $\mathfrak{C}$ of frames is $L(\mathfrak{C}) = \bigcap\{L(\mathfrak{F}) : \mathfrak{F} \in \mathfrak{C}\}$. The following characterizes the logics of interest (proofs of these well-known facts can for example be found in Chagrov & Zakharyaschev, 1997; see also Bezhanishvili & Harding, 2012, prop. 2.5).

LEMMA 2.1.

1. S4 *is the modal logic of finite qtrees.*
2. S4.1 *is the modal logic of finite qtrees* $\mathfrak{F}$ *satisfying* $\mathrm{qmax}(\mathfrak{F}) = \max(\mathfrak{F})$.
3. S4.Grz *is the modal logic of finite trees.*
4. S4.Grz$_n$ *is the modal logic of finite trees of height* $\leq n$.

Let $\mathfrak{F} = (W, R)$ and $\mathfrak{G} = (V, S)$ be frames. Then $\mathfrak{G}$ is a *subframe* of $\mathfrak{F}$ provided $V \subseteq W$ and S is the restriction of R to V. We call $\mathfrak{G}$ a *generated subframe* if $w \in V$ and wRv imply $v \in V$. A map $f : W \to V$ is a *p-morphism* provided wRw' implies $f(w)Sf(w')$ and $f(w)Sv$ implies there is $w' \in R(w)$ such that $f(w') = v$. The first condition is usually referred to as the forth condition and the second one as the back condition of a p-morphism. If there is a p-morphism from $\mathfrak{F}$ onto $\mathfrak{G}$, then we call $\mathfrak{G}$ a *p-morphic image* of $\mathfrak{F}$. It is well known that generated subframes and p-morphic images are truth-preserving operations (see, e.g., Chagrov & Zakharyaschev, 1997). Therefore, if $\mathfrak{G}$ is either a generated subframe or a p-morphic image of $\mathfrak{F}$ and $\mathfrak{G} \not\models \varphi$, then $\mathfrak{F} \not\models \varphi$.

The next construction can be found in Bezhanishvili & Harding (2012, def. 3.10). Let $\mathfrak{G} = (V, S)$ be a finite qtree and let $C_1, \ldots, C_n$ be the maximal clusters of $\mathfrak{G}$ (i.e. each C_i is maximal in the skeleton of $\mathfrak{G}$). Let $m_1, \ldots, m_n$ be distinct and not in V. Define a new qtree $\mathfrak{F} = (W, R)$ by putting each m_i on top of C_i. More precisely, set $W = V \cup \{m_1, \ldots, m_n\}$ and let R be the least quasi-order on W containing S and (w, m_i) for each $w \in C_i$, $i = 1, \ldots, n$. We call $\mathfrak{F}$ obtained in this manner a *top thin quasi-tree* or *tt-qtree*; see Figure 1.

Note that any map $f : \mathfrak{F} \to \mathfrak{G}$ extending the identity on V and satisfying $f(m_i) \in C_i$ for $i = 1, \ldots, n$ is onto and satisfies the forth condition of a p-morphism. In addition, if each cluster C_i consists of a single node (that is, $\mathrm{qmax}(\mathfrak{G}) = \max(\mathfrak{G})$), then f also satisfies the back condition, and hence is an onto p-morphism. For a tt-qtree $\mathfrak{F} = (W, R)$, we denote by $\mathfrak{F}^-$ the subframe of $\mathfrak{F}$ whose underlying set is $W - \max(\mathfrak{F})$. Thus, $\mathfrak{F}$ is obtained from $\mathfrak{F}^-$ by the construction described above. Moreover, if $\mathrm{qmax}(\mathfrak{F}^-) = \max(\mathfrak{F}^-)$, then $\mathfrak{F}^-$ is a p-morphic image of $\mathfrak{F}$.

LEMMA 2.2 (Bezhanishvili & Harding, 2012). S4.1 *is the modal logic of tt-qtrees.*

Proof. Clearly each tt-qtree $\mathfrak{F}$ satisfies $\mathrm{qmax}(\mathfrak{F}) = \max(\mathfrak{F})$, so $\mathfrak{F} \models \Box\Diamond p \to \Diamond\Box p$. Let S4.1 $\nvdash \varphi$. By Lemma 2.1(2), there is a finite qtree $\mathfrak{G}$ such that $\mathrm{qmax}(\mathfrak{G}) = \max(\mathfrak{G})$ and $\mathfrak{G} \not\models \varphi$. Let $\mathfrak{F}$ be the tt-qtree obtained from $\mathfrak{G}$. Then $\mathfrak{G}$ is a p-morphic image of $\mathfrak{F}$. Thus, $\mathfrak{F}$ refutes φ, and the result follows. □

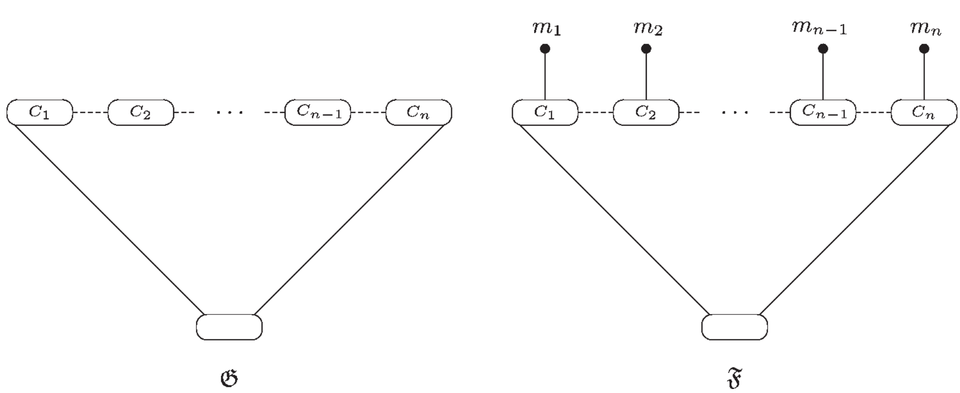

Fig. 1. Constructing a tt-qtree $\mathfrak{F}$ from $\mathfrak{G}$.

2.3. Topological semantics. Topological semantics generalizes relational semantics for S4. We assume that all topological spaces are nonempty. We interpret modal formulas as subsets of a topological space, the classical connectives as Booleans, $\Box$ as interior, and $\Diamond$ as closure. Then for a point x in a topological space X,

$$x \vDash \Box\varphi \quad \text{iff} \quad \text{there is an open neighborhood } U_x \text{ of } x \text{ such that } y \vDash \varphi \text{ for all } y \in U_x,$$
$$x \vDash \Diamond\varphi \quad \text{iff} \quad \text{for all open neighborhoods } U_x \text{ of } x \text{ there is } y \in U_x \text{ such that } y \vDash \varphi.$$

In fact, S4-frames are special topological spaces, called *Alexandrov spaces*, where the intersection of an arbitrary family of opens is open (alternatively, each point has a least open neighborhood). Indeed, for an S4-frame $\mathfrak{F} = (W, R)$, generated subframes of $\mathfrak{F}$ serve as (nonempty) opens of the Alexandrov topology of $\mathfrak{F}$, $\{R(w) : w \in W\}$ is a basis, and R^{-1} is the closure operator of the topology.

Truth and validity of a formula in a topological space is defined as in relational semantics. For a topological space X, we denote by $L(X)$ the modal logic of X (the set of formulas valid in X). The modal logic of a class $\mathfrak{C}$ of spaces is $L(\mathfrak{C}) = \bigcap\{L(X) : X \in \mathfrak{C}\}$.

In topological semantics the role of generated subframes and p-morphisms is played by open subspaces and interior maps. We recall that a map $f : X \to Y$ between topological spaces is *continuous* if V open in Y implies $f^{-1}(V)$ is open in X, and f is *open* if U open in X implies $f(U)$ is open in Y. We call f *interior* if it is both continuous and open. If f is an onto interior map, then we call Y an *interior image* of X. It is well known (see, e.g., van Benthem *et al.*, 2003, prop. 2.9) that open subspaces and interior images are truth-preserving operations on topological spaces. Therefore, to prove the MTRS-theorem that S4 is the modal logic of each dense-in-itself metric space X, by Lemma 2.1(1), it is sufficient to show that each finite qtree is an interior image of X. In fact, the main construction in the MTRS-theorem proves that each finite rooted S4-frame is an interior image of X. We will repeatedly utilize this fact.

2.4. Cantor-Bendixson decomposition. Let X be a topological space. We denote the interior and closure operators of X by $\mathbf{i}$ and $\mathbf{c}$, respectively. We recall that the *derivative* (or *limit point*) operator, denoted $\mathbf{d}$, is defined as follows: $x \in \mathbf{d}(A)$ iff each open neighborhood U_x of x contains a point of A different from x. The interior, closure, and derivative operators in a subspace Y of X are denoted $\mathbf{i}_Y$, $\mathbf{c}_Y$, and $\mathbf{d}_Y$, respectively. A subset A of X is *dense* if $\mathbf{c}(A) = X$, *dense-in-itself* if $A \subseteq \mathbf{d}(A)$ (equivalently $a \in \mathbf{c}(A - \{a\})$ for each $a \in A$), and *discrete* if $A \cap \mathbf{d}(A) = \varnothing$ (equivalently $a \notin \mathbf{c}(A - \{a\})$ for each $a \in A$).

A point $x \in X$ is *isolated* if $\{x\}$ is open in X. Let $\mathrm{iso}(X)$ be the set of isolated points of X. Then $\mathrm{iso}(X) = X - \mathbf{d}(X)$, so X is dense-in-itself iff $\mathrm{iso}(X) = \varnothing$. A space X is *scattered* if $\mathrm{iso}(Y) \neq \varnothing$ for each nonempty subspace Y of X. This immediately yields that if X is a scattered space, then $\mathrm{iso}(X)$ is dense in X. But there exist non-scattered spaces in which the isolated points are dense. We call a space X *weakly scattered* if $\mathrm{iso}(X)$ is dense in X.

By the Cantor-Bendixson theorem, each space X can be decomposed into the disjoint union of an open scattered and closed dense-in-itself subspaces. For $A \subseteq X$ and an ordinal α, define $\mathbf{d}^\alpha(A)$ by setting

$$\begin{aligned}
\mathbf{d}^0(A) &= A, \\
\mathbf{d}^\alpha(A) &= \mathbf{d}(\mathbf{d}^\beta(A)) \text{ if } \alpha = \beta + 1 \text{ is a successor}, \\
\mathbf{d}^\alpha(A) &= \bigcap\{\mathbf{d}^\beta(A) : \beta < \alpha\} \text{ if } \alpha \text{ is a limit}.
\end{aligned}$$

It is well known that X is scattered iff $\mathbf{d}^\alpha(X) = \varnothing$ for some ordinal α. Since $\mathbf{d}^\alpha(X) \subseteq \mathbf{d}^\beta(X)$ for any ordinals $\alpha \geq \beta$, there is an ordinal α such that $\mathbf{d}^\alpha(X) = \mathbf{d}^{\alpha+1}(X)$. The *rank* of X is $\varrho = \min\{\alpha : \mathbf{d}^\alpha(X) = \mathbf{d}^{\alpha+1}(X)\}$. Putting $D = \mathbf{d}^\varrho(X)$ and $S = X - D$ delivers the Cantor-Bendixson decomposition of X.

We note that the rank of X is 0 iff X is dense-in-itself, which happens iff $S = \varnothing$. Therefore, if X is scattered, then its rank is ≥ 1 (because we are only concerned with nonempty spaces). We view S as dissected into levels, the 'number' of levels being the rank ϱ of X. Specifically, set $X_\alpha = \mathbf{d}^\alpha(X) - \mathbf{d}^{\alpha+1}(X)$ for $\alpha < \varrho$. It is easy to see that each X_α is discrete in X and that X_α is the set of isolated points of the subspace $\mathbf{d}^\alpha(X)$ of X; that is, $X_\alpha = \mathrm{iso}(\mathbf{d}^\alpha X)$. See Figure 2.

2.5. Metric spaces. Let X be a metric space with the distance function $d : X \times X \to \mathbb{R}$. Then the open balls $B_\varepsilon(x) = \{y \in X : d(x, y) < \varepsilon\}$ form a basis for X. Therefore, for $x \in X$ and $A \subseteq X$, $x \in \mathbf{c}(A)$ iff there is a sequence $\{x_n\}$ in A converging to x. The distance from A to x is $d(A, x) = \inf\{d(a, x) : a \in A\}$. So $d(A, x) = 0$ iff $x \in \mathbf{c}(A)$. The next simple lemma will be of use for us.

LEMMA 2.3. *Let X be a metric space. If A is a discrete subset of X, then there is a pairwise disjoint family of open balls $\{B_{r_a}(a) : a \in A\}$.*

Proof. Let $a \in A$. Since A is discrete, $a \notin \mathbf{c}(A - \{a\})$, giving $d(A - \{a\}, a) = \varepsilon_a > 0$. Set $r_a = \frac{\varepsilon_a}{2}$ and consider $\{B_{r_a}(a) : a \in A\}$. Let $x \in B_{r_a}(a) \cap B_{r_a}(b)$ for some $a, b \in A$. Then

$$d(a, b) \leq d(a, x) + d(x, b) < r_a + r_b = \frac{\varepsilon_a}{2} + \frac{\varepsilon_b}{2} \leq \max(\varepsilon_a, \varepsilon_b).$$

By symmetry, we may proceed by considering only the case $\max(\varepsilon_a, \varepsilon_b) = \varepsilon_a$, so $d(a, b) < \varepsilon_a$. Since $b \in A$ and $d(A - \{a\}, a) = \varepsilon_a$, it follows that $b \in A - (A - \{a\}) = \{a\}$. This yields $a = b$, and hence $\{B_{r_a}(a) : a \in A\}$ is indeed pairwise disjoint. □

We recall that a subset A of a topological space X is a *zero-set* if there is a continuous function $f : X \to [0, 1]$ such that $A = f^{-1}(0)$. Complements of zero-sets are called *cozero-sets*. A Hausdorff space is *completely regular* if cozero-sets form a basis for the topology.

A subset A of X is *clopen* if it is closed and open, and a Hausdorff space is *zero-dimensional* if it has a basis of clopens. Given two covers $\mathcal{U} = \{U_i : i \in I\}$ and $\mathcal{V} = \{V_j : j \in J\}$ of X, we say that $\mathcal{V}$ is a *refinement* of $\mathcal{U}$ provided for each $i \in I$ there is $j \in J$ such that $V_j \subseteq U_i$.

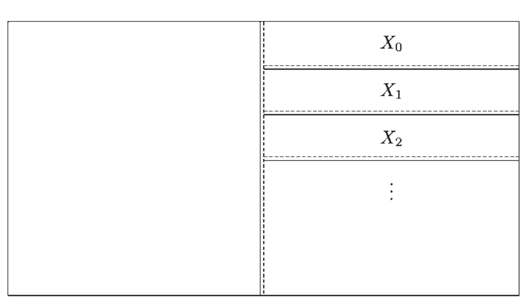

Fig. 2. Cantor-Bendixson decomposition of $X = S \cup D$ and the levels X_α of S.

A completely regular space X is *strongly zero-dimensional* provided every finite cover of X consisting of cozero-sets has a finite pairwise disjoint open refinement. (Clearly the refinement consists of clopen sets.) It is well known that each strongly zero-dimensional space is zero-dimensional, but the converse is not true in general.

In what follows we will use essentially Telgarsky's theorem, Telgársky (1968), that each scattered metric space is strongly zero-dimensional.

REMARK 2.4. In fact, Telgarsky proves (see Telgársky, 1968, cor. 3) that if X is a scattered paracompact space, then $\dim(X) = 0$. Since each metric space is paracompact (Engelking, 1989, theorem 5.1.3) and $\dim(X) = 0$ iff X is strongly zero-dimensional (Engelking, 1989, p. 385), it follows that each scattered metric space is strongly zero-dimensional.

§3. Main results. In this section we axiomatize the modal logic of each metric space. Let X be a metric space. Either X is weakly scattered or not. If X is not weakly scattered, then we use the MTRS-theorem to show that $L(X) = \mathsf{S4}$. If X is weakly scattered, then either X is scattered or not. If X is not scattered, then we use the Cantor-Bendixson decomposition and Telgarsky's theorem to show that $L(X) = \mathsf{S4.1}$. Finally, if X is scattered, then either the rank of X is finite or infinite. If the rank of X is infinite, then we show that $L(X) = \mathsf{S4.Grz}$, and if the rank of X is $n \in \omega - \{0\}$, then we show that $L(X) = \mathsf{S4.Grz}_n$.

3.1. The non-weakly scattered case. Let X be a non-weakly scattered metric space. Set $U = X - \mathbf{c}(\mathrm{iso}(X))$. Then U is a nonempty dense-in-itself open subspace of X. Therefore, U is a dense-in-itself metric space. By the MTRS-theorem, $L(U) = \mathsf{S4}$. But since U is an open subspace of X, we have $L(X) \subseteq L(U)$. Thus, $\mathsf{S4} \subseteq L(X) \subseteq L(U) = \mathsf{S4}$, and we arrive at the following.

THEOREM 3.1. *If X is a non-weakly scattered metric space, then $L(X) = \mathsf{S4}$.*

3.2. The weakly scattered non-scattered case. We prove that each tt-qtree is an interior image of a weakly scattered non-scattered metric space. For this we require the following lemma, which will also be useful in the scattered case.

LEMMA 3.2. *Let $k \in \omega - \{0\}$. For any finite pairwise disjoint family $\{F_i : i < k\}$ of nonempty closed subsets of a strongly zero-dimensional normal space X, there is a clopen partition $\{U_i : i < k\}$ of X such that $F_i \subseteq U_i$ for each $i < k$.*

Proof. By induction on k. If $k = 1$, then the family consists of a single nonempty closed set F, and we take the clopen partition $\{X\}$. Clearly $F \subseteq X$.

Next suppose the result is true for $k \geq 1$. Let $\{F_i : i \leq k\}$ be a pairwise disjoint family of nonempty closed subsets of X. Then $\{F_i,\ F_{k-1} \cup F_k : i < k-1\}$ is a pairwise disjoint family consisting of k nonempty closed subsets of X. The inductive hypothesis delivers a clopen partition $\{U_i, U : i < k-1\}$ of X such that $F_i \subseteq U_i$ when $i < k-1$ and $F_{k-1} \cup F_k \subseteq U$. Since X is normal and the closed sets F_{k-1} and F_k are disjoint, Urysohn's lemma yields a continuous function $f : X \to [0, 1]$ such that $F_{k-1} \subseteq f^{-1}(0)$ and $F_k \subseteq f^{-1}(1)$. Since X is strongly zero-dimensional, by Engelking (1989, lem. 6.2.2), there is a clopen set $V \subseteq X$ such that $F_{k-1} \subseteq V \subseteq X - F_k$. Setting $U_{k-1} = U \cap V$ and $U_k = U - V$ gives a clopen partition $\{U_i : i \leq k\}$ of X such that $F_i \subseteq U_i$ for each $i \leq k$. ☐

LEMMA 3.3. *Each tt-qtree is an interior image of any weakly scattered non-scattered metric space.*

Proof. Let $\mathfrak{F}$ be a tt-qtree. Then $\mathfrak{F}^-$ is a finite qtree. Let X be a weakly scattered non-scattered metric space and let $X = S \cup D$ be the Cantor-Bendixson decomposition of X, where S is an open scattered subspace, D is a closed dense-in-itself subspace, and $S \cap D = \varnothing$. Since X is weakly scattered, $S \neq \varnothing$; and since X is non-scattered, $D \neq \varnothing$. As D is a dense-in-itself metric space, it follows from the MTRS-theorem that there is an onto interior map $g : D \to \mathfrak{F}^-$. We show there exists an onto interior map $f : X \to \mathfrak{F}$ extending g such that $f(S) = \max(\mathfrak{F})$. We proceed by (strong) induction on the height of $\mathfrak{F} = (W, R)$. Since $\mathfrak{F}$ is a tt-qtree, its smallest height is 2.

Base Case: Suppose the height of $\mathfrak{F}$ is 2. Since $\mathfrak{F}$ is a tt-qtree of height 2, $\max(\mathfrak{F}) = \{m\}$ and $\mathfrak{F}^-$ is the root cluster C_r of $\mathfrak{F}$. We extend $g : D \to \mathfrak{F}^-$ to $f : X \to \mathfrak{F}$ by setting $f(x) = m$ for each $x \in S$, see Figure 3.

Clearly f is a well-defined onto map extending g such that $f(S) = \max(\mathfrak{F})$. Since $\{m\}$ is the only nonempty proper generated subframe of $\mathfrak{F}$ and $f^{-1}(m) = S$, we see that f is continuous. To see that f is open, let U be a nonempty open in X. If $U \subseteq S$, then $f(U) = \{m\}$. Otherwise, since $X = S \cup D$ is weakly scattered, both $U \cap D$ and $U \cap \mathrm{iso}(X) \subseteq U \cap S$ are nonempty. Therefore, because g is interior,

$$f(U) = f(U \cap S) \cup f(U \cap D) = f(U \cap S) \cup g(U \cap D) = \{m\} \cup C_r = W.$$

Thus, f is open.

Inductive Step: Suppose the height of $\mathfrak{F}$ is greater than 2. Assume that for any finite tt-qtree $\mathfrak{G}$ of lesser height than $\mathfrak{F}$, for any weakly scattered non-scattered metric space Y whose Cantor-Bendixson decomposition is $Y = S' \cup D'$, and for any onto interior map $h' : D' \to \mathfrak{G}^-$, there is an onto interior map $h : Y \to \mathfrak{G}$ extending h' and satisfying $h(S') = \max(\mathfrak{G})$.

We must extend $g : D \to \mathfrak{F}^-$ to an onto interior map $f : X \to \mathfrak{F}$ such that $f(S) = \max(\mathfrak{F})$. Let $C_1, \dots, C_k$ be the children clusters of the root cluster C_r of $\mathfrak{F}$. For $i = 1, \dots, k$, let $\mathfrak{F}_i$ be the generated subframe of $\mathfrak{F}$ whose underlying set is $R(C_i)$. Then $\mathfrak{F}_i$ is a finite tt-qtree and $\mathfrak{F}_i$ has lesser height than $\mathfrak{F}$. Set $F = g^{-1}(C_r)$ and $D_i = g^{-1}(R_{\mathfrak{F}^-}(C_i))$, where $R_{\mathfrak{F}^-}$ is the restriction of R to $\mathfrak{F}^-$. Then $D - F = D_1 \cup \cdots \cup D_k$. Put $Y = S \cup (D - F)$, see Figure 4.

Since D is closed in X and $g : D \to \mathfrak{F}^-$ is interior, F is closed in X and each D_i is open in D. Moreover, Y is open in X as $Y = X - F$. Furthermore, D_i is closed in Y. To see this, observe that

$$\begin{aligned}\mathbf{c}(D_i) &= \mathbf{c}_D(D_i) = \mathbf{c}_D(g^{-1}(R_{\mathfrak{F}^-}(C_i))) \\ &= g^{-1}(R^{-1}(R_{\mathfrak{F}^-}(C_i))) = g^{-1}(R_{\mathfrak{F}^-}(C_i) \cup C_r) = D_i \cup F.\end{aligned}$$

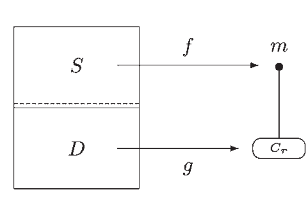

Fig. 3. Extending g to f when the height of $\mathfrak{F}$ is 2.

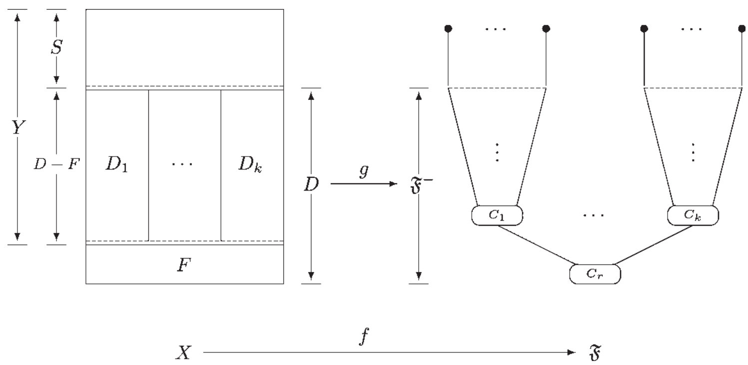

Fig. 4. Extending g to f when the height of $\mathfrak{F}$ is greater than 2.

Therefore, $D_i = \mathbf{c}(D_i) \cap Y$, and so D_i is closed in Y. Since Y is a metric space and $D_1, \ldots, D_k$ are disjoint closed subsets of Y, (Rasiowa & Sikorski, 1963, chap. III, theorem 6.1) yields open subsets $U_1, \ldots, U_k$ of Y such that $D_i \subseteq U_i$ for each i and $\mathbf{c}_Y(U_1), \ldots, \mathbf{c}_Y(U_k)$ are also disjoint.

Because Y is open in X, each U_i is open in X. Clearly $D_i \subseteq U_i \cap D$. To see the converse, let $x \in U_i \cap D$. Then $x \in D = X - S$ and $x \in U_i \subseteq Y = S \cup (D - F)$, giving that $x \in D - F$. So $x \in D_j$ for some j. Therefore, $x \in D_j \subseteq U_j \subseteq \mathbf{c}_Y(U_j)$, yielding $\mathbf{c}_Y(U_i) \cap \mathbf{c}_Y(U_j) \neq \varnothing$. Thus, $i = j$, so $x \in D_i$, and hence $D_i = U_i \cap D$.

Set $F_i = \mathbf{c}_Y(U_i) \cap S$. Then $F_1, \ldots, F_k$ are disjoint nonempty closed subsets in S. Since S is a scattered metric space, it follows from Telgarsky's theorem that S is strongly zero-dimensional. By Lemma 3.2, there is a clopen partition $S_1, \ldots, S_k$ of S such that $F_i \subseteq S_i$ for each i. Since S is open in X, each S_i is open in X.

Set $Y_i = S_i \cup U_i$ for each i. Since both U_i and S_i are open in X, Y_i is open in X. Moreover, as $U_i \cap S \subseteq F_i \subseteq S_i$ and $U_i \cap D = D_i$, we have

$$\begin{aligned} Y_i &= S_i \cup U_i = S_i \cup (U_i \cap X) = S_i \cup (U_i \cap (S \cup D)) \\ &= S_i \cup (U_i \cap S) \cup (U_i \cap D) = S_i \cup D_i. \end{aligned}$$

Figure 5 depicts the clopen partition $S_1, \ldots, S_k$ of S and demonstrates the sets Y_i.

The Cantor-Bendixson decomposition of Y_i is realized through D_i and S_i. Indeed, D_i is dense-in-itself since D_i is an open subset of the dense-in-itself space D, and S_i is scattered

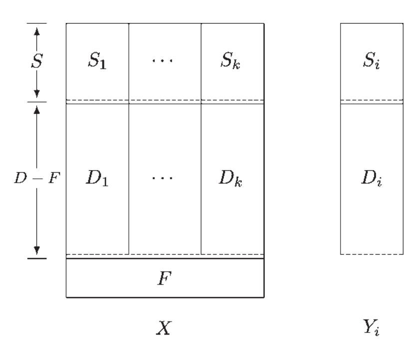

Fig. 5. Clopen partition of S and $Y_i = S_i \cup D_i$.

as S_i is a subset of the scattered space S. Therefore, if ϱ is the rank of S, then for any $\alpha \geq \varrho$, we have

$$(\mathbf{d}_{Y_i})^\alpha(Y_i) = (\mathbf{d}_{Y_i})^\alpha(S_i \cup D_i) = (\mathbf{d}_{Y_i})^\alpha(S_i) \cup (\mathbf{d}_{Y_i})^\alpha(D_i) = D_i.$$

Since an open subspace of a weakly scattered space is weakly scattered, Y_i is weakly scattered. Furthermore, Y_i is not scattered since $D_i \neq \varnothing$. So Y_i is a weakly scattered metric space that is not scattered.

Let $g_i = g|_{D_i}$. Clearly $g_i : D_i \to \mathfrak{F}_i^-$ is onto and continuous. It is also open because D_i is open in $D = \mathrm{dom}(g)$. Therefore, g_i is an onto interior map. By the inductive hypothesis, there is an onto interior map $f_i : Y_i \to \mathfrak{F}_i$ extending g_i such that $f_i(S_i) = \max(\mathfrak{F}_i)$. Define $f : X \to W$ by setting $f(x) = g(x)$ on $F = g^{-1}(C_r)$ and $f(x) = f_i(x)$ on Y_i. Since $Y_1, \ldots, Y_k, F$ is a partition of X, f is well defined. Moreover,

$$\begin{aligned} f(X) &= f(Y_1) \cup \cdots \cup f(Y_k) \cup f(F) = f_1(Y_1) \cup \cdots \cup f_k(Y_k) \cup g(F) \\ &= R(C_1) \cup \cdots \cup R(C_k) \cup C_r = W, \end{aligned}$$

showing that f is onto. Clearly f extends g on F. Let $x \in D_i$. Then

$$f(x) = f_i(x) = g_i(x) = g(x),$$

showing that f does indeed extend g. Furthermore,

$$\begin{aligned} f(S) &= f(S_1 \cup \cdots \cup S_k) = f(S_1) \cup \cdots \cup f(S_k) = f_1(S_1) \cup \cdots \cup f_k(S_k) \\ &= \max(\mathfrak{F}_1) \cup \cdots \cup \max(\mathfrak{F}_k) = \max(\mathfrak{F}). \end{aligned}$$

It remains to prove that f is interior. Let $w \in W$. If $w \in R(C_i)$ for some i, then $R(w) \subseteq R(C_i)$, so $f^{-1}(R(w)) = (f_i)^{-1}(R(w))$ is open in Y_i. Since Y_i is open in X, we conclude that $f^{-1}(R(w))$ is open in X. If $w \notin R(C_i)$ for all i, then $w \in C_r$, so $R(w) = W$, and hence $f^{-1}(R(w)) = f^{-1}(W) = X$ is open in X. Thus, f is continuous.

Let U be open in X. If $U \cap F = \varnothing$, then $U \subseteq Y_1 \cup \cdots \cup Y_k$, so

$$\begin{aligned} f(U) &= f((U \cap Y_1) \cup \cdots \cup (U \cap Y_k)) = f(U \cap Y_1) \cup \cdots \cup f(U \cap Y_k) \\ &= f_1(U \cap Y_1) \cup \cdots \cup f_k(U \cap Y_k). \end{aligned}$$

Since $U \cap Y_i$ is open in Y_i and $f_i : Y_i \to \mathfrak{F}_i$ is interior, $f_i(U \cap Y_i)$ is a generated subframe of $\mathfrak{F}_i$. But $\mathfrak{F}_i$ is a generated subframe of $\mathfrak{F}$, so each $f_i(U \cap Y_i)$ is a generated subframe of $\mathfrak{F}$, and hence $f(U)$ is a generated subframe of $\mathfrak{F}$.

Suppose that $U \cap F \neq \varnothing$. Then $U \cap D \neq \varnothing$, so $g(U \cap D) \neq \varnothing$. Since $g(F) = C_r$, we obtain $g(U \cap D) \cap C_r \neq \varnothing$. As $U \cap D$ is open in D and $g : D \to \mathfrak{F}^-$ is interior, $g(U \cap D) = W - \max(\mathfrak{F})$. Since the height of $\mathfrak{F}$ is greater than 2, each C_i belongs to $\mathfrak{F}^-$. Therefore, $C_i \subseteq g(U \cap D) \subseteq f(U)$. We show that $C_i \subseteq f(U \cap Y_i)$. Let $x \in U$ with $f(x) \in C_i$. Then $x \notin S$ because $f(S) = \max(\mathfrak{F})$. This implies $x \in D$ and $f(x) = g(x)$. Thus, $g(x) \in C_i$, giving $x \in D_i$. This yields $x \in Y_i$, so $C_i \subseteq f(U \cap Y_i)$. Therefore, $C_i \subseteq f_i(U \cap Y_i)$. Since f_i is interior, this implies $R(C_i) \subseteq f_i(U \cap Y_i) \subseteq f(U)$. We also have $f(U) \supseteq g(U \cap D) \supseteq C_r$. Thus, $f(U) = W$. Consequently, f is open, and hence interior. □

THEOREM 3.4. *If X is a weakly scattered non-scattered metric space, then $L(X) =$* S4.1.

Proof. Let X be a weakly scattered non-scattered metric space. Since X is weakly scattered, it is well-known (see, e.g., Bezhanishvili & Harding, 2012, sec. 2) that $\mathsf{S4.1} \subseteq L(X)$. Conversely, if $\mathsf{S4.1} \nvdash \varphi$, then by Lemma 2.2, φ is refuted on a finite tt-qtree $\mathfrak{F}$. By Lemma 3.3, $\mathfrak{F}$ is an interior image of X. Therefore, X refutes φ. Thus, $\mathsf{S4.1} = L(X)$. □

3.3. The scattered case. For a scattered space X and $n \in \omega$, let $X_n^{\uparrow} = \bigcup_{m \le n} X_m$, where we recall that $X_n = \mathbf{d}^n(X) - \mathbf{d}^{n+1}(X) = \text{iso}(\mathbf{d}^n(X))$ and each X_n is a discrete subset of X.

LEMMA 3.5. *Let $n \in \omega$, $\mathfrak{T}$ be a finite tree of height at most $n+1$, and let X be a scattered metric space such that $X_n \neq \varnothing$. Then there is an onto interior map $f : X_n^{\uparrow} \to \mathfrak{T}$ such that $f(x)$ is the root of $\mathfrak{T}$ for each $x \in X_n$.*

Proof. The proof is by induction on $n \in \omega$.

Base case: If $n = 0$, then $\mathfrak{T}$ consists of a single reflexive root r, and $X_0^{\uparrow} = X_0$ is a discrete space. Clearly sending the entire X_0 to r yields an onto interior map $f : X_0^{\uparrow} \to \mathfrak{T}$ satisfying $f(x)$ is the root of $\mathfrak{T}$ for each $x \in X_0$.

Inductive case: Let $n \in \omega - \{0\}$. Suppose that for each finite tree of height at most n and each scattered metric space Y with $Y_{n-1} \neq \varnothing$, there is an interior map from $Y_{n-1}^{\uparrow}$ onto the tree which maps Y_{n-1} to the root of the tree. Let $\mathfrak{T}$ be a finite tree of height $n+1$ and let X be a scattered space such that $X_n \neq \varnothing$. We must show that there is an interior map $f : X_n^{\uparrow} \to \mathfrak{T}$ such that $f(x)$ is the root of $\mathfrak{T}$ for each $x \in X_n$.

Let r be the root of $\mathfrak{T}$ and let $c_1, \ldots, c_k$ be the children of r. For each i let $\mathfrak{T}_i$ be the subtree of $\mathfrak{T}$ generated by c_i. Then the underlying set of $\mathfrak{T}_i$ is $R(c_i)$, and the height of $\mathfrak{T}_i$ is at most n.

Since X_n is discrete, Lemma 2.3 delivers a pairwise disjoint family of open balls $\{B(x) : x \in X_n\}$. Because X is a metric space, for each $x \in X_n$, there is a sequence $\{a_i^x\} \subseteq X_{n-1}$ converging to x. By taking an appropriate tail of the sequence, without loss of generality we may assume that $\{a_i^x\} \subseteq B(x)$.

We partition each $\{a_i^x\}$ into k subsequences, $\sigma_1^x, \ldots, \sigma_k^x$, each converging to x. For each $i \le k$, let $F_i = \bigcup_{x \in X_n} \sigma_i^x$. Then $X_n \subseteq \mathbf{c}(F_i)$ for each $i \le k$. Moreover, $F_1, \ldots, F_k$ are pairwise disjoint and closed in $X_{n-1}^{\uparrow}$. Since $X_{n-1}^{\uparrow}$ is a scattered metric space (because it is a subspace of X), by Telgarsky's theorem, $X_{n-1}^{\uparrow}$ is strongly zero-dimensional. Therefore, Lemma 3.2 delivers a clopen partition $A_1, \ldots, A_k$ of $X_{n-1}^{\uparrow}$ such that $F_i \subseteq A_i$ for each i. Clearly each A_i is a scattered metric space. Since each A_i is open in X, we have

$$\mathbf{d}_{A_i}^{n-1}(A_i) - \mathbf{d}_{A_i}^{n}(A_i) = \left(\mathbf{d}^{n-1}(A_i) - \mathbf{d}^n(A_i)\right) \cap A_i = A_i \cap X_{n-1} \supseteq F_i \neq \varnothing.$$

Therefore, the inductive hypothesis applies, by which, for each $i \le k$ there is $f_i : A_i \to \mathfrak{T}_i$ such that $f_i(A_i \cap X_{n-1}) = \{c_i\}$.

Define $f : X_n^{\uparrow} \to \mathfrak{T}$ by setting $f(x) = r$ if $x \in X_n$ and $f(x) = f_i(x)$ if $x \in A_i$. Clearly f is well-defined because $A_1, \ldots, A_k, X_n$ is a partition of $X_n^{\uparrow}$. Moreover, f is onto since

$$\begin{aligned} f\left(X_n^{\uparrow}\right) &= f(A_1) \cup \cdots \cup f(A_k) \cup f(X_n) = f_1(A_1) \cup \cdots \cup f_k(A_k) \cup f(X_n) \\ &= \mathfrak{T}_1 \cup \cdots \cup \mathfrak{T}_k \cup \{r\} = \mathfrak{T}. \end{aligned}$$

It is left to prove that f is interior. Let $t \in \mathfrak{T}$ and consider $f^{-1}(R(t))$. If $t = r$, then $f^{-1}(R(t)) = f^{-1}(\mathfrak{T}) = X_n^{\uparrow}$ is open in $X_n^{\uparrow}$. If $t \neq r$, then $t \in \mathfrak{T}_i$ for some $i \leq k$. Therefore, $f^{-1}(R(t)) = (f_i)^{-1}(R(t))$ is open in A_i, and since A_i is open in $X_n^{\uparrow}$, it follows that $f^{-1}(R(t))$ is open in $X_n^{\uparrow}$. Thus, f is continuous.

Let U be open in $X_n^{\uparrow}$. Let $i \leq k$. Then $U \cap A_i$ is open in A_i. Since f_i is interior, $f_i(U \cap A_i)$ is a generated subframe of $\mathfrak{T}_i$. But $\mathfrak{T}_i$ is a generated subframe of $\mathfrak{T}$. Therefore, $f_i(U \cap A_i)$ is a generated subframe of $\mathfrak{T}$. If $U \cap X_n = \varnothing$, then

$$\begin{aligned} f(U) &= f((U \cap A_1) \cup \cdots \cup (U \cap A_k)) = f(U \cap A_1) \cup \cdots \cup f(U \cap A_k) \\ &= f_1(U \cap A_1) \cup \cdots \cup f_k(U \cap A_k), \end{aligned}$$

showing that $f(U)$ is a union of generated subframes of $\mathfrak{T}$, hence a generated subframe of $\mathfrak{T}$.

Suppose $U \cap X_n \neq \varnothing$. Let $x \in U \cap X_n$. Since a_i^x converges to x, a tail of the sequence is in U. Therefore, $U \cap A_i \cap X_{n-1} \neq \varnothing$, and so $f(U \cap A_i) = f_i(U \cap A_i)$ is a generated subframe of $\mathfrak{T}_i$ containing the root c_i of $\mathfrak{T}_i$. This yields $f(U \cap A_i) = \mathfrak{T}_i$ for each $i \leq k$. Thus,

$$\begin{aligned} f(U) &= f((U \cap A_1) \cup \cdots \cup (U \cap A_k) \cup (U \cap X_n)) \\ &= f(U \cap A_1) \cup \cdots \cup f(U \cap A_k) \cup f(U \cap X_n) \\ &= \mathfrak{T}_1 \cup \cdots \cup \mathfrak{T}_k \cup \{r\} = \mathfrak{T}. \end{aligned}$$

Consequently, f is open, and hence interior. □

LEMMA 3.6. *Let X be a scattered space and let V be a valuation. For each $n \in \omega$, we have $X_n^{\uparrow} \subseteq V(\mathsf{bd}_{n+1})$.*

Proof. By induction on $n \in \omega$. For the base case, suppose that $n = 0$. Let $A_1 = V(p_1)$. Then $V(\mathsf{bd}_1) = (X - \mathbf{ci}(A_1)) \cup A_1$. Since $X_0^{\uparrow} = X_0$, we must show that $X_0 \subseteq V(\mathsf{bd}_1)$. Let $x \in X_0$. Then $\{x\}$ is open in X. If $x \notin X - \mathbf{ci}(A_1)$, then $x \in \mathbf{ci}(A_1)$. Since $\{x\}$ is open, $\{x\} \cap \mathbf{i}(A_1) \neq \varnothing$. Therefore, $x \in \mathbf{i}(A_1) \subseteq A_1$. Thus, $X_0 \subseteq V(\mathsf{bd}_1)$.

Let $n \geq 0$. Suppose that $X_n^{\uparrow} \subseteq V(\mathsf{bd}_{n+1})$. We must show that $X_{n+1}^{\uparrow} \subseteq V(\mathsf{bd}_{n+2})$. Let $V(p_{n+2}) = A_{n+2}$. Then

$$V(\mathsf{bd}_{n+2}) = (X - \mathbf{c}(\mathbf{i}(A_{n+2}) - V(\mathsf{bd}_{n+1}))) \cup A_{n+2}.$$

Let $x \in X_{n+1}^{\uparrow}$. If $x \notin X - \mathbf{c}(\mathbf{i}(A_{n+2}) - V(\mathsf{bd}_{n+1}))$, then $x \in \mathbf{c}(\mathbf{i}(A_{n+2}) - V(\mathsf{bd}_{n+1}))$. Consider the set $U = \{x\} \cup X_n^{\uparrow}$. Since $X_n^{\uparrow}$ is open and X_{n+1} is discrete, U is open in X. As $x \in \mathbf{c}(\mathbf{i}(A_{n+2}) - V(\mathsf{bd}_{n+1}))$, there is $y \in U \cap (\mathbf{i}(A_{n+2}) - V(\mathsf{bd}_{n+1}))$. Because $y \notin V(\mathsf{bd}_{n+1})$ and $X_n^{\uparrow} \subseteq V(\mathsf{bd}_{n+1})$, we have $y \notin X_n^{\uparrow}$. But $y \in U = \{x\} \cup X_n^{\uparrow}$, so $x = y$. Therefore, $x \in \mathbf{i}(A_{n+2}) \subseteq A_{n+2}$. Thus, $X_{n+1}^{\uparrow} \subseteq V(\mathsf{bd}_{n+2})$. □

THEOREM 3.7. *Let X be a scattered metric space.*

1. *If the rank of X is infinite, then $L(X) = \mathsf{S4.Grz}$.*
2. *If the rank of X is $n \in \omega - \{0\}$, then $L(X) = \mathsf{S4.Grz}_n$.*

Proof. (1) It is well known (see, e.g., Esakia, 1981) that if X is scattered, then $\mathsf{S4.Grz} \subseteq L(X)$. Conversely, suppose that $\mathsf{S4.Grz} \nvdash \varphi$. By Lemma 2.1(3), there is a finite tree $\mathfrak{T}$ refuting φ. Let $n \geq 1$ be the height of $\mathfrak{T}$. Since the rank of X is infinite, X_{n-1} is a nonempty subset of X. By Lemma 3.5, $\mathfrak{T}$ is an interior image of $X_{n-1}^{\uparrow}$. Therefore, $X_{n-1}^{\uparrow}$ refutes φ. Since $X_{n-1}^{\uparrow}$ is an open subspace of X, we conclude that X also refutes φ. Thus, $L(X) =$ S4.Grz.

(2) By (1), S4.Grz $\subseteq L(X)$. By Lemma 3.6, $X_{n-1}^{\uparrow} \vDash \mathsf{bd}_n$. Since the rank of X is n, we have $X = X_{n-1}^{\uparrow}$. Therefore, $X \vDash \mathsf{bd}_n$. Thus, $\mathsf{S4.Grz}_n \subseteq L(X)$. Conversely, suppose that $\mathsf{S4.Grz}_n \nvdash \varphi$. By Lemma 2.1(4), there is a tree $\mathfrak{T}$ of height at most n refuting φ. By Lemma 3.5, $\mathfrak{T}$ is an interior image of X. Therefore, X refutes φ. Thus, $L(X) = \mathsf{S4.Grz}_n$. □

Combining Theorems 3.1, 3.4, and 3.7 yields:

THEOREM 3.8 (Main Theorem). *Let X be a metric space.*

1. *If X is not weakly scattered, then* $L(X) =$ S4.
2. *If X is weakly scattered but not scattered, then* $L(X) =$ S4.1.
3. *If X is scattered and has infinite rank, then* $L(X) =$ S4.Grz.
4. *If X is scattered and has rank* $n \in \omega - \{0\}$, *then* $L(X) = \mathsf{S4.Grz}_n$.

Since each of these logics is finitely axiomatizable, has the finite model property, and hence is decidable, we obtain:

COROLLARY 3.9. *The modal logic of an arbitrary metric space is finitely axiomatizable, has the finite model property, and hence is decidable.*

REMARK 3.10. Our results have an immediate application in the setting of superintuitionistic logics. Let IPC be the intuitionistic propositional calculus and let IPC_n be the superintuitionistic logics $\mathsf{IPC} + \mathsf{ibd}_n$, where:

$$\begin{aligned} \mathsf{ibd}_1 &= p_1 \vee \neg p_1, \\ \mathsf{ibd}_{n+1} &= p_{n+1} \vee (p_{n+1} \to \mathsf{ibd}_n). \end{aligned}$$

The formula ibd_n is the intuitionistic version of the modal formula bd_n. The well-known Gödel translation associates with each superintuitionistic logic L a normal extension S of S4, called a *modal companion* of L. It is well known (see, e.g., Chagrov & Zakharyaschev, 1997, sec. 9.6) that modal companions of IPC are the logics in the interval [S4, S4.Grz]. In particular, S4, S4.1, and S4.Grz are modal companions of IPC. Also, $\mathsf{S4.Grz}_n$ is a modal companion of IPC_n for each $n \geq 1$. Thus, our Main Theorem yields that the superintuitionistic logics of metric spaces form the following chain which is order-isomorphic (with respect to the $\supset$ relation) to the ordinal $\omega + 1$:

$$\mathsf{IPC}_1 \supset \mathsf{IPC}_2 \supset \mathsf{IPC}_3 \supset \cdots \ \mathsf{IPC}.$$

More precisely, the superintuitionistic logic of a metric space X is IPC_n iff X is a scattered metric space of rank n, and it is IPC otherwise.

REMARK 3.11. Since Telgarsky's theorem holds for scattered paracompact spaces, it is natural to seek to generalize our results from metric spaces to paracompact spaces. For this, however, it is necessary to extend the MTRS-theorem to the paracompact setting.

§4. Acknowledgments. The work of the first two authors was partially supported by the Rustaveli National Science Foundation of Georgia grant #FR/489/5-105/11.

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