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Funayama's theorem revisited

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Abstract. Funayama's theorem states that there is an embedding e of a lattice L into a complete Boolean algebra B such that e preserves all existing joins and meets in L iff L satisfies the join infinite distributive law (JID) and the meet infinite distributive law (MID). More generally, there is a lattice embedding $e \colon L \to B$ preserving all existing joins in L iff L satisfies (JID), and there is a lattice embedding $e \colon L \to B$ preserving all existing meets in L iff L satisfies (MID). Funayama's original proof is quite involved. There are two more accessible proofs in case L is complete. One was given by Grätzer by means of free Boolean extensions and MacNeille completions, and the other by Johnstone by means of nuclei and Booleanization.

We show that Grätzer's proof has an obvious generalization to the non-complete case, and that in the complete case the complete Boolean algebras produced by Grätzer and Johnstone are isomorphic. We prove that in the non-complete case, the class of lattices satisfying (JID) properly contains the class of Heyting algebras, and we characterize lattices satisfying (JID) and (MID) by means of their Priestley duals. Utilizing duality theory, we give alternative proofs of Funayama's theorem and of the isomorphism between the complete Boolean algebras produced by Grätzer and Johnstone. We also show that unlike Grätzer's proof, there is no obvious way to generalize Johnstone's proof to the non-complete case.

1. Introduction

Let L be a lattice. We recall that L satisfies the *join infinite distributive law* (JID) if for each $a \in L$ and $S \subseteq L$, whenever $\bigvee S$ exists, so does $\bigvee\{a \wedge s : s \in S\}$ and $a \wedge \bigvee S = \bigvee\{a \wedge s : s \in S\}$.

Similarly, L satisfies the *meet infinite distributive law* (MID) if whenever $\bigwedge S$ exists, so does $\bigwedge\{a \vee s : s \in S\}$ and $a \vee \bigwedge S = \bigwedge\{a \vee s : s \in S\}$.

Obviously, each lattice that satisfies either (JID) or (MID) is distributive. A classic result in lattice theory is Funayama's theorem [12] stating that there is an embedding e of L into a complete Boolean algebra B that preserves all existing joins and meets in L iff L satisfies both (JID) and (MID). Since each complete Boolean algebra satisfies both (JID) and (MID), one direction of Funayama's theorem is obvious: if there is an embedding e of L into a complete Boolean algebra B that preserves all existing joins and meets in L,

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then L satisfies both (JID) and (MID). Thus, the main challenge is in proving the other direction of Funayama's theorem, that if a lattice L satisfies both (JID) and (MID), then there is an embedding e of L into a complete Boolean algebra B such that e preserves all existing joins and meets in L.

We can state Funayama's theorem more generally as follows. There is a lattice embedding e of a lattice L into a complete Boolean algebra B that preserves all existing joins in L iff L satisfies (JID), and there is a lattice embedding e of a lattice L into a complete Boolean algebra B that preserves all existing meets in L iff L satisfies (MID). Of course, the proofs of these two statements are dual to each other, so it is sufficient to be able to prove only one of the two, and we will mainly concentrate on the first one.

Funayama's original proof is quite involved. In [14, Sec. II.4], Grätzer gave a more accessible proof in case L is complete by showing that if L satisfies both (JID) and (MID), then the embedding of L into its free Boolean extension $B(L)$ is a complete lattice embedding. Then taking the MacNeille completion $\overline{B(L)}$ of $B(L)$ produces a complete Boolean algebra and the embedding $B(L) \hookrightarrow \overline{B(L)}$ preserves all existing joins and meets in $B(L)$. Thus, the composition $L \hookrightarrow B(L) \hookrightarrow \overline{B(L)}$ is a complete lattice embedding.

A different proof of Funayama's theorem in case L is complete was given in Johnstone [16, Sec. II.2]. Let L be a complete lattice satisfying (JID). Then L is a frame. Therefore, the poset $N(L)$ of all nuclei on L is also a frame, and the embedding $L \hookrightarrow N(L)$ is a frame homomorphism. Let $N(L)_{\neg\neg}$ be the Booleanization of $N(L)$; that is, the Boolean frame of regular nuclei on L. Thus, $N(L)_{\neg\neg}$ is a complete Boolean algebra and the composition $L \hookrightarrow N(L) \twoheadrightarrow N(L)_{\neg\neg}$ is a frame embedding. If in addition L satisfies (MID), then the embedding $L \hookrightarrow N(L)_{\neg\neg}$ is a complete lattice embedding.

In this paper, we show that Grätzer's proof has an obvious generalization to the case when L is not necessarily complete, thus providing an accessible proof of Funayama's theorem in its full generality. If L is complete, we show that the complete Boolean algebras $\overline{B(L)}$ and $N(L)_{\neg\neg}$ produced by Grätzer and Johnstone, respectively, are isomorphic. This confirms a conjecture made by Leo Esakia in the early 1990s. In the non-complete case, we show that the class of lattices satisfying (JID) properly contains the class of Heyting algebras, and we characterize lattices satisfying (JID) and (MID) by means of their Priestley spaces. Utilizing duality theory, we give alternative proofs of Funayama's theorem and of the isomorphism between $\overline{B(L)}$ and $N(L)_{\neg\neg}$. We also show that unlike Grätzer's proof, there is no obvious way to generalize Johnstone's proof to the non-complete case.

2. Free Boolean extensions, MacNeille completions, and a generalization of Grätzer's proof

Let L be a lattice. As we pointed out in the introduction, if there is a lattice embedding e of L into a complete Boolean algebra B that preserves all existing joins in L, then as B satisfies (JID), so does L. (Dually if e preserves all existing meets in L, then L satisfies (MID).) To prove the converse, let L be a lattice satisfying (JID). Then L is a distributive lattice. But L may not be bounded. As a first step, we add the missing bounds to L as follows. If L does not have 1, then we add on top of L a decreasing chain $C = \{1 > c_1 > c_2 > \cdots\}$; that is, we take the disjoint union of L and C and set $a \le c$ for each $a \in L$ and $c \in C$. Dually, if L does not have 0, then we add below L an increasing chain $0 < d_1 < d_2 < \cdots$. In case L has neither 0 nor 1, we add both chains to L. The resulting poset M is obviously a bounded distributive lattice. Moreover, a subset of L has a join (resp. meet) in L iff it has it in M, and when these two joins (resp. meets) exist, they coincide. Therefore, the embedding $L \hookrightarrow M$ preserves all existing joins (and meets). In addition, since L satisfies (JID), so does M. Thus, without loss of generality, we may assume that L is bounded.

Let BDL be the category of bounded distributive lattices and bounded lattice homomorphisms and let BA be the category of Boolean algebras and Boolean homomorphisms. Then it is well known that BA is a full subcategory of BDL and that the embedding $\mathsf{BA} \hookrightarrow \mathsf{BDL}$ has a left adjoint (see, e.g., [1, 7, 11, 13, 14, 18, 23]). This left adjoint sends each bounded distributive lattice L to its free Boolean extension $B(L)$, which can be constructed as follows. Let X_L be the prime spectrum of L; that is, the set of prime filters of L ordered by inclusion. For $a \in L$, let $\varphi(a) = \{x \in X_L : a \in x\}$. Then φ is a bounded lattice embedding of L into the bounded lattice of all up-sets of X_L. Let $B(\varphi[L])$ be the Boolean subalgebra of the powerset of X_L generated by $\varphi[L]$. The Boolean algebra $B(\varphi[L])$ is (isomorphic to) the free Boolean extension $B(L)$ of L. In fact, each bounded lattice homomorphism from L to a Boolean algebra B extends uniquely to a Boolean homomorphism from $B(L)$ to B.

Lemma 2.1. *Let L be a bounded distributive lattice, let $B(L)$ be the free Boolean extension of L, and let $e\colon L \hookrightarrow B(L)$ be the canonical lattice embedding.*

(1) *If L satisfies* (JID), *then e preserves all existing joins in L.*
(2) *If L satisfies* (MID), *then e preserves all existing meets in L.*

Proof. We only prove (1) as (2) is proved similarly. Let L satisfy (JID), $S \subseteq L$, and $\bigvee_L S$ exist. Let $x \in B(L)$ be an upper bound of S in $B(L)$. It is sufficient to show that $\bigvee_L S \le x$. As $x \in B(L)$, there exist $a_1, \dots, a_n, b_1, \dots, b_n \in L$ such that $x = \bigwedge_{i=1}^n (\neg a_i \vee b_i)$. Let $s \in S$. As x is an upper bound of S, we have $s \le \neg a_i \vee b_i$ for each i. Therefore, $s \wedge a_i \le b_i$ for each i. Since L satisfies (JID), $\bigvee_L \{s \wedge a_i : s \in S\}$ exists and $\bigvee_L \{s \wedge a_i : s \in S\} = a_i \wedge \bigvee_L S$. Thus, $a_i \wedge \bigvee_L S \le b_i$ for each i, so $\bigvee_L S \le \neg a_i \vee b_i$ for each i, and so $\bigvee_L S \le x$. Consequently, e preserves all existing joins in L. □

Let B be a Boolean algebra. We recall (see, e.g., [1]) that the MacNeille completion of B is a complete Boolean algebra $\overline{B}$ such that there is a Boolean embedding $e\colon B \hookrightarrow \overline{B}$ that is join-dense in $\overline{B}$ (equivalently, e is meet-dense in $\overline{B}$). It is well known that the embedding e preserves all existing joins and meets in B.

Remark 2.2. The explicit construction of $\overline{B}$ is by means of normal ideals of B, where an ideal N of B is normal if $N^{ul} = N$. (Here $(-)^u$ and $(-)^l$ are the operations of taking the upper bounds and lower bounds, respectively.) As was shown in [14, Exercises to Sec. II.4], normal ideals of B turn out to be the regular elements of the frame of ideals of B. More precisely, let $\mathrm{Idl}(B)$ be the frame of ideals of B, and let $\mathrm{RIdl}(B)$ be the Boolean frame of regular elements of $\mathrm{Idl}(B)$; that is, those $I \in \mathrm{Idl}(B)$ that satisfy $\neg\neg I = I$, where $\neg I$ is the pseudocomplement of I in $\mathrm{Idl}(B)$. Then $\mathrm{RIdl}(B)$ is isomorphic to $\overline{B}$, which can be seen by observing that N is a normal ideal of B iff $N \in \mathrm{RIdl}(B)$.

Theorem 2.3 (Funayama's Theorem)**.** *Let L be a lattice.*

(1) *L satisfies* (JID) *iff there exists a lattice embedding e of L into a complete Boolean algebra B that preserves all existing joins in L.*
(2) *L satisfies* (MID) *iff there exists a lattice embedding e of L into a complete Boolean algebra B that preserves all existing meets in L.*
(3) *L satisfies* (JID) *and* (MID) *iff there exists an embedding e of L into a complete Boolean algebra B that preserves all existing joins and meets in L.*

Proof. It suffices to prove (1) as (2) is proved similarly and (3) follows from (1) and (2). For (1), it is sufficient to show that if L satisfies (JID), then there exists a complete Boolean algebra B and a lattice embedding $e\colon L \to B$ that preserves all existing joins in L. As we already pointed out, we may assume without loss of generality that L is bounded. Let $B(L)$ be the free Boolean extension of L, and let $\overline{B(L)}$ be the MacNeille completion of $B(L)$. By Lemma 2.1(1), the lattice embedding $L \hookrightarrow B(L)$ preserves all existing joins in L, and it is well known that the Boolean embedding $B(L) \hookrightarrow \overline{B(L)}$ preserves all existing joins (and meets) in $B(L)$. Thus, the composition $L \hookrightarrow B(L) \hookrightarrow \overline{B(L)}$ is a lattice embedding preserving all existing joins in L. □

When L is complete, Theorem 2.3 yields Grätzer's proof of Funayama's theorem. Theorem 2.3 also has an obvious corollary for Heyting algebras. It is well known that if L is complete, then L satisfies (JID) iff L is a Heyting algebra (dually, L satisfies (MID) iff L is a co-Heyting algebra). It is easy to see that one direction of this correspondence also holds when L is not complete. Namely, if L is a Heyting algebra, then L satisfies (JID), and dually, if L is a co-Heyting algebra, then L satisfies (MID). However, there exist lattices satisfying (JID) that are not Heyting algebras, as well as lattices satisfying (MID) that are not co-Heyting algebras. This is demonstrated in the next example.

Example 2.4. Let L be the lattice depicted in Figure 1. It is obvious that L is not a Heyting algebra. For example, $\neg a_0$ does not exist in L. On the other hand, L satisfies (JID) because no non-trivial infinite joins exist in L.

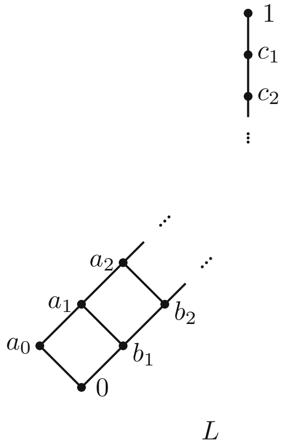

FIGURE 1

Dualizing this example produces a lattice L satisfying (MID) that is not a co-Heyting algebra.

We conclude this section by the following obvious consequence of Theorem 2.3.

Corollary 2.5. *Let L be a lattice.*

(1) *If L is a Heyting algebra, then there exists a lattice embedding e of L into a complete Boolean algebra B that preserves all existing joins in L.*
(2) *If L is a co-Heyting algebra, then there exists a lattice embedding e of L into a complete Boolean algebra B that preserves all existing meets in L.*
(3) *If L is a bi-Heyting algebra, then there exists an embedding e of L into a complete Boolean algebra B that preserves all existing joins and meets in L.*

3. Nuclei, Booleanization, and Johnstone's proof

We recall that a *nucleus* on a meet-semilattice M is a map $j\colon M \to M$ satisfying

$$a \leq ja, \quad jja = ja, \quad j(a \wedge b) = ja \wedge jb,$$

for all $a, b \in M$. We will be mostly interested in nuclei on Heyting algebras and frames. If j is a nucleus on a frame L, then the set $L^j = \{a \in L : ja = a\}$ of its fixed points is a frame and $j\colon L \to L^j$ is an onto frame homomorphism whose right adjoint is the inclusion $L^j \hookrightarrow L$ [16, Sec. II.2].

For a frame L, let $N(L)$ be the set of all nuclei on L. If we order $N(L)$ pointwise, then it is well known that $N(L)$ is a frame and that $a \mapsto a \vee (-)$ is a frame embedding of L into $N(L)$ [16, Sec. II.2]. We call $j \in N(L)$ *regular* if $\neg\neg j = j$, where $\neg\neg$ is taken in $N(L)$. It is well known [16, Sec. II.2] that the set $N(L)_{\neg\neg}$ of all regular nuclei on L is a Boolean frame (complete Boolean algebra). Following the terminology of Banaschewski and Pultr [2], we call $N(L)_{\neg\neg}$ the *Booleanization* of $N(L)$.

Johnstone [16, Sec. II.2] proves that if L is a frame, then the composition $L \hookrightarrow N(L) \twoheadrightarrow N(L)_{\neg\neg}$ is a frame embedding. In addition, the embedding $L \hookrightarrow N(L)$ preserves arbitrary meets iff L satisfies (MID). As $N(L)_{\neg\neg}$ is closed under arbitrary meets in $N(L)$, the composition $L \hookrightarrow N(L) \twoheadrightarrow N(L)_{\neg\neg}$ preserves all meets in L iff L satisfies (MID). Therefore, the composition $L \hookrightarrow N(L) \twoheadrightarrow N(L)_{\neg\neg}$ is a complete lattice embedding iff L satisfies both (JID) and (MID). This yields another proof of Funayama's theorem for complete lattices.

We show that the complete Boolean algebras $N(L)_{\neg\neg}$ and $\overline{B(L)}$ are isomorphic, thus showing that the Grätzer and Johnstone proofs of Funayama's theorem for complete L, although different, produce the same complete Boolean algebra in which there is a frame embedding of L.

Theorem 3.1. *For each frame L, the complete Boolean algebras $\overline{B(L)}$ and $N(L)_{\neg\neg}$ are isomorphic.*

Proof. Following [16, Sec. II.2], for $a \in L$, let $c_a = a \vee (-)$ and $u_a = a \to (-)$. By [16, Sec. II.2], $c_a, u_a \in N(L)$ and are each other's complements in $N(L)$. Therefore, c_a, u_a belong to the center of $N(L)$. As the center of $N(L)$ is contained in $N(L)_{\neg\neg}$, we have $c_a, u_a \in N(L)_{\neg\neg}$. In fact, as shown in [16, Sec. II.2], the map $h\colon L \to N(L)_{\neg\neg}$ given by $h(a) = c_a$ is a frame embedding. (This is in essence Johnstone's proof of Funayama's theorem for frames.) As $B(L)$ is the free Boolean extension of L and $N(L)_{\neg\neg}$ is a (complete) Boolean algebra, the frame embedding $h\colon L \to N(L)_{\neg\neg}$ extends to a unique Boolean homomorphism $\overline{h}\colon B(L) \to N(L)_{\neg\neg}$, which can be defined as follows. For $x \in B(L)$, there exist $a_1, \dots, a_n, b_1, \dots, b_n \in L$ such that $x = \bigwedge_{i=1}^n (\neg a_i \vee b_i)$. Then $\overline{h}(x) = \bigwedge_{i=1}^n (\neg c_{a_i} \vee c_{b_i})$. If $\overline{h}(x) = 1$, then $c_{a_i} \le c_{b_i}$ for each i, so $a_i \le b_i$ for each i, yielding $x = 1$. Therefore, $\overline{h}$ is 1–1, and so up to isomorphism, $B(L)$ is a Boolean subalgebra of $N(L)_{\neg\neg}$. (In fact, $B(L)$ is isomorphic to a Boolean subalgebra of the center of $N(L)$.) By [16, Sec. II.2], each $j \in N(L)$ is a join of nuclei of the form $c_a \wedge u_b$, so each j is a join of nuclei of the form $c_a \wedge \neg c_b$. This means that up to isomorphism $B(L)$ is join-dense in $N(L)$, hence up to isomorphism, $B(L)$ is a join-dense subalgebra of $N(L)_{\neg\neg}$. Thus, $N(L)_{\neg\neg}$ is isomorphic to the MacNeille completion $\overline{B(L)}$ of $B(L)$. $\square$

Consequently, in case L is complete, Grätzer's proof of Funayama's theorem can be summarized by the diagram: $L \hookrightarrow B(L) \hookrightarrow \overline{B(L)}$, while Johnstone's proof can be summarized by the diagram: $L \hookrightarrow N(L) \twoheadrightarrow N(L)_{\neg\neg}$. Although the constructions are different, by Theorem 3.1, the complete Boolean algebras produced in both cases are isomorphic. This confirms a conjecture made by Leo Esakia in the early 1990s.

4. Dual characterization of lattices satisfying (JID) and (MID) and Funayama's theorem

In this section, we characterize lattices satisfying (JID) and (MID) by means of their Priestley spaces. This yields alternative proofs of Funayama's theorem and Theorem 3.1 by means of Priestley duality. We conclude the paper by showing that unlike Grätzer's proof, Johnstone's proof has no obvious generalization to the non-complete case.

We assume the reader's familiarity with Stone duality for Boolean algebras [26] and Priestley duality for bounded distributive lattices [20, 21]. We recall that the Priestley space of a bounded distributive lattice L is constructed as follows: Let X_L be the prime spectrum (the set of prime filters) of L ordered by inclusion. For $a \in L$, recall that $\varphi(a) = \{x \in X_L : a \in x\}$, and generate the topology on X_L by the basis $\{\varphi(a) - \varphi(b) : a, b \in L\}$. Then X_L is the Priestley space of L. Also, the bounded distributive lattice of a Priestley space X is the lattice L_X of clopen up-sets of X.

Esakia duality for Heyting algebras [8] is a restricted version of Priestley duality. We recall that an Esakia space is a Priestley space X in which the down-set $\downarrow U$ of each clopen $U \subseteq X$ is clopen, and that the constructions of the Esakia space of a Heyting algebra and of the Heyting algebra of an Esakia space are the same as in Priestley duality.

Remark 4.1. The Priestley space X_L of the lattice L of Example 2.4 is depicted in Figure 2, where the only limit point is $\{1, c_1, \dots\}$; all other points are isolated. Clearly, X_L is not an Esakia space because $\uparrow a_0$ is an isolated point of X_L, hence the singleton $\{\uparrow a_0\}$ is clopen in X_L, but its down-set is not clopen in X_L.

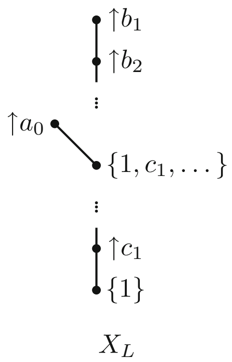

Figure 2

From now on, in order to use Priestley duality freely, we restrict ourselves to bounded distributive lattices. If L is not bounded, then as we already saw, without loss of generality we may always add bounds to L. Recall that BDL is the class of bounded distributive lattices. Let JID be the class of bounded lattices satisfying (JID) and let HA be the class of Heyting algebras. Then $\mathsf{HA} \subsetneqq \mathsf{JID} \subsetneqq \mathsf{BDL}$. Consequently, Priestley spaces corresponding to bounded lattices satisfying (JID) satisfy a condition that is weaker than the Esakia condition. To describe this condition, we first recall several conditions equivalent to the Esakia condition. The next lemma can be extracted from [11, Sec. III.1], but since the extraction requires a little effort and also since [11] is in Russian, we give a full proof of it.

Lemma 4.2. *The following are equivalent for a Priestley space X:*

(1) *X is an Esakia space.*
(2) *The closure of each up-set is an up-set.*
(3) *The closure of each open up-set is an up-set.*

Proof. (1)⇒(2): Let U be an up-set, $x \in \overline{U}$, and $x \leq y$. If $y \notin \overline{U}$, then as X is a Stone space, there exists a clopen set V such that $y \in V$ and $V \cap \overline{U} = \emptyset$. As X is an Esakia space, $\downarrow V$ is clopen. Since $x \leq y$ and $y \in V$, we have $x \in \downarrow V$. As $V \cap U = \emptyset$ and U is an up-set, $\downarrow V \cap U = \emptyset$. Thus, we have an open neighborhood $\downarrow V$ of x that does not intersect U. This means that $x \notin \overline{U}$. The obtained contradiction proves that $y \in \overline{U}$, so $\overline{U}$ is an up-set.

(2)⇒(3): This is obvious.

(3)⇒(1): Let U be clopen. As X is a Priestley space, $\downarrow U$ is closed, so it is sufficient to show that $\downarrow U$ is open. We have $X - \downarrow U$ is an open up-set. By (3), $\overline{X - \downarrow U}$ is an up-set. As $U \cap (X - \downarrow U) = \emptyset$ and U is open, $U \cap \overline{X - \downarrow U} = \emptyset$. As $\overline{X - \downarrow U}$ is an up-set, this implies $\downarrow U \cap \overline{X - \downarrow U} = \emptyset$. Therefore, $\overline{X - \downarrow U} \subseteq X - \downarrow U$, so $X - \downarrow U = \overline{X - \downarrow U}$. Thus, $X - \downarrow U$ is closed, so $\downarrow U$ is open, and so X is an Esakia space. □

Remark 4.3. A condition equivalent to condition (2) of Lemma 4.2 is that the interior of each down-set is a down-set, and a condition equivalent to condition (3) is that the interior of each closed down-set is a down-set.

Remark 4.4. Dual spaces of co-Heyting algebras are Priestley spaces satisfying that U clopen implies $\uparrow U$ is clopen [9, 10]. We call these spaces *co-Esakia spaces*. An argument dual to the proof of Lemma 4.2 then gives that for a Priestley space to be a co-Esakia space is equivalent to either of the following two conditions: (i) the closure of each down-set is a down-set (equivalently, the interior of each up-set is an up-set); (ii) the closure of each open down-set is a down-set (equivalently, the interior of each closed up-set is an up-set).

Let X be a Priestley space and let $S \subseteq X$. Following [15], we let

$$\mathbf{J}S = X - \downarrow(X - \mathrm{int}S) \text{ and } \mathbf{D}S = \uparrow\overline{S}.$$

It is easy to see [15] that $\mathbf{J}S$ is the largest open up-set contained in S and $\mathbf{D}S$ is the smallest closed up-set containing S. Using this notation, being an Esakia space means that if U is an open up-set, then $\mathbf{D}U = \overline{U}$.

Definition 4.5. Let X be a Priestley space.

(1) We call X a *J-space* if for each open up-set U, whenever $\mathbf{D}U$ is clopen, $\mathbf{D}U = \overline{U}$.
(2) We call X an *M-space* if for each closed up-set F, whenever $\mathbf{J}F$ is clopen, $\mathbf{J}F = \mathrm{int}F$.
(3) We call X a *JM-space* if X is both a J-space and an M-space.

Obviously, the condition defining a J-space weakens the condition defining an Esakia space. (Similarly, the condition defining an M-space weakens the condition defining a co-Esakia space. If we call a Priestley space a *bi-Esakia space* whenever it is both an Esakia space and a co-Esakia space, then the condition defining a JM-space weakens the condition defining a bi-Esakia space.) We show that the conditions of Definition 4.5 give dual characterizations of bounded lattices satisfying (JID) and (MID).

Theorem 4.6. *Let L be a bounded distributive lattice and let X_L be the Priestley space of L.*

(1) *L satisfies* (JID) *iff X_L is a J-space.*
(2) *L satisfies* (MID) *iff X_L is an M-space.*
(3) *L satisfies* (JID) *and* (MID) *iff X_L is a JM-space.*

Proof. (1): First suppose that X_L is a J-space. Let $a \in L$, $S \subseteq L$, and $\bigvee S$ exist in L. It is sufficient to show that $a \wedge \bigvee S$ is a least upper bound of $\{a \wedge s : s \in S\}$. Let u be an upper bound of $\{a \wedge s : s \in S\}$. Then $\varphi(a) \cap \varphi(s) \subseteq \varphi(u)$ for each $s \in S$. Therefore, $\varphi(a) \cap \bigcup\{\varphi(s) : s \in S\} \subseteq \varphi(u)$. We let $U = \bigcup\{\varphi(s) : s \in S\}$. Then U is an open up-set of X_L and $\varphi(a) \cap U \subseteq \varphi(u)$. So $U \subseteq (X_L - \varphi(a)) \cup \varphi(u)$. As $(X_L - \varphi(a)) \cup \varphi(u)$ is clopen, this implies $\overline{U} \subseteq (X_L - \varphi(a)) \cup \varphi(u)$, so $\varphi(a) \cap \overline{U} \subseteq \varphi(u)$. Since $\bigvee S$ exists in L, by [21, Prop. 15] (see also [22, Thm. 1.5] or [4, Lem. 2.3.1]), $\varphi(\bigvee S) = \mathbf{D}U$, so $\mathbf{D}U$ is clopen. As X_L is a J-space, $\mathbf{D}U = \overline{U}$. Thus, $\varphi(a) \cap \mathbf{D}U \subseteq \varphi(u)$, implying that $\varphi(a \wedge \bigvee S) \subseteq \varphi(u)$. This yields $a \wedge \bigvee S \leq u$. Consequently, $a \wedge \bigvee S = \bigvee\{a \wedge s : s \in S\}$, and so L satisfies (JID).

Conversely, suppose that L satisfies (JID). Let U be an open up-set of X_L such that $\mathbf{D}U$ is clopen. We must show that $\mathbf{D}U = \overline{U}$. If not, then there exists $x \in \mathbf{D}U - \overline{U}$. Therefore, there exists $y \in \overline{U}$ with $y \leq x$. As $x \notin \overline{U}$, we have $x \notin {\downarrow}x \cap \overline{U}$. Since X_L is a Priestley space, there exists a clopen up-set $\varphi(a)$ such that $x \in \varphi(a)$ and $\varphi(a) \cap {\downarrow}x \cap \overline{U} = \emptyset$. As U is an open up-set, there is $S \subseteq L$ such that $U = \bigcup\{\varphi(s) : s \in S\}$. Since $\mathbf{D}U$ is clopen, by [21, Prop. 15] (see also [22, Thm. 1.5] or [4, Lem. 2.3.1]), $\bigvee S$ exists and $\varphi(\bigvee S) = \mathbf{D}U$. As L satisfies (JID), $\bigvee\{a \wedge s : s \in S\}$ exists and $a \wedge \bigvee S = \bigvee\{a \wedge s : s \in S\}$.

Clearly, $x \in \varphi(a) \cap \mathbf{D}U$, so $x \in \varphi(a \wedge \bigvee S)$. On the other hand,

$$\begin{aligned}\varphi(\bigvee\{a \wedge s : s \in S\}) &= \mathbf{D}\bigcup\{\varphi(a) \cap \varphi(s) : s \in S\} \\ &= \mathbf{D}(\varphi(a) \cap U) \subseteq \mathbf{D}(\varphi(a) \cap \overline{U}) = {\uparrow}(\varphi(a) \cap \overline{U}).\end{aligned}$$

But ${\downarrow}x \cap \varphi(a) \cap \overline{U} = \emptyset$ implies $x \notin {\uparrow}(\varphi(a) \cap \overline{U})$. So, $x \notin \varphi(\bigvee\{a \wedge s : s \in S\})$, which yields $a \wedge \bigvee S \neq \bigvee\{a \wedge s : s \in S\}$. The obtained contradiction proves that $\mathbf{D}U = \overline{U}$. Thus, X_L is a J-space.

(2): This can be proved by an argument dual to (1).

(3): This follows from (1) and (2). □

Remark 4.7. Theorem 4.6 can be used to give an alternative proof of the nontrivial implication in Funayama's theorem by means of Priestley duality. Indeed, let L satisfy (JID) and without loss of generality assume that L is bounded. Let X_L be the Priestley space of L. By Theorem 4.6, X_L is a J-space. Let $S \subseteq L$ be such that $\bigvee_L S$ exists. We let U be the open up-set $\bigcup\{\varphi(s) : s \in S\}$. Then $\varphi(\bigvee_L S) = \mathbf{D}U$, so $\mathbf{D}U$ is clopen, and as X_L is a J-space, $\mathbf{D}U = \overline{U}$. Therefore, $\overline{U}$ is clopen. Since $B(L)$ is isomorphic to the Boolean algebra of clopen subsets of X_L, $\overline{U}$ being clopen implies the join of the image of S in $B(L)$ exists and is equal to $\overline{U}$. Thus, the canonical embedding $\varphi\colon L \hookrightarrow B(L)$ preserves all existing joins in L. A similar argument gives that if L satisfies (MID), then X_L is an M-space, and so φ preserves all existing meets in L. Thus, if L satisfies both (JID) and (MID), then X_L is a JM-space, and so φ preserves all existing joins and meets in L. Taking the MacNeille completion of $B(L)$ then completes the proof.

Remark 4.8. Priestley spaces can be viewed from the bitopological point of view [5, 17, 19, 21]. (There is a large intersection among the results of [19] and [5]. Unfortunately, we were unaware of [19] when we wrote [5].) Each Priestley space carries the topology τ_1 of open up-sets and the topology τ_2 of open down-sets, and the Priestley topology τ is the join of these two topologies. In this setting, $\mathbf{J}$ is the interior in the topology τ_1 and $\mathbf{D}$ is the closure in the topology τ_2. From this perspective, being an Esakia space means that if U is τ_1-open, then the τ_2-closure coincides with the τ-closure; and being a J-space means that for each τ_1-open U, if τ_2-closure of U is τ-open, then it coincides with the τ-closure of U. Similarly, being a co-Esakia space means that if U is τ_2-closed, then the τ_1-interior coincides with the τ-interior; and being an M-space means that for each τ_2-closed F, if τ_1-interior of F is τ-closed, then it coincides with the τ-interior of F.

Next we show how to use duality theory to obtain an alternative proof of Theorem 3.1. As frames are complete Heyting algebras, we will use Esakia duality. It is well known (see, e.g., [4, 15, 22]) that a Heyting algebra A is complete iff in its Esakia space the closure of each open up-set is clopen. Following the nomenclature in Priestley spaces [21], we call an Esakia space X *extremally order-disconnected* if the closure of each open up-set is clopen.

It follows that a Heyting algebra is a frame iff its Esakia space is extremally order-disconnected.

We recall the dual characterization of nuclei on Heyting algebras given in [6]. Let X be an Esakia space. A closed subset S of X is a *subframe* of X if for each clopen U of X, the set ${\downarrow}(U \cap S)$ is clopen in X. Let A be a Heyting algebra and let X be its Esakia space. If j is a nucleus on A, then the Esakia space of A^j can be identified with the subframe $S_j = \{x \in X : j^{-1}[x] = x\}$ of X. Conversely, if S is a subframe of X, then the induced partial order on S turns it into an Esakia space, and the lattice homomorphism $S \cap (-) \colon L_X \to L_S$ has a right adjoint r_S given by $r_S(U) = X - {\downarrow}(S - U)$. The composite $j_S = r_S \circ (S \cap (-))$ is then a nucleus on L_X. Therefore, the assignments $j \mapsto S_j$ and $S \mapsto j_S$ establish an order-reversing one-to-one correspondence between nuclei on A and subframes of X. We show that if X is an extremally order-disconnected Esakia space, then each regular closed subset of X is a subframe of X.

Lemma 4.9. *Let X be an extremally order-disconnected Esakia space. If F is a regular closed subset of X, then ${\downarrow}F$ is clopen.*

Proof. As F is closed, so is ${\downarrow}F$. Since X is extremally order-disconnected, the closure of each open up-set is clopen. This is equivalent to the statement that the interior of each closed down-set is clopen. Therefore, $\operatorname{int}{\downarrow}F$ is clopen. But $F \subseteq {\downarrow}F$, so $\operatorname{int} F \subseteq \operatorname{int}{\downarrow}F$, and so $\overline{\operatorname{int} F} \subseteq \overline{\operatorname{int}{\downarrow}F}$. As F is regular closed, $\overline{\operatorname{int} F} = F$. Since $\operatorname{int}{\downarrow}F$ is clopen, $\overline{\operatorname{int}{\downarrow}F} = \operatorname{int}{\downarrow}F$. So $\overline{\operatorname{int} F} \subseteq \overline{\operatorname{int}{\downarrow}F}$ implies $F \subseteq \operatorname{int}{\downarrow}F$. As X is an Esakia space, by Remark 4.3, the interior of each closed down-set is a down-set. Therefore, $\operatorname{int}{\downarrow}F$ is a down-set, so $F \subseteq \operatorname{int}{\downarrow}F$ implies ${\downarrow}F \subseteq \operatorname{int}{\downarrow}F$, and so ${\downarrow}F$ is open. Thus, ${\downarrow}F$ is clopen. □

Lemma 4.10.

(1) *Let A be a Heyting algebra, $a \in A$ complemented, and $b \in A$ regular. Then $a \vee b$ is regular.*
(2) *Let X be a topological space, $C \subseteq X$ clopen, and $U \subseteq X$ regular open. Then $C \cup U$ is regular open.*
(3) *Let X be a topological space, $C \subseteq X$ clopen, and $F \subseteq X$ regular closed. Then $C \cap F$ is regular closed.*

Proof. (1): We have $\neg\neg(a \vee b) = \neg(\neg a \wedge \neg b) = \neg a \to \neg\neg b = \neg a \to b$ as b is regular. Since a is complemented, so is $\neg a$. Therefore, $\neg a \to b = \neg\neg a \vee b = a \vee b$. Thus, $\neg\neg(a \vee b) = a \vee b$, and so $a \vee b$ is regular.

(2): This follows from (1) by taking A to be the (complete) Heyting algebra of open subsets of X.

(3): This is the dual statement of (2). □

Proposition 4.11. *Let X be an extremally order-disconnected Esakia space. Then each regular closed subset of X is a subframe of X.*

Proof. Let S be a regular closed subset of X and let U be clopen in X. By Lemma 4.10(3), $S \cap U$ is regular closed in X. By Lemma 4.9, ${\downarrow}(S \cap U)$ is clopen in X. Thus, S is a subframe of X. □

For an Esakia space X, let $S(X)$ be the poset of subframes of X ordered by inclusion. If X is the Esakia space of a Heyting algebra A, then $S(X)$ is dually isomorphic to $N(A)$ [6, Thm. 30]. If A is a frame, then $N(A)$ is a frame [16, Sec. II.2]. Therefore, if X is extremally order-disconnected, then $S(X)$ is dually isomorphic to the frame $N(A)$, so $S(X)$ is a complete lattice satisfying (MID). In particular, $S(X)$ is a co-Heyting algebra, so possesses co-implication, which we denote by $\leftarrow$. We also denote co-negation by $\sim$. Clearly, for $F \in S(X)$, we have $\sim F = X \leftarrow F$. By analogy with $N(A)$, we call an element F of $S(X)$ regular if $\sim\sim F = F$.

Proposition 4.12. *Let L be a frame and X its Esakia space. Then regular closed subsets of X are precisely the regular elements of $S(X)$.*

Proof. Let $S \in S(X)$. We show that $\sim S = \overline{X - S}$. We have $\overline{X - S}$ is the smallest among those closed subsets F of X for which $S \cup F = X$. As $\overline{X - S}$ is regular closed, by Proposition 4.11, $\overline{X - S} \in S(X)$. Therefore, $\sim S = \overline{X - S}$. Thus, S is a regular element of $S(X)$ iff $S = \sim\sim S$ iff $S = \overline{X - \overline{X - S}}$ iff $S = \overline{\text{int} S}$ iff S is regular closed. □

We are ready to give an alternative proof of Theorem 3.1.

Theorem 4.13. *For a frame L, the MacNeille completion $\overline{B(L)}$ of $B(L)$ is isomorphic to the complete Boolean algebra $N(L)_{\neg\neg}$ of regular elements of $N(L)$.*

Proof. Let X be the Esakia space of L. Then $B(L)$ is isomorphic to the Boolean algebra of clopen subsets of X. Therefore, the MacNeille completion $\overline{B(L)}$ of $B(L)$ is isomorphic to the complete Boolean algebra $\mathcal{RC}(X)$ of regular closed subsets of X. (This well-known fact about Boolean algebras can for example be found in [24].) As $N(L)$ is dually isomorphic to $S(X)$, the Boolean algebra $N(L)_{\neg\neg}$ of regular elements of $N(L)$ is isomorphic to the Boolean algebra of regular elements of $S(X)$. By Proposition 4.12, the latter is $\mathcal{RC}(X)$. Thus, $\overline{B(L)}$ is isomorphic to $N(L)_{\neg\neg}$. □

We can also obtain a dual characterization of when $N(L)$ is a Boolean algebra.

Theorem 4.14. *Let L be a frame and let X be the Esakia space of L. Then $N(L)$ is Boolean iff $S(X) = \mathcal{RC}(X)$.*

Proof. We have $N(L)$ is Boolean iff $N(L) = N(L)_{\neg\neg}$. By the dual isomorphism between $N(L)$ and $S(X)$ and by Proposition 4.12, this happens iff $S(X) = \mathcal{RC}(X)$. □

Remark 4.15. A purely algebraic characterization of when $N(L)$ is Boolean was given in [3]. In [25], it was shown that if L is the frame of open subsets of a T_0-space X, then $N(L)$ is Boolean iff X is scattered. To these, Theorem 4.14 adds a characterization of when $N(L)$ is Boolean in terms of the Esakia space of L. In [6, Cor. 33.1], it is stated erroneously that $N(L)$ is Boolean iff $S(X_L)$ is the Boolean algebra of clopen subsets of X_L. The mistake comes from the fact that although clopen subsets of X_L give rise to complemented elements of $N(L)$, there exist nuclei on L such that their corresponding subframes are not clopen. Of course, by Theorem 4.14, they are regular closed.

We conclude the paper by showing that unlike Grätzer's proof, Johnstone's proof has no obvious generalization to the non-complete case. If L is not complete, then it is known that $N(L)$ may not even be a lattice. For a simple such example, let L be the linearly ordered lattice shown in Figure 3 together with its dual space X_L.

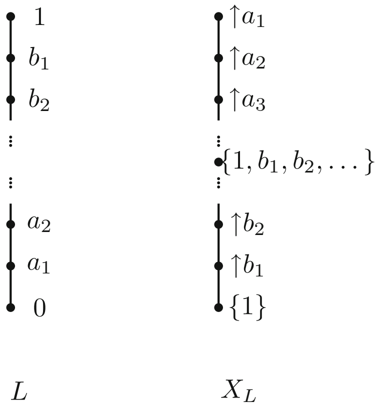

FIGURE 3

As L is linearly ordered, L is a Heyting algebra, so X_L is an Esakia space. By [6, Thm. 30], the poset $N(L)$ of nuclei of L is dually isomorphic to the poset $S(X_L)$ of subframes of X_L. To describe $S(X_L)$, let X_+ denote the "upper part" and let X_- denote the "lower part" of X_L; that is, $X_+ = \{\uparrow a_1, \uparrow a_2, \uparrow a_3, \dots\}$ and $X_- = \{\{1\}, \uparrow b_1, \uparrow b_2, \dots\}$. We also let $\omega = \{1, b_1, b_2, \dots\}$. Clearly ω is the only limit point of X_L, all other points of X_L are isolated, and X_L is the one-point compactification of $X_+ \cup X_-$.

Lemma 4.16. *Let $S \subseteq X_L$. Then $S \in S(X_L)$ iff S is finite and $\omega \notin S$, or $S \cap X_+$ is infinite and $\omega \in S$.*

Proof. First, suppose that S is finite and $\omega \notin S$. Then S is clopen, so $S \in S(X_L)$. Next, suppose that $S \cap X_+$ is infinite and $\omega \in S$. Then S is closed. Let U be clopen in S. If $U \cap X_+ = \varnothing$, then U is a finite subset of X_-. Therefore, $\downarrow U$ is also a finite subset of X_-, so is clopen in X_L. If $U \cap X_+ \neq \varnothing$, let s be the largest element of $U \cap X_+$. Then $\downarrow U = \downarrow s$ is clopen in X_L. In either case we see that $S \in S(X_L)$.

Conversely, suppose $S \in S(X_L)$. As S is closed, if $\omega \notin S$, then S must be finite, so S is finite and $\omega \notin S$. Suppose that $\omega \in S$. If $S \cap X_+$ is finite, say $S \cap X_+ = \{s_1, \dots, s_n\}$, then $U = S - \{s_1, \dots, s_n\}$ is clopen in S, whereas ${\downarrow}U = X_- \cup \{\omega\}$ is not clopen in X_L, which contradicts that $S \in S(X_L)$. Therefore, $S \cap X_+$ is infinite and $\omega \in S$. □

Proposition 4.17. *Let L and X_L be as in Figure* 3.

(1) *$S(X_L)$ is not a $\wedge$-semilattice, so $S(X_L)$ is not a lattice.*
(2) *$N(L)$ is not a $\vee$-semilattice, so $N(L)$ is not a lattice.*

Proof. (1): Let A_1, A_2 be any infinite disjoint subsets of X_+; let $S_1 = A_1 \cup \{\omega\} \cup X_-$, and let $S_2 = A_2 \cup \{\omega\} \cup X_-$. By Lemma 4.16, $S_1, S_2 \in S(X_L)$. On the other hand, $S_1 \cap S_2 = \{\omega\} \cup X_- \notin S(X_L)$. In fact, the set $\{T \in S(X_L) : T \subseteq S_1, S_2\}$ consists of all finite subsets of X_-, and so $\{S_1, S_2\}$ does not have a greatest lower bound in $S(X_L)$. Thus, $S(X_L)$ is not a $\wedge$-semilattice, so $S(X_L)$ is not a lattice.

(2): As $N(L)$ is dually isomorphic to $S(X_L)$ (see [6, Thm. 30]), this follows from (1). □

This shows that if L is not complete, then the definition of $N(L)_{\neg\neg}$ is problematic because $\neg a$ may not exist for some $a \in N(L)$. We could nevertheless try to define $N(L)_{\neg\neg}$ as the set of those $a \in N(L)$ for which $\neg\neg a$ exists in $N(L)$ and $a = \neg\neg a$. In terms of $S(X_L)$, this means that $S(X_L)_{\sim\sim}$ could be defined as those $S \in S(X_L)$ for which $\sim\sim S$ exists in $S(X_L)$ and $S = \sim\sim S$. Unfortunately, so defined $N(L)_{\neg\neg}$ and $S(X_L)_{\sim\sim}$ may still not be lattices, let alone Boolean algebras. Indeed, consider the sets S_1 and S_2 from the proof of Proposition 4.17. It is easy to see that $\sim S_1 = (X_+ - A_1) \cup \{\omega\}$, so $\sim S_1$ exists in $S(X_L)$ and $\sim\sim S_1 = S_1$. Similarly, $\sim S_2$ exists in $S(X_L)$ and $\sim\sim S_2 = S_2$. Therefore, $S_1, S_2 \in S(X_L)_{\sim\sim}$. But, as we have already seen, $S_1 \wedge S_2$ does not exist in $S(X_L)$, and the same argument shows that $S_1 \wedge S_2$ does not exist in $S(X_L)_{\sim\sim}$. Thus, $S(X_L)_{\sim\sim}$ is not a lattice. So neither is $N(L)_{\neg\neg}$. In particular, the complete Boolean algebra $\mathcal{RC}(X_L)$ of regular closed subsets of X_L is not isomorphic to $S(X_L)_{\sim\sim}$. In fact, $S(X_L)$ is properly contained in $\mathcal{RC}(X_L)$. To see this, as X_L is the one-point compactification of $X_L - \{\omega\}$, it is clear that $S \in \mathcal{RC}(X_L)$ iff S is closed and S is infinite or $\omega \notin S$. That $S(X_L) \subseteq \mathcal{RC}(X_L)$ now follows from Lemma 4.16. On the other hand, as $X_- \cup \{\omega\} \in \mathcal{RC}(X_L) - S(X_L)$, this inclusion is proper.

All this indicates that there is no obvious generalization of Johnstone's proof of Funayama's theorem to the non-complete case.

Remark 4.18. Nevertheless, there is a possibility of embedding Heyting algebras into Boolean algebras so that all existing joins and finite meets are preserved by means of regular operators of Wilson [27]. We recall that given a Heyting algebra A, a map $r\colon A \to A$ is a *regular operator* if $r(a \to b) = a \to rb$ for all $a, b \in A$. Wilson [27] shows that if A is a frame, then the set $\mathcal{R}(A)$ of regular operators on A is a complete Boolean algebra which is isomorphic to $N(N(A))_{\neg\neg}$. If A is not complete, then $\mathcal{R}(A)$ is still a Boolean algebra and the embedding of A into $\mathcal{R}(A)$ preserves all existing joins and finite meets in A. However, $\mathcal{R}(A)$ may fail to be complete. For example, if A is a non-complete Boolean algebra, then $\mathcal{R}(A)$ is isomorphic to A. Moreover, even if A is complete, $\mathcal{R}(A)$ may be strictly larger than $N(A)_{\neg\neg}$. Furthermore, this method does not work for lattices satisfying (JID) that are not Heyting algebras because regular operators cannot be defined for these lattices.

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References

[1] Balbes, R., Dwinger, Ph.: Distributive Lattices. University of Missouri Press, Columbia (1974)
[2] Banaschewski, B., Pultr, A.: Booleanization. Cahiers Topologie Géom. Différentielle Catég. **37(1)**, 41–60 (1996)
[3] Beazer, R., Macnab, D.S.: Modal extensions of Heyting algebras. Colloq. Math. **41**, 1–12 (1979)
[4] Bezhanishvili, G., Bezhanishvili, N.: Profinite Heyting algebras. Order **25(3)**, 211–227 (2008)
[5] Bezhanishvili, G., Bezhanishvili, N., Gabelaia, D., Kurz, A.: Bitopological duality for distributive lattices and Heyting algebras. Math. Structures Comput. Sci. **20**, 359–393 (2010)
[6] Bezhanishvili, G. and Ghilardi, S.: An algebraic approach to subframe logics. Intuitionistic case. Ann. Pure Appl. Logic **147**, 84–100 (2007)
[7] Blok, W.J., Dwinger, Ph.: Equational classes of closure algebras. I. Indag. Math. **37**, 189–198 (1975)
[8] Esakia, L.: Topological Kripke models. Soviet Math. Dokl. **15**, 147–151 (1974)
[9] Esakia, L.: The problem of dualism in the intuitionistic logic and Browerian lattices. In: V Inter. Congress of Logic, Methodology and Philosophy of Science, pp. 7–8. Canada (1975)
[10] Esakia, L.: Semantical analysis of bimodal (tense) systems. In: Logic, Semantics and Methodology, pp. 87–99. Metsniereba Press, Tbilisi (1978) (Russian)
[11] Esakia, L.: Heyting Algebras I. Duality Theory. Metsniereba Press, Tbilisi (1985) (Russian)
[12] Funayama, N.: Imbedding infinitely distributive lattices completely isomorphically into Boolean algebras. Nagoya Math. J. **15**, 71–81 (1959)
[13] Gehrke, M.: Canonical extensions, Esakia spaces, and universal models. In: Leo Esakia on Modal and Intuitionistic Logics. Springer (in press)
[14] Grätzer, G.: General Lattice Theory. Birkhäuser, Basel (1978)
[15] Harding, J., Bezhanishvili, G.: MacNeille completions of Heyting algebras. Houston J. Math. **30**, 937–952 (2004)
[16] Johnstone, P.T.: Stone Spaces. Cambridge University Press, Cambridge (1982)
[17] Jung, A., Moshier, M.A.: On the bitopological nature of Stone duality (2006, Technical Report CSR-06-13, University of Birmingham, 110 pages)
[18] McKinsey, J.C.C., Tarski, A.: On closed elements in closure algebras. Ann. of Math. **47**, 122–162 (1946)
[19] Picado, J.: Join-continuous frames, Priestley's duality and biframes. Appl. Categ. Structures **2(3)**, 297–313 (1994)
[20] Priestley, H.A.: Representation of distributive lattices by means of ordered Stone spaces. Bull. London Math. Soc. **2**, 186–190 (1970)
[21] Priestley, H.A.: Ordered topological spaces and the representation of distributive lattices. Proc. London Math. Soc. **24**, 507–530 (1972)
[22] Pultr, A., Sichler, J.: Frames in Priestley's duality. Cahiers Topologie Géom. Différentielle Catég. **29**, 193–202 (1988)

[23] Rasiowa, H., Sikorski, R.: The Mathematics of Metamathematics. Monografie Matematyczne, Tom 41, Państwowe Wydawnictwo Naukowe, Warsaw (1963)
[24] Sikorski, R.: Boolean algebras. Ergebnisse der Mathematik und ihrer Grenzgebiete, N. F., Heft 25, Springer, Berlin (1960)
[25] Simmons, H.: Spaces with Boolean assemblies. Colloq. Math. **43(1)**, 23–39 (1981)
[26] Stone, M.H.: The theory of representations for Boolean algebras. Trans. Amer. Math. Soc. **40**, 37–111 (1936)
[27] Wilson, J.T.: The Assembly Tower and Some Categorical and Algebraic Aspects of Frame Theory. PhD thesis, Carnegie Mellon University (1994)

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