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Connected modal logics

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Abstract We introduce the concept of a connected logic (over **S4**) and show that each connected logic with the finite model property is the logic of a subalgebra of the closure algebra of all subsets of the real line **R**, thus generalizing the McKinsey-Tarski theorem. As a consequence, we obtain that each intermediate logic with the finite model property is the logic of a subalgebra of the Heyting algebra of all open subsets of **R**.

Keywords Modal logic · Topology · Closure algebra · Connectedness

Mathematics Subject Classification (2000) 03B45 · 03B55

1 Introduction

It is a fundamental result of McKinsey and Tarski [16] that if we interpret modal diamond as the closure in a topological space, then **S4** is the logic of the closure algebra of all subsets of any dense-in-itself metrizable separable space. As a consequence, we obtain that **S4** is the logic of the closure algebra $\mathbf{R}^+ = (\wp(\mathbf{R}), \mathbf{cl})$ of all subsets of the real line **R**. (We use $\wp(\mathbf{R})$ to denote the powerset of **R** and **cl** to denote the closure

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in **R**.) This result was sharpened in [2], where it was shown that **S4** is in fact the logic of the closure algebra $B(C^{\omega}(\mathbf{R}))$ of Boolean combinations of countable unions of convex subsets of **R**. On the other hand, the logic of the closure algebra $C^{<\omega}(\mathbf{R})$ of finite unions of convex subsets of **R** is much stronger than **S4** (see [1, Sect. 5.1] or [20, Sect. 3]). There is a lot of room in between $C^{<\omega}(\mathbf{R})$ and $B(C^{\omega}(\mathbf{R}))$, and it is only natural to seek a hierarchy of modal logics which can be obtained as logics of closure algebras in the interval $[C^{<\omega}(\mathbf{R}), B(C^{\omega}(\mathbf{R}))]$. One obvious closure algebra in this interval is the closure algebra $B(\mathrm{Op}(\mathbf{R}))$ of Boolean combinations of open subsets of **R**. As follows from [2, Rem. 10], the logic of $B(\mathrm{Op}(\mathbf{R}))$ is **S4.Grz**—the well-known logic of Grzegorczyk, which is the extension of **S4** by the Grzegorczyk axiom $\Box(\Box(p \to \Box p) \to p) \to p$. It was left as an open problem in [2, Sect. 5] to classify the logics of closure algebras in the above interval. More generally, it is an open problem to classify the logics of subalgebras of $B(C^{\omega}(\mathbf{R}))$, or more generally yet, of subalgebras of $\mathbf{R}^+$. This paper is the first step in this direction. Based on the fact that **R** and hence every subalgebra of $\mathbf{R}^+$ is connected, we introduce the concept of a *connected logic* over **S4** and show that each connected logic with the finite model property (fmp for short) is the logic of a subalgebra of $\mathbf{R}^+$. Since **S4** itself is a connected logic with the fmp, the McKinsey-Tarski result follows. In fact, one way to look at the McKinsey-Tarski result is that **S4** is a connected logic with the fmp. There are, of course, many other connected logics with the fmp. The list includes such well-known modal systems as **S4.1**, **S4.2**, **S4.1.2**, **S4.Grz.2**, and **S5**. We describe the subalgebras of $\mathbf{R}^+$ that give rise to these logics. In addition, we show that each logic over **S4.1** is a connected logic. It follows that there are continuum many connected logics with the fmp and continuum many without the fmp. Our main result implies that subalgebras of $\mathbf{R}^+$ give rise to continuum many different connected logics. It remains an open problem whether there are subalgebras of $\mathbf{R}^+$ that give rise to connected logics without the fmp.

It is a consequence of our results and the Blok-Esakia theorem that each intermediate logic is a connected logic, and that each intermediate logic with the fmp is the logic of a subalgebra of the Heyting algebra $\mathrm{Op}(\mathbf{R})$ of all open subsets of **R**.

The McKinsey-Tarski theorem also implies that **S4** is the logic of the closure algebra $\mathbf{Q}^+ = (\wp(\mathbf{Q}), \mathbf{cl})$ of all subsets of the rational line **Q**, as well as the logic of the closure algebra $\mathbf{C}^+ = (\wp(\mathbf{C}), \mathbf{cl})$ of all subsets of the Cantor space **C**. Unlike **R**, both **Q** and **C** are highly disconnected. Based on this difference, as well as on the topological structure of **Q** and **C**, we show that each logic over **S4** with the fmp is the logic of a subalgebra of $\mathbf{Q}^+$ and also the logic of a subalgebra of $\mathbf{C}^+$. Consequently, each intermediate logic with the fmp is the logic of a subalgebra of the Heyting algebra $\mathrm{Op}(\mathbf{Q})$ of all open subsets of **Q** and also the logic of a subalgebra of the Heyting algebra $\mathrm{Op}(\mathbf{C})$ of all open subsets of **C**.

2 Preliminaries

We assume the reader's familiarity with the basics of modal logic and its relational, topological, and algebraic semantics. We use [6] and [5] as our main references on

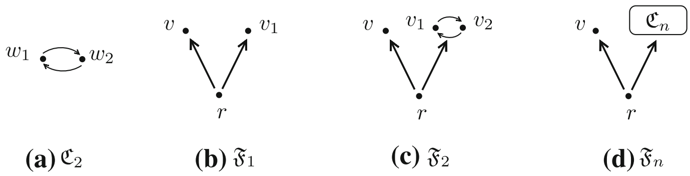


Fig. 1 The 2-cluster, the 1-fork, the 2-fork, and the n-fork

modal logic and its relational and algebraic semantics, and [19] as our main reference on the topological semantics of modal logic.

We recall that **S4** is the least set of formulas containing all classical tautologies, the axiom schemata: (1) $\Box\varphi \to \varphi$, (2) $\Box\varphi \to \Box\Box\varphi$, (3) $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$, and closed under modus ponens $(\varphi, \varphi \to \psi/\psi)$ and necessitation $(\varphi/\Box\varphi)$. The diamond $\Diamond$ is the usual abbreviation of $\neg\Box\neg$.

We also recall that a *closure algebra* is a pair $(B, \Diamond)$ where B is a Boolean algebra and $\Diamond : B \to B$ is a unary function satisfying the following four Kuratowski axioms: (i) $a \le \Diamond a$, (ii) $\Diamond a = \Diamond\Diamond a$, (iii) $\Diamond(a \vee b) = \Diamond a \vee \Diamond b$, and (iv) $\Diamond 0 = 0$. As usual, the dual operator $\Box : B \to B$ is defined by $\Box a = \neg\Diamond\neg a$.

It is well known that closure algebras are algebraic models of **S4**. Typical examples of closure algebras come from topology. If X is a topological space, then the powerset $\wp(X)$ of X together with the closure operator **cl** forms a closure algebra. In fact, each closure algebra is represented as a subalgebra of the closure algebra $(\wp(X), \mathbf{cl})$ for some topological space X [16, Thm. 2.4].

Another source of examples of closure algebras comes from the relational semantics of **S4**. Given an **S4**-frame $\mathfrak{F} = (W, R)$ (that is, a quasi-ordered set), the powerset $\wp(W)$ together with R^{-1} forms a closure algebra, where we recall that $R^{-1}[A] = \{w \in W : \exists a \in A \text{ with } wRa\}$. Again, we have a representation theorem: each closure algebra is represented as a subalgebra of the closure algebra $(\wp(W), R^{-1})$ for some **S4**-frame $\mathfrak{F}$ [15, Thms. 3.10, 3.14].

We will unify the relational and topological semantics of **S4** by viewing each **S4**-frame $\mathfrak{F} = (W, R)$ as a topological space with the topology τ_R consisting of upsets of W, where we recall that $U \subseteq W$ is an *upset* of W if $u \in U$ and uRw imply $w \in U$. From this point of view, **S4**-frames form a subclass of topological spaces, known as *Alexandroff spaces*, and are characterized topologically by the property that the intersection of any family of open sets is again open. In such a space the closure of a subset A is given by $\mathbf{cl}(A) = R^{-1}[A]$ and the interior is given by $\mathbf{int}(A) = -R^{-1}[-A]$, where $-$ denotes set-theoretic complement.

Let $\mathfrak{F} = (W, R)$ be an **S4**-frame. As usual, we call a subset C of W a *cluster* if wRv and vRw for each $w, v \in C$. A cluster C is *simple* if $C = \{w\}$ for some $w \in W$. A cluster containing a point w will be denoted by $C[w]$. The *n-cluster* is a pair $\mathfrak{C}_n = (W_n, R_n)$, where $W_n = \{w_1, \dots, w_n\}$ and $R_n = W_n \times W_n$. The 2-cluster is shown in Fig. 1a.

A point $w \in W$ is called *quasi-maximal* if $R[w] = C[w]$, in which case $C[w]$ is called a *maximal cluster*. A *maximal point* is a quasi-maximal point w such that $C[w]$ is a simple cluster. The notions of *quasi-minimal point*, *minimal point*, and *minimal cluster* are dual.

A frame $\mathfrak{F}$ is called *rooted* if $W = R[w]$ for some $w \in W$. In such a case w (and indeed any $v \in C[w]$) is called a *root* of $\mathfrak{F}$. If $C[r]$ is simple, then r is called *the root*. The *n-fork* is a pair $\mathfrak{F}_n = (V_n, Q_n)$, where $V_n = \{r, v, v_1, \ldots, v_n\}$, $rQ_n w$ for each $w \in V_n$, $vQ_n v$, and $v_i Q_n v_j$ for each $i, j \leq n$. The 1-fork will simply be called the *fork*. The fork, the 2-fork, and the n-fork are shown in Fig. 1b–d.

Note that the subframe $W_n = \{v_1, \ldots, v_n\}$ of the n-fork $\mathfrak{F}_n$ is exactly the n-cluster. It is a maximal cluster, all $v_i \in W_n$ are quasi-maximal points, $v \in V_n$ is the only maximal point, and $r \in V_n$ is the root of $\mathfrak{F}_n$.

3 Connected closure algebras

Let A be a closure algebra. Following [16, Sect. 1], we call an element a of A *closed* if $a = \Diamond a$ and *open* if $a = \Box a$. Also, following the standard topological terminology, we call $a \in A$ *clopen* if $\Box a = a = \Diamond a$ and *regular open* if $a = \Box\Diamond a$. The next lemma is a simple yet useful tool for our considerations. A proof of (i) and (ii) is an obvious generalization of similar statements for topological spaces, and (iii) is a consequence of [16, Cor. 1.8].

Lemma 3.1 *Let A be a closure algebra and $a, b \in A$. Then:*

(i) $\Box\Diamond\Box\Diamond a = \Box\Diamond a$.
(ii) *If a, b are regular open, then $a \wedge b$ is regular open.*
(iii) $\Diamond a \wedge \Box b \leq \Diamond(a \wedge \Box b)$.

Definition 3.2 [16, Def. 1.9] A closure algebra A is called *connected* if 0 and 1 are the only clopen elements of A.

Clearly if A is a connected algebra, then each subalgebra S of A is also connected.

Theorem 3.3 *A closure algebra is connected iff it is isomorphic to a subalgebra of the closure algebra $X^+ = (\wp(X), \mathbf{cl})$ of all subsets of a connected space X.*

Proof Clearly if X is connected, then X^+ is connected and so each closure algebra isomorphic to a subalgebra of X^+ is also connected. Conversely, let A be connected. It follows from [16, Thm. 2.4] that there is a topological space X such that A is isomorphic to a subalgebra of X^+. We recall that X can be taken to be the set of all ultrafilters of A and the topology on X is defined by the basis $\{\varphi(\Box a) : a \in A\}$, where $\varphi(a) = \{x \in X : a \in x\}$ is the Stone map. It follows that a subset U of X is open if $U = \bigcup\{\varphi(\Box a) : \varphi(\Box a) \subseteq U\}$. We show that X is connected. For this we observe that X is compact, which is easy to see because the Stone topology on X, which has $\{\varphi(a) : a \in A\}$ as a basis, is a finer topology. Since the Stone topology is compact, so is X (for details we refer to [1, Fact 3.6] or [3, Thm. 2.12]). To finish the proof, let U be a clopen subset of X. Then $U = \bigcup\{\varphi(\Box a) : \varphi(\Box a) \subseteq U\}$ and also U is closed. Since X is compact and U is closed, U is compact. Therefore, there exist $a_1, \ldots, a_n$ such that $U = \varphi(\Box a_1) \cup \cdots \cup \varphi(\Box a_n)$. Let $a = \Box a_1 \vee \cdots \vee \Box a_n$. Then a is an open element of A. As φ commutes with $\cup$, we obtain $U = \varphi(a)$. Because $\varphi(a)$ is closed, $\varphi(a) = \mathbf{cl}(\varphi(a)) = \varphi(\Diamond a)$. Therefore, $a = \Diamond a$, so $\Box a = a = \Diamond a$, and so a is clopen in A. Since A is connected, either $a = 0$ or $a = 1$. Therefore, either $U = \varphi(0) = \emptyset$ or $U = \varphi(1) = X$, which implies that X is connected. □

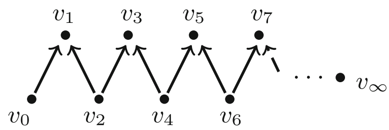


Fig. 2 The descriptive **S4**-frame $\mathfrak{F}$

Let $\mathfrak{F} = (W, R)$ be an **S4**-frame and $w, v \in W$. We recall that a *path* between $w, v \in W$ is a finite sequence $w_0, \dots, w_n \in W$ such that $w_0 = w, w_n = v$ and for all $i < n$ either $w_i R w_{i+1}$ or $w_{i+1} R w_i$. We call $\mathfrak{F}$ *path-connected* if there is a path between any two points of $\mathfrak{F}$. It turns out that viewing $\mathfrak{F}$ as a topological space, path-connectedness simply means connectedness. This is a well-known fact. We only give a sketch of the proof.

Lemma 3.4 *Let $\mathfrak{F} = (W, R)$ be an* **S4***-frame. Then (W, R) is path-connected iff (W, τ_R) is a connected space.*

Proof ($\Leftarrow$) Suppose that (W, R) is not path-connected. Then there exists no path between some $w, v \in W$. Consider the set C of all points of W which are connected by a path to v. Clearly C is an upset and a downset, thus C is clopen in τ_R. As C is nonempty (because $v \in C$) and $C \neq X$ (because $w \notin C$), we obtain that (W, τ_R) is disconnected.

[$\Rightarrow$] Suppose that (W, τ_R) is disconnected. Then there exists a nonempty clopen $C \subsetneq X$. Therefore, C is an upset and a downset. Take $w \in C$ and $v \notin C$. Clearly there is no path connecting w with v. Thus, (W, R) is not path-connected. $\square$

Let A be a closure algebra. We recall that the standard construction of the **S4**-frame $\mathfrak{F} = (W, R)$ such that A is isomorphic to a subalgebra of $\mathfrak{F}^+ = (\wp(W), R^{-1})$ is as follows: W is the set of ultrafilters of A and

$$wRv \text{ iff } \Box a \in w \text{ implies } a \in v \text{ for each } a \in A.$$

The frame $\mathfrak{F}$ is usually referred to as the *ultrafilter frame* of A. Based on Theorem 3.3 and Lemma 3.4, it is natural to expect that $\mathfrak{F}$ is path-connected. However, this is not the case in general. In order to give a counterexample, we recall Esakia duality between closure algebras and descriptive **S4**-frames.

A *descriptive **S4**-frame* is a Stone (that is, compact, Hausdorff, and zero-dimensional) space X together with a quasi-order $R \subseteq X \times X$ such that (a) $R[x]$ is closed for each $x \in X$ and (b) $R^{-1}[A]$ is clopen for each clopen $A \subseteq X$. It follows from [8] that the category of closure algebras is dually equivalent to the category of descriptive **S4**-frames.

Let $\mathfrak{F} = (V, S, P)$ be the descriptive **S4**-frame shown in Fig. 2, where P is the Boolean algebra of finite subsets of V (without v_∞) and cofinite subsets of V (with v_∞). Clearly (V, S) is not path-connected. On the other hand, it is easy to see that (P, S^{-1}) is a connected closure algebra. By Esakia duality, (V, S) is isomorphic to the ultrafilter frame of (P, S^{-1}). Consequently, the ultrafilter frame of a connected closure algebra A may not be path-connected. Nevertheless, we can still embed (P, S^{-1}) in the closure algebra of all subsets of a path-connected frame. Let $V_0 = V - \{v_\infty\}$ and let S_0 be the restriction of S to V_0. Clearly $\mathfrak{F}_0 = (V_0, S_0)$ is path-connected. Moreover, it is easy to see that $f : P \to \wp(V_0)$, given by $f(E) = E \cap V_0$ for each $E \in P$, is a closure algebra embedding. The following remains an open problem:

Open Problem 1: Is each connected closure algebra isomorphic to a subalgebra of the closure algebra of all subsets of some path-connected **S4**-frame?

4 Connected logics

For a closure algebra A, we let $L(A)$ denote the logic of A; that is, the set of formulas valid in A. Clearly $L(A)$ is a logic over **S4**. Similarly, for a topological space X, we let $L(X)$ denote the logic of X, and for an **S4**-frame $\mathfrak{F}$, we let $L(\mathfrak{F})$ denote the logic of $\mathfrak{F}$. Clearly $L(X) = L(X^+)$ and $L(\mathfrak{F}) = L(\mathfrak{F}^+)$, so both $L(X)$ and $L(\mathfrak{F})$ are logics over **S4**.

Definition 4.1 We call a modal logic L over **S4** *connected* if $L = L(A)$ for some connected closure algebra A.

Our main theorem can now be formulated as follows:

Main Theorem *Let L be a modal logic over* **S4** *with the fmp. Then the following conditions are equivalent:*

(1) *L is connected.*
(2) *$L = L(\mathfrak{F})$ for some path-connected* **S4**-*frame $\mathfrak{F}$.*
(3) *$L = L(X)$ for some connected space X.*
(4) *$L = L(A)$ for some subalgebra A of $\mathbf{R}^+$.*

In order to prove our main theorem, we will require a series of technical lemmas, which uncover the structure of the rooted finite frames of connected logics (independent of whether the logic is generated by its finite frames or not).

Before plunging into the technical details, we say a couple of words about the technique we will use. We will rely heavily on the splitting technique, developed by Jankov [14] for intermediate logics, and adapted by Rautenberg [17] to modal logics. In particular, the splitting theorem implies that if a finite subdirectly irreducible closure algebra A belongs to the variety generated by a closure algebra B, then A is a homomorphic image of a subalgebra of B. We will also use the Jankov-type frame formulas for **S4** developed by Fine [13] (and others), and Esakia duality between closure algebras and descriptive **S4**-frames. This duality yields the dual equivalence between the category of finite closure algebras and the category of finite **S4**-frames (see, e.g., [12]), which, of course, is isomorphic to the category of finite topological spaces. In particular, a finite **S4**-frame $\mathfrak{F}$ is rooted iff the corresponding closure algebra $\mathfrak{F}^+$ is subdirectly irreducible. Finally, the n-clusters and n-forks described in the preliminary section will play a fundamental role in our considerations.

To aid the reader in following the proof of the first of our key lemmas, we consider a guiding example. Consider the fork $\mathfrak{F}_1 = (V, S)$ where $V = \{r, v, v_1\}$, the two-cluster $\mathfrak{C}_2 = (C, T)$ where $C = \{c_1, c_2\}$, and the two-fork $\mathfrak{F}_2 = (W, R)$ where $W = \{w_0, w_1, w_2, w_3\}$ (see Fig. 3).

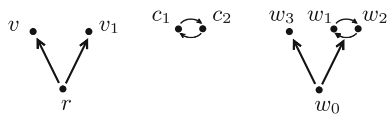


Fig. 3 The fork, the 2-cluster, and the 2-fork

(a) $\mathfrak{F}_1$ **(b)** $\mathfrak{C}_2$ **(c)** $\mathfrak{F}_2$

Given onto interior maps $f : \mathbf{R} \to \mathfrak{F}_1$ and $g : \mathbf{R} \to \mathfrak{C}_2$, we show how to construct an onto interior map $h : \mathbf{R} \to \mathfrak{F}_2$. To aid the imagination, the reader may consider a more concrete example, where f is given by sending the interval $(-\infty, 0)$ to v, the interval $(0, \infty)$ to v_1, and 0 to r; while g is given by sending all the rational numbers to c_1 and all the irrational numbers to c_2. Let $C_1 = g^{-1}(c_1)$ and $C_2 = g^{-1}(c_2)$. Since g is an interior map, it is clear that C_1 and C_2 are disjoint dense subsets of $\mathbf{R}$, so $\mathbf{cl}(C_1) = \mathbf{cl}(C_2) = \mathbf{R}$. Let $U = f^{-1}(v_1)$. Since f is interior, U is a regular open subset of $\mathbf{R}$. Now let $B_1 = C_1 \cap U$ and $B_2 = C_2 \cap U$. It is easy to see that both B_1 and B_2 are dense in U, that $B_1 \cap B_2 = \emptyset$, and that $B_1 \cup B_2 = U$. Let $B_3 = \mathbf{int}(-U)$. Since f is interior, it is clear that $B_3 = f^{-1}(v)$. We let $B_0 = -(B_3 \cup U)$. Obviously $B_0 = f^{-1}(r)$. Since f is an interior map, $B_0 \subseteq \mathbf{cl}(B_i)$ and $B_i \subseteq -\mathbf{cl}(B_0)$ for $1 \leq i \leq 3$. We define $h : \mathbf{R} \to \mathfrak{F}_2$ by sending all the points from B_i to the point $w_i \in W$ (where $0 \leq i \leq 3$). Then h is an onto interior map. This is easy to check for our concrete choices of f and g because $B_0 = \{0\}$, B_1 is the set of all positive rational numbers, B_2 is the set of all positive irrational numbers, and $B_3 = (-\infty, 0)$.

The next lemma generalizes this idea to connected closure algebras.

Lemma 4.2 *Let L be a connected logic over* **S4**. *If $\mathfrak{F}_1 \models L$ and $\mathfrak{C}_n \models L$, then $\mathfrak{F}_n \models L$.*

Proof Suppose that $\mathfrak{F}_1 \models L$ and $\mathfrak{C}_n \models L$. Since L is connected, $L = L(A)$ for some connected closure algebra A. Since $\mathfrak{F}_1$ and $\mathfrak{C}_n$ are rooted frames, the closure algebras $\mathfrak{F}_1^+$ and $\mathfrak{C}_n^+$ are subdirectly irreducible. Therefore, by the splitting theorem, $\mathfrak{F}_1^+$ and $\mathfrak{C}_n^+$ are homomorphic images of subalgebras of A. Thus, there exist subalgebras B and C of A and onto homomorphisms $g : B \twoheadrightarrow \mathfrak{F}_1^+$ and $h : C \twoheadrightarrow \mathfrak{C}_n^+$. We show that $\mathfrak{F}_n^+$ is a homomorphic image of a subalgebra of A.

Claim 1 *There exist $c_1, \ldots, c_n, u \in C$ such that:*

(i) $c_i \wedge c_j = 0$ *whenever* $i \neq j$,
(ii) $\bigvee c_i = u > 0$,
(iii) $u \leq \Diamond c_i$ *for each* $i \leq n$,
(iv) u *is regular open.*

Proof of claim: We recall that $\mathfrak{C}_n = (W_n, R_n)$, where $W_n = \{w_1, \ldots, w_n\}$ and $R_n = W_n \times W_n$. Choose $a_1, \ldots, a_n \in C$ such that $h(a_i) = \{w_i\}$. Let $i \neq j$. If $a_i \wedge a_j \neq 0$, then we take $a_i - \bigvee_{i \neq j} a_j$ in place of a_i. Since

$$h(a_i - \bigvee_{i \neq j} a_j) = h(a_i) - \bigvee_{i \neq j} h(a_j) = \{w_i\} - \{w_j : j \neq i\} = \{w_i\} = h(a_i),$$

we may assume without loss of generality that $a_i \wedge a_j = 0$.

Set $u = \Box \bigwedge \Diamond a_i$ and $c_i = a_i \wedge u$. Clearly $c_i \leq u$ and $c_i \wedge c_j = 0$ if $i \neq j$. Since $u = \bigwedge \Box\Diamond a_i$, by Lemma 3.1(i,ii), u is regular open. We show that $u \leq \Diamond c_i$ for each i. By Lemma 3.1(iii), we have:

$$\Diamond c_i = \Diamond(a_i \wedge \Box \bigwedge_j \Diamond a_j) \geq \Diamond a_i \wedge \Box \bigwedge_j \Diamond a_j$$
$$\geq \Box\Diamond a_i \wedge \bigwedge_j \Box\Diamond a_j = \bigwedge_j \Box\Diamond a_j = u.$$

We also have $u > 0$ since $h(u) = \Box \bigwedge_j \Diamond(h(a_j)) = W_n$. Finally, if $\bigvee c_i < u$, then by replacing c_n with $c'_n = u - \bigvee_{i=1}^{n-1} c_i$, we ensure that $\bigvee c_i = u$. Clearly $c'_n \geq c_n$, and so $\Diamond c'_n \geq \Diamond c_n \geq u$. Lastly, by construction of c'_n, we have $c'_n \wedge c_i = 0$ for each $i \neq n$. □

Our next task is to refine Claim 1 and show that u can be chosen so that $u \neq 1$. For this we have to go beyond C and use some elements of B as well. Suppose that $u = 1$. Let w denote a maximal point of $\mathfrak{F}_1$. Since $g : B \twoheadrightarrow \mathfrak{F}_1^+$ is onto, there is $v_1 \in B$ such that $g(v_1) = \{w\}$. We have

$$g(\Box\Diamond v_1) = \Box\Diamond g(v_1) = \Box\Diamond\{w\} = \{w\}$$

as $\{w\}$ is regular open in $\mathfrak{F}_1$. This implies that $\Box\Diamond v_1 \neq 0$. Set $v = \Box\Diamond v_1$. By Lemma 3.1(i), v is regular open. Let $b_i = c_i \wedge v$. Then $b_1, \ldots, b_n, v$ satisfy the conditions of Claim 1. To see this observe that the only nontrivial clause to check is $v \leq \Diamond b_i$. But since v is open, by Lemma 3.1(iii), we obtain

$$\Diamond b_i = \Diamond(c_i \wedge v) \geq \Diamond c_i \wedge v = v$$

since $\Diamond c_i \geq u = 1$. Consequently, we have found $b_1, \ldots, b_n, v \in A$ such that they satisfy the conditions of Claim 1 and in addition $v < 1$. Set $b_{n+1} = \Box\neg v$ and $b_0 = \neg(v \vee b_{n+1})$.

Claim 2 *The elements $b_0, \ldots, b_{n+1} \in A$ satisfy the following conditions:*

(i) $b_i \wedge b_j = 0$ *whenever* $i \neq j$,
(ii) $\bigvee b_i = 1$,
(iii) $0 < b_0 \leq \Diamond b_i$ *for each* $i \leq n+1$,
(iv) $b_i \leq \Diamond b_j$ *for each* $1 \leq i, j \leq n$,
(v) $b_i \leq \neg(\Diamond b_0 \vee \Diamond b_{n+1})$ *for each* $1 \leq i \leq n$ *and* $b_{n+1} \leq \neg\Diamond b_0$.

Proof of claim: By the choice of $b_0, \ldots, b_{n+1}$, (i), (ii), and (iv) are easily seen to be satisfied. We proceed to show (iii). It follows from the definition of b_0 that b_0 is closed. Moreover,

$$b_0 = \neg(v \vee b_{n+1}) = \neg v \wedge \neg\Box\neg v = \neg\Box\Diamond v \wedge \Diamond v = \Diamond\Box\neg v \wedge \Diamond v = \Diamond b_{n+1} \wedge \Diamond v.$$

Therefore, $b_0 \leq \Diamond b_{n+1}$ and $b_0 \leq \Diamond v$. We also have $v \leq \Diamond b_i$ for each $1 \leq i \leq n$. Thus, $\Diamond v \leq \Diamond\Diamond b_i = \Diamond b_i$ for each $1 \leq i \leq n$. It follows that $b_0 \leq \Diamond b_i$ for each $1 \leq i \leq n$. Suppose that $b_0 = 0$. Then $v \vee b_{n+1} = 1$, so $v \vee \Box\neg v = 1$, and so $\neg\Box\neg v \leq v$. Therefore, $\Diamond v \leq v$, and so $v = \Diamond v$, which is impossible because A has no nontrivial clopens. Consequently, $0 < b_0 \leq \Diamond b_i$ for each $i \leq n+1$.

Finally, we show (v). Since b_0 is closed and $b_0 \leq \Diamond b_{n+1}$, for each $1 \leq i \leq n$ we have:

$$\neg(\Diamond b_0 \vee \Diamond b_{n+1}) = \neg(b_0 \vee \Diamond b_{n+1}) = \neg\Diamond b_{n+1} = \neg\Diamond\Box\neg v = \Box\Diamond v = v \geq b_i.$$

Moreover, $\neg\Diamond b_0 = \neg b_0 = \neg\neg(v \vee b_{n+1}) = (v \vee b_{n+1}) \geq b_{n+1}$. Thus, $b_{n+1} \leq \neg\Diamond b_0$ and $b_i \leq \neg(\Diamond b_0 \vee \Diamond b_{n+1})$ for each $1 \leq i \leq n$. □

We are now one step away from completing the proof of the lemma. Recall that the n-fork is the frame $\mathfrak{F}_n = (V_n, Q_n)$, where $V_n = \{w_0, w_1, w_2, \ldots, w_{n+1}\}$, $w_0 Q_n w$ for each $w \in V_n$, $w_i Q_n w_j$ for all $1 \leq i, j \leq n$, and $w_{n+1} Q_n w_{n+1}$. Let $\chi(\mathfrak{F}_n)(p_0 \ldots, p_{n+1})$ be the Fine formula of $\mathfrak{F}_n$; that is,

$$\chi(\mathfrak{F}_n)(p_0, \ldots, p_{n+1}) = p_0 \wedge \Box \bigvee_{0\leq i\leq n+1} p_i \wedge \Box \bigwedge_{i\neq j} (p_i \to \neg p_j) \wedge \bigwedge_{w_i R w_j} \Box(p_i \to \Diamond p_j) \wedge \bigwedge_{w_i \not R w_j} \Box(p_i \to \neg\Diamond p_j)$$

We show that $\chi(\mathfrak{F}_n)[b_0, \ldots, b_{n+1}] = b_0$. By Claim 2(i, ii), $\bigwedge_{i\neq j} (b_i \to \neg b_j) = 1$ and $\bigvee_{0\leq i\leq n+1} b_i = 1$. Since

$$R = \{(w_0, w_i) : 0 \leq i \leq n+1\} \cup \{(w_i, w_j) : 1 \leq i \leq n\} \cup \{(w_{n+1}, w_{n+1})\},$$

by Claim 2(iii, iv), $w_i R w_j$ implies $b_i \leq \Diamond b_j$. Thus, $\bigwedge_{w_i R w_j} \Box(b_i \to \Diamond b_j) = 1$. As $w_{n+1} \not R w_i$ for $0 \leq i \leq n$ and $w_i \not R w_j$ for $1 \leq i \leq n$ and $j = 0, n+1$, by Claim 2(v), $b_i \leq \neg\Diamond b_j$ whenever $w_i \not R w_j$. Therefore, $\bigwedge_{w_i \not R w_j} \Box(b_i \to \neg\Diamond b_j) = 1$, and so we have $\chi(\mathfrak{F}_n)[b_0, \ldots, b_{n+1}] = b_0$. Because $b_0 \neq 0$, the formula $\neg\chi(\mathfrak{F}_n)$ is refutable on A. Therefore, by the Fine theorem, $\mathfrak{F}_n^+$ is a subalgebra of a homomorphic image of A.[1] Consequently, $\mathfrak{F}_n \models L$. □

Now we introduce the operation of "gluing" of the n-fork with an **S4**-frame that has a maximal n-cluster.

Definition 4.3 Let $\mathfrak{F} = (W, R)$ be an **S4**-frame with a maximal n-cluster $C = \{w_1, \ldots, w_n\}$. Define the frame $\mathfrak{F}\natural\mathfrak{F}_n = (W_1, R_1)$ as follows (see Fig. 4): $W_1 = W \cup$

[1] Actually, Fine's theorem in its original formulation applies to **S4**-frames. The version for descriptive **S4**-frames can be found in [21, Lem. 3.9]. This together with the duality between descriptive **S4**-frames and closure algebras yields the algebraic reformulation of Fine's theorem used here.

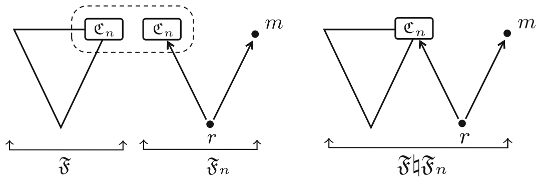


Fig. 4 The 'gluing' of $\mathfrak{F}$ and $\mathfrak{F}_n$

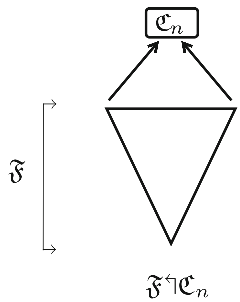


Fig. 5 The ordered sum of $\mathfrak{F}$ and $\mathfrak{C}_n$

$\{r, m\}$, where $r, m \notin W$, $R_1 \cap (W \times W) = R$, rR_1r, mR_1m, rR_1m, and rR_1w_i for all $i \leq n$.

Lemma 4.4 *Let L be a connected logic and $\mathfrak{F}_1 \not\models L$. If $\mathfrak{F}$ is an* **S4***-frame with a maximal n-cluster such that $\mathfrak{F} \models L$, then $\mathfrak{F}\natural\mathfrak{F}_n \models L$.*

Proof Let $C = \{w_1, \ldots, w_n\}$ be a maximal n-cluster of $\mathfrak{F}$. Since C is a maximal cluster, the **S4**-frame (C, R_C) is a generated subframe of $\mathfrak{F}$, where R_C is the restriction of R to C. But (C, R_C) is isomorphic to $\mathfrak{C}_n$. Therefore, $\mathfrak{C}_n \models L$. Thus, by Lemma 4.2, $\mathfrak{F}_n \models L$. Let $\mathfrak{F} \sqcup \mathfrak{F}_n$ denote the disjoint union of $\mathfrak{F}$ and $\mathfrak{F}_n$. Then $\mathfrak{F} \sqcup \mathfrak{F}_n \models L$. Since "gluing" the points of respective maximal n-clusters (see Fig. 4) produces an onto p-morphism $f : \mathfrak{F} \sqcup \mathfrak{F}_n \twoheadrightarrow \mathfrak{F}\natural\mathfrak{F}_n$, we obtain $\mathfrak{F}\natural\mathfrak{F}_n \models L$. □

If $\mathfrak{F}_1 \not\models L$, then it is well-known (see, e.g., [22, Sect. 6.1]) that **S4.2** $\subseteq L$, where **S4.2** $=$ **S4** $+ \Diamond\Box p \to \Box\Diamond p$. The finite rooted **S4.2**-frames are precisely the finite **S4**-frames with a unique maximal cluster (see, e.g., [6, Sect. 3.5]). We define another operation on the **S4**-frames which will always produce an **S4.2**-frame.

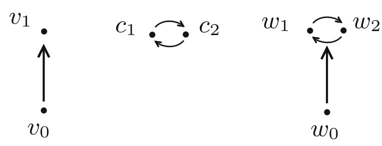
Definition 4.5 Let $\mathfrak{F} = (W, R)$ be an **S4**-frame and $\mathfrak{C}_n = (W_n, R_n)$ the n-cluster. Without loss of generality we assume that $W \cap W_n = \emptyset$, and define the *ordered sum* $\mathfrak{F}\Lsh\mathfrak{C}_n = (W_1, R_1)$ as follows (see Fig. 5):

$$\begin{aligned}
&W_1 = W \cup W_n,\\
&R_1 \cap (W \times W) = R,\\
&R_1 \cap (W_n \times W_n) = R_n\\
&wR_1w_i \text{ for each } w \in W \text{ and } w_i \in W_n.
\end{aligned}$$

That is, $\mathfrak{F}\Lsh\mathfrak{C}_n$ is obtained by putting the n-cluster $\mathfrak{C}_n$ on top of $\mathfrak{F}$. Clearly $\mathfrak{F}\Lsh\mathfrak{C}_n$ has a unique maximal cluster which is isomorphic to $\mathfrak{C}_n$.

To aid the reader in following the proof of the second of our key lemmas, we again consider a guiding example. Consider the frame $\mathfrak{G}_1 = (V, S)$ where $V = \{v_0, v_1\}$,

Fig. 6 Guiding example to Lemma 4.6



(a) $\mathfrak{G}_1$ **(b)** $\mathfrak{C}_2$ **(c)** $\mathfrak{G}_2$

the two-cluster $\mathfrak{C}_2 = (C, T)$ where $C = \{c_1, c_2\}$, and the frame $\mathfrak{G}_2 = (W, R)$ where $W = \{w_0, w_1, w_2\}$ (see Fig. 6).

Given onto interior maps $f : \mathbf{R} \to \mathfrak{G}_1$ and $g : \mathbf{R} \to \mathfrak{C}_2$, we show how to construct an onto interior map $h : \mathbf{R} \to \mathfrak{G}_2$. To aid the imagination, the reader may consider a more concrete example, where f is given by sending 0 to v_0 and $\mathbf{R} - \{0\}$ to v_1; while g is given by sending all the rational numbers to c_1 and all the irrational numbers to c_2. Let $C_1 = g^{-1}(c_1)$ and $C_2 = g^{-1}(c_2)$. Since g is an interior map, it is clear that C_1 and C_2 are dense subsets of $\mathbf{R}$. Let $B_0 = f^{-1}(v_0)$ and $B_1 = f^{-1}(v_1)$. Since f is interior, B_1 is an open dense subset of $\mathbf{R}$. Now let $D_0 = B_0$, $D_1 = C_1 \cap B_1$, and $D_2 = C_2 \cap B_1$. It is easy to see that both D_1 and D_2 are dense in B_1, that $D_1 \cap D_2 = \emptyset$, and that $D_1 \cup D_2 = B_1$. Since f is an interior map, $D_0 \subseteq \mathbf{cl}(D_1)$ and $D_0 \subseteq \mathbf{cl}(D_2)$. We define $h : \mathbf{R} \to \mathfrak{F}_2$ by sending all the points from D_i to the point $w_i \in W$ (where $0 \leq i \leq 2$). Then h is an onto interior map. This is easy to check for our concrete choices of f and g because $D_0 = \{0\}$, D_1 is the set of all nonzero rational numbers, and D_2 is the set of all irrational numbers.

The next lemma generalizes this idea to connected closure algebras.

Lemma 4.6 *Let L be a connected logic over* **S4.2** *and let $\mathfrak{F}$ be a rooted* **S4***-frame. For $n, k < \omega$, if $\mathfrak{F}\urcorner\mathfrak{C}_n \models L$ and $\mathfrak{C}_k \models L$, then $\mathfrak{F}\urcorner\mathfrak{C}_k \models L$.*

Proof We first observe that $\mathfrak{F}\urcorner\mathfrak{C}_1$ is a p-morphic image of $\mathfrak{F}\urcorner\mathfrak{C}_n$. Therefore, $\mathfrak{F}\urcorner\mathfrak{C}_n \models L$ implies $\mathfrak{F}\urcorner\mathfrak{C}_1 \models L$. As L is connected, $L = L(A)$ for some connected closure algebra A. Because $\mathfrak{F}\urcorner\mathfrak{C}_1 \models L$, $\mathfrak{C}_k \models L$, and $(\mathfrak{F}\urcorner\mathfrak{C}_1)^+$, $\mathfrak{C}_k^+$ are subdirectly irreducible algebras, it follows from the splitting theorem that both $(\mathfrak{F}\urcorner\mathfrak{C}_1)^+$ and $\mathfrak{C}_k^+$ are homomorphic images of subalgebras of A. Therefore, there exist subalgebras B and C of A and onto homomorphisms $g : B \twoheadrightarrow (\mathfrak{F}\urcorner\mathfrak{C}_1)^+$ and $h : C \twoheadrightarrow \mathfrak{C}_k^+$. We show that $(\mathfrak{F}\urcorner\mathfrak{C}_k)^+$ is also a homomorphic image of a subalgebra of A.

An argument similar to the proof of Claim 1 of Lemma 4.2 produces $c_1, \ldots, c_k, u \in C$ such that:

(i) $c_i \wedge c_j = 0$ whenever $i \neq j$,
(ii) $\bigvee c_i = u > 0$,
(iii) $u \leq \Diamond c_i$ for each $i \leq k$,
(iv) u is regular open.

Since A is an **S4.2**-algebra, we have $\Diamond\Box a \leq \Box\Diamond a$ for all $a \in A$. Therefore,

$$\Diamond u = \Diamond\Box u \leq \Box\Diamond u = u$$

Consequently, u is closed. As u is also open, $u > 0$ is clopen, which by connectedness of A implies $u = 1$. Thus, the conditions for $c_1, \ldots, c_k$ can be rewritten as follows:

(i) $c_i \wedge c_j = 0$ whenever $i \neq j$,
(ii) $\bigvee c_i = 1$,
(iii) $\Diamond c_i = 1$ for each $i \leq k$.

Now consider the frame $\mathfrak{F}^{\urcorner}\mathfrak{C}_1 = (W, R)$. Suppose $W = \{w_0, \ldots, w_l, w_{l+1}\}$ where w_0 is a root and w_{l+1} is the maximal point coming from $\mathfrak{C}_1$. Choose $b_0, \ldots, b_l, b_{l+1} \in B$ so that $g(b_i) = \{w_i\}$ for each $i \leq l+1$ and $b_i \wedge b_j = 0$ if $i \neq j$. We may also assume that $b_{l+1} = \Box b_{l+1}$ since $g(\Box b_{l+1}) = \Box g(b_{l+1}) = \Box\{w_{l+1}\} = \{w_{l+1}\} = g(b_{l+1})$ and $\Box b_{l+1} \leq b_{l+1}$. Let $\chi(\mathfrak{F}^{\urcorner}\mathfrak{C}_1)(p_0, \ldots, p_{l+1})$ be the Fine formula of $\mathfrak{F}^{\urcorner}\mathfrak{C}_1$. We have:

$$\begin{aligned} g[\chi(\mathfrak{F}^{\urcorner}\mathfrak{C}_1)(b_0, \ldots, b_{l+1})] &= \chi(\mathfrak{F}^{\urcorner}\mathfrak{C}_1)(g(b_0), \ldots, g(b_{l+1})) \\ &= \chi(\mathfrak{F}^{\urcorner}\mathfrak{C}_1)(\{w_0\}, \ldots, \{w_{l+1}\}) = \{w_0\} \end{aligned}$$

Therefore, $\chi(\mathfrak{F}^{\urcorner}\mathfrak{C}_1)[b_0, \ldots, b_{l+1}] > 0$.

Define $d_0, \ldots, d_{l+k}$ by $d_i = b_i$ for $0 \leq i \leq l$ and $d_{l+j} = c_j \wedge b_{l+1}$ for $1 \leq j \leq k$. Since $\bigvee c_i = 1$, the distributive law gives us $\bigvee_{j=1}^{k} d_{l+j} = b_{l+1}$. Moreover, since b_{l+1} is open, by Lemma 3.1(iii), we have:

$$\Diamond d_{l+j} = \Diamond(c_i \wedge b_{l+1}) \geq \Diamond c_i \wedge b_{l+1} = 1 \wedge b_{l+1} = b_{l+1}.$$

Thus, $\Diamond b_{l+1} \leq \Diamond d_{l+j}$ for $1 \leq j \leq k$.

Now we take a closer look at the formula $\chi(\mathfrak{F}^{\urcorner}\mathfrak{C}_k)[d_0, \ldots, d_l, d_{l+1}, \ldots, d_{l+k}]$. By assuming that $\mathfrak{F}^{\urcorner}\mathfrak{C}_k = (W', R')$, where $W' = \{w_0, \ldots, w_l, w_{l+1}, \ldots, w_{l+k}\}$, w_0 is a root, $w_{l+1}, \ldots, w_{l+k}$ are the points from the maximal cluster $\mathfrak{C}_k$, R' coincides with R on the points $\{w_0, \ldots, w_l, w_{l+1}\}$, and $w_{l+i} R' w_{l+j}$ for all $i, j \leq k$, we can write $\chi(\mathfrak{F}^{\urcorner}\mathfrak{C}_k)[d_0, \ldots, d_l, d_{l+1}, \ldots, d_{l+k}]$ as the meet $\varphi_1 \wedge \varphi_2 \wedge \varphi_3$, where:

$$\varphi_1 = d_0 \wedge \Box \bigvee d_i \wedge \Box \bigwedge_{i \neq j} (d_i \to \neg d_j)$$

$$\varphi_2 = \bigwedge_{w_i R' w_j} \Box(d_i \to \Diamond d_j)$$

$$\varphi_3 = \bigwedge_{w_i \not R' w_j} \Box(d_i \to \neg\Diamond d_j)$$

Since $d_i \wedge d_j = 0$ for all $i, j \leq l+k$, we obtain $\Box \bigwedge_{i \neq j} (d_i \to \neg d_j) = \Box \bigwedge 1 = 1$. Moreover,

$$\bigvee d_i = \bigvee_{i=1}^{l} d_i \vee \bigvee_{j=1}^{k} d_{l+j} = \bigvee_{i=1}^{l} b_i \vee b_{l+1} = \bigvee_{i=1}^{l+1} b_i.$$

Thus, $\varphi_1 = b_0 \wedge \Box \bigvee_{i=1}^{l+1} b_i$.

For $i, j \leq l$ we have $w_i R' w_j$ iff $w_i R w_j$, and for $j > l$ and $i \leq l + k$ we have $w_i R' w_j$. Also, $\Diamond d_j \geq \Diamond b_{l+1}$ for all $j > l$; $d_i = b_i$ for all $i \leq l$; and $(w_i, w_j) \notin R'$ whenever $i > l$, $j \leq l$. Consequently,

$$\begin{aligned}
\bigwedge_{w_i R' w_j} (d_i \to \Diamond d_j) &= \bigwedge_{\substack{w_i R' w_j \\ i,j \leq l}} (d_i \to \Diamond d_j) \wedge \bigwedge_{\substack{i \leq l \\ j > l}} (d_i \to \Diamond d_j) \wedge \bigwedge_{i,j > l} (d_i \to \Diamond d_j) \\
&= \bigwedge_{\substack{w_i R w_j \\ i,j \leq l}} (b_i \to \Diamond b_j) \wedge \bigwedge_{\substack{i \leq l \\ j > l}} (b_i \to \Diamond d_j) \wedge \bigwedge_{i,j > l} (d_i \to \Diamond d_j) \\
&\geq \bigwedge_{\substack{w_i R w_j \\ i,j \leq l}} (b_i \to \Diamond b_j) \wedge \bigwedge_{i \leq l} (b_i \to \Diamond b_{l+1}) \wedge \bigwedge_{i > l} (d_i \to \Diamond b_{l+1}).
\end{aligned}$$

Since $\bigvee_{j=1}^{k} d_{l+j} = b_{l+1}$, we have $d_i \leq b_{l+1} \leq \Diamond b_{l+1}$ for each $i > l$. Therefore, $\bigwedge_{i>l} (d_i \to \Diamond b_{l+1}) = 1$. Further, as $w_i R w_{l+1}$ for all $i \leq l$ and $b_{l+1} \to \Diamond b_{l+1} = 1$, we obtain $\bigwedge_{w_i R w_j} (b_i \to \Diamond b_j) = \bigwedge_{\substack{w_i R w_j \\ i,j \leq l}} (b_i \to \Diamond b_j) \wedge \bigwedge_{i \leq l} (b_i \to \Diamond b_{l+1})$. Thus, $\bigwedge_{w_i R' w_j} (d_i \to \Diamond d_j) \geq \bigwedge_{w_i R w_j} (b_i \to \Diamond b_j)$, and so $\varphi_2 \geq \Box \bigwedge_{w_i R w_j} (b_i \to \Diamond b_j)$.

Furthermore, for $i, j \leq l$ we have $(w_i, w_j) \notin R'$ iff $(w_i, w_j) \notin R$. In addition, $(w_{l+1}, w_i) \notin R$ for all $i < l + 1$; and $w_i R w_j$ whenever $j > l$. Therefore, taking into account that $d_i \leq b_{l+1}$ for all $i > l$, we obtain:

$$\begin{aligned}
\bigwedge_{w_i \not R' w_j} (d_i \to \neg\Diamond d_j) &= \bigwedge_{\substack{w_i \not R' w_j \\ i,j \leq l}} (d_i \to \neg\Diamond d_j) \wedge \bigwedge_{\substack{i > l \\ j \leq l}} (d_i \to \neg\Diamond d_j) \\
&= \bigwedge_{\substack{w_i \not R w_j \\ i,j \leq l}} (b_i \to \neg\Diamond b_j) \wedge \bigwedge_{\substack{i > l \\ j \leq l}} (d_i \to \neg\Diamond b_j) \\
&\geq \bigwedge_{\substack{w_i \not R w_j \\ i,j \leq l}} (b_i \to \neg\Diamond b_j) \wedge \bigwedge_{j \leq l} (b_{l+1} \to \neg\Diamond b_j) \\
&= \bigwedge_{w_i \not R w_j} (b_i \to \neg\Diamond b_j).
\end{aligned}$$

It follows that $\varphi_3 \geq \Box \bigwedge_{w_i \not R w_j} (b_i \to \neg\Diamond b_j)$. To sum up:

$$\begin{aligned}
\chi(\mathfrak{F} \urcorner \mathfrak{C}_k)[d_0, \ldots, d_{l+k}] &= \varphi_1 \wedge \varphi_2 \wedge \varphi_3 \\
&\geq b_0 \wedge \Box \bigvee_{i=1}^{l+1} b_i \wedge \Box \bigwedge_{w_i R w_j} (b_i \to \Diamond b_j) \wedge \Box \bigwedge_{w_i \not R w_j} (b_i \to \neg\Diamond b_j) \\
&= \chi(\mathfrak{F} \urcorner \mathfrak{C}_1)[b_0, \ldots, b_{l+1}] > 0.
\end{aligned}$$

By Fine's theorem, $(\mathfrak{F} \urcorner \mathfrak{C}_k)^+$ is a homomorphic image of a subalgebra of A. □

We will require a slight generalization of the above lemma. Let $\mathfrak{C}_\omega = (W_\omega, R_\omega)$ denote the countable ω-cluster. Let P be a subset of the set of all propositional letters. We will utilize the concept of P-bisimulation, which is a modification of the standard notion of bisimulation (see, e.g., [5, Defn. 2.16]). Namely, a relation between two **S4**-models is a *P-bisimulation* if it satisfies the back and forth conditions and if bisimilar points satisfy the same propositional letters from P. It follows from the proof of [5, Thm. 2.20] that modal formulas in P-variables are invariant under P-bisimulations.

Lemma 4.7 *Let L be a connected logic over* **S4.2** *and let $\mathfrak{F}$ be a rooted* **S4**-*frame. If $\mathfrak{F}\urcorner\mathfrak{C}_n \models L$ for some $n < \omega$ and $\mathfrak{C}_k \models L$ for all $k < \omega$, then $\mathfrak{F}\urcorner\mathfrak{C}_\omega \models L$.*

Proof Let $\mathfrak{F}\urcorner\mathfrak{C}_n \models L$ for some $n < \omega$ and $\mathfrak{C}_k \models L$ for all $k < \omega$. Then it follows from Lemma 4.6 that $\mathfrak{F}\urcorner\mathfrak{C}_k \models L$ for all $k < \omega$. Suppose that $\mathfrak{F}\urcorner\mathfrak{C}_\omega \not\models \varphi$ for some $\varphi \in L$. Let $\mathrm{Sub}(\varphi)$ denote the set of subformulas of φ, and define an equivalence relation $\equiv$ on $\mathfrak{C}_\omega$ by $w \equiv v$ iff $(\forall\psi \in \mathrm{Sub}(\varphi))(w \models \psi \Leftrightarrow v \models \psi)$. Clearly $\equiv$ has only finitely many equivalence classes, which we denote by $c_1, \ldots c_m$. Define $f : \mathfrak{F}\urcorner\mathfrak{C}_\omega \to \mathfrak{F}\urcorner\mathfrak{C}_m$ by

$$f(w) = \begin{cases} w & \text{if } w \in \mathfrak{F} \\ w_i & \text{if } w \in c_i \end{cases}$$

It is easy to see that f is an onto p-morphism. Let P be the set of propositional letters occurring in φ. We define $\models$ on $\mathfrak{F}\urcorner\mathfrak{C}_m$ by $w \models p$ iff there exists $v \in \mathfrak{F}\urcorner\mathfrak{C}_\omega$ with $f(v) = w$ and $v \models p$. Then f is a P-bisimulation between the models $(\mathfrak{F}\urcorner\mathfrak{C}_\omega, \models)$ and $(\mathfrak{F}\urcorner\mathfrak{C}_m, \models)$. Since modal formulas in P-variables are invariant under P-bisimulations, we obtain that $\mathfrak{F}\urcorner\mathfrak{C}_m \not\models \varphi$. Therefore, $\mathfrak{F}\urcorner\mathfrak{C}_m \not\models L$, a contradiction. Thus, $\mathfrak{F}\urcorner\mathfrak{C}_\omega \models L$. □

Finally, we are ready to prove the main result of the paper.

Theorem 4.8 (Main Theorem) *Let L be a modal logic over* **S4** *with fmp. Then the following conditions are equivalent:*

(1) *L is connected.*
(2) *$L = L(\mathfrak{F})$ for some path-connected* **S4**-*frame $\mathfrak{F}$.*
(3) *$L = L(X)$ for some connected space X.*
(4) *$L = L(A)$ for some subalgebra A of $\mathbf{R}^+$.*

Proof The implications (2) ⇒ (3) ⇒ (1) and (4) ⇒ (1) are obvious. Therefore, it is sufficient to show (1) ⇒ (2) ⇒ (4).

(1) ⇒ (2) ⇒ (4): Our strategy will be as follows. Firstly we build a path-connected frame $\mathfrak{G}$ such that $L = L(\mathfrak{G})$, thus establishing (1) ⇒ (2). Secondly, we show that $\mathfrak{G}$, viewed as a topological space, is an interior image of $\mathbf{R}$. This will show that $\mathfrak{G}^+$ is isomorphic to a subalgebra of $\mathbf{R}^+$, thus establishing (2) ⇒ (4).

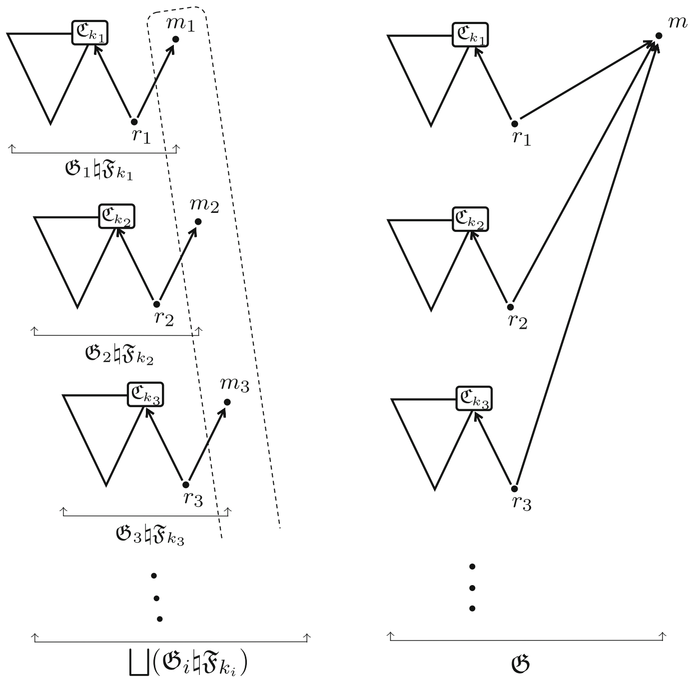


Fig. 7 A p-morphism from $\bigsqcup(\mathfrak{G}_i \natural \mathfrak{F}_{k_i})$ onto $\mathfrak{G}$

Let $\mathfrak{G}_1, \mathfrak{G}_2, \ldots$ be the list of all finite rooted non-isomorphic L-frames. Then L is the logic of $\{\mathfrak{G}_1, \mathfrak{G}_2, \ldots\}$. We have two possible cases: either $\mathfrak{F}_1 \models L$ or $\mathfrak{F}_1 \not\models L$.

Case 1: Suppose that $\mathfrak{F}_1 \models L$. For each i we choose a maximal cluster of $\mathfrak{G}_i$. Let k_i denote the size of this cluster. By Lemma 4.4, $\mathfrak{G}_i \natural \mathfrak{F}_{k_i}$ is an L-frame. Let m_i denote the maximal point and r_i denote the root of $\mathfrak{F}_{k_i}$ in $\mathfrak{G}_i \natural \mathfrak{F}_{k_i}$. We let $\mathfrak{G}$ be the p-morphic image of the disjoint union $\bigsqcup(\mathfrak{G}_i \natural \mathfrak{F}_{k_i})$ under the map that glues all the m_i into a single point m and is the identity on the rest of the points (see Fig. 7).

It follows that $\mathfrak{G} \models L$. Moreover, $\mathfrak{G} = (W, R)$ is path-connected. To see this, let $w, v \in W$. Then w is from some $\mathfrak{G}_i \natural \mathfrak{F}_{k_i}$ and v is from some $\mathfrak{G}_j \natural \mathfrak{F}_{k_j}$. Since both of these frames are connected, there is a path $w, x_1, \ldots, x_k, m_i$ and a path $m_j, y_1, \ldots, y_l, v$. Then the sequence $w, x_1, \ldots, x_k, m, y_1, \ldots, y_l, v$ is a path in $\mathfrak{G}$, and so $\mathfrak{G}$ is path-connected. Furthermore, since each $\mathfrak{G}_i$ is a generated subframe of $\mathfrak{G}$, we obtain $L = L(\mathfrak{G})$.

Case 2: Suppose that $\mathfrak{F}_1 \not\models L$. Then **S4.2** $\subseteq L$. We associate $\alpha \leq \omega$ with L as follows. If there exists $n < \omega$ such that $\mathfrak{C}_n \models L$ and $\mathfrak{C}_{n+1} \not\models L$, then we set $\alpha = n$. Otherwise we set $\alpha = \omega$. Since **S4.2** $\subseteq L$, each $\mathfrak{G}_i$ has the form $\mathfrak{H}_i \urcorner \mathfrak{C}_{k_i}$, where $k_i \leq \alpha$. By Lemmas 4.6 and 4.7, for each i, we have $\mathfrak{H}_i \urcorner \mathfrak{C}_\alpha \models L$. Let $\mathfrak{G}$ be the p-morphic image of the disjoint union $\bigsqcup(\mathfrak{H}_i \urcorner \mathfrak{C}_\alpha)$ that glues together all the maximal clusters $\mathfrak{C}_\alpha$ (see Fig. 8). Then $\mathfrak{G} \models L$. Moreover, since $\mathfrak{G}$ has a unique maximal cluster, $\mathfrak{G}$ is path-connected. Furthermore, as each $\mathfrak{G}_i$ is a generated subframe of $\mathfrak{G}$, we obtain that $L = L(\mathfrak{G})$. This establishes (1) $\Rightarrow$ (2).

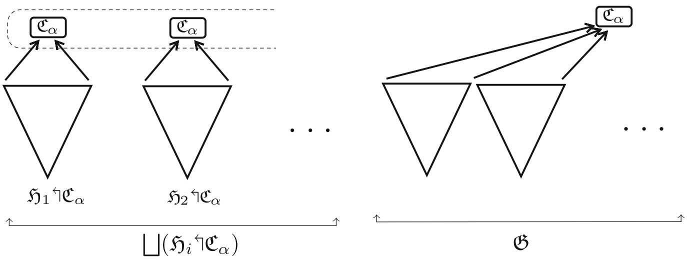


Fig. 8 A p-morphism from $\bigsqcup(\mathfrak{H}_i \Lsh \mathfrak{C}_\alpha)$ onto $\mathfrak{G}$

Our final goal is to view $\mathfrak{G}$ as a topological space and build an interior map from $\mathbf{R}$ onto $\mathfrak{G}$. We will need the following claim. We point out that part (1) of the claim follows from [18, Lem. 17], where the setting is more general. Our proof below, however, provides a more explicit construction of the map.

Claim 3 *Let $[a, b]$ be a closed interval of the real line $\mathbf{R}$, let $\mathfrak{F}$ be a finite rooted **S4**-frame with a maximal n-cluster, let $\mathfrak{F}_n$ be the n-fork with the root r, and let $\mathfrak{C}_\alpha = (W_\alpha, R_\alpha)$ be the α-cluster, where $\alpha \leq \omega$.*

(1) *There is an onto interior map $f : [a, b] \to \mathfrak{F} \natural \mathfrak{F}_n$ such that $a, b \in f^{-1}(r)$.*
(2) *There is an onto interior map $g : [a, b] \to \mathfrak{F} \Lsh \mathfrak{C}_\alpha$.*

Proof of claim: (1) Since all closed intervals of the real line are homeomorphic, we will prove the claim for a concrete interval $[0, 5]$. The general claim will follow. The required map will be built using [2, Sect. 3] which provides an onto interior map from any bounded interval of the real line onto any finite rooted **S4**-frame. Thus, there exist onto interior maps $g : [2, 3] \to \mathfrak{F}$, $g_1 : [0, 1] \to \mathfrak{F}_n$, $g_2 : [4, 5] \to \mathfrak{F}_n$, $h_1 : (1, 2) \to \mathfrak{C}_n$, and $h_2 : (3, 4) \to \mathfrak{C}_n$. It follows from the construction in [2, Sect. 3] that $0, 1 \in g_1^{-1}(r), 4, 5 \in g_2^{-1}(r)$, and $2, 3 \in g^{-1}(C_r)$, where C_r denotes the minimal cluster of $\mathfrak{F}$. We take f to be the union $g_1 \cup h_1 \cup g \cup h_2 \cup g_2$, where $\mathfrak{F}$, $\mathfrak{F}_n$, and $\mathfrak{C}_n$ are viewed as generated subframes of $\mathfrak{F} \natural \mathfrak{F}_n$. It is clear that $0, 5 \in f^{-1}(r)$. It is also easy to see that f is a well-defined onto map. To see that f is open, consider any open interval $I \subseteq [4, 5]$. Obviously, $f(I)$ is a union of upsets in $\mathfrak{F} \natural \mathfrak{F}_n$, hence $f(I)$ is an upset. Therefore, f is an open map. It is left to be shown that f is continuous. Let U be an upset of $\mathfrak{F} \natural \mathfrak{F}_n$. Then either $\mathfrak{C}_n \subseteq U$ or $\mathfrak{C}_n \cap U = \emptyset$. Suppose that $\mathfrak{C}_n \cap U = \emptyset$. Then U misses both r and C_r. Therefore, $0, 1, 2, 3, 4, 5 \notin f^{-1}(U)$, and so $f^{-1}(U) = U_1 \cup U_2 \cup U_3$, where U_1 is open in $(0, 1)$, U_2 is open in $(2, 3)$, and U_3 is open in $(4, 5)$. Thus, $f^{-1}(U)$ is open in $[0, 5]$. Now suppose that $\mathfrak{C}_n \subseteq U$. Then $f^{-1}(U) = U_1 \cup (1, 2) \cup U_2 \cup (3, 4) \cup U_3$, where U_1 is open in $[0, 1]$, U_2 is open in $[2, 3]$, and U_3 is open in $[4, 5]$. Therefore, $f^{-1}(U)$ is open in $[0, 5]$, and so f is continuous. Consequently, f is an onto interior map.

(2) If $\alpha < \omega$, then we can apply [2, Cor. 14], by which $\mathfrak{F} \Lsh \mathfrak{C}_\alpha$ is an interior image of $[a, b]$. In particular, let h denote an interior map from $[a, b]$ onto $\mathfrak{F} \Lsh \mathfrak{C}_1$, where $W_1 = \{w\}$. Suppose now that $\alpha = \omega$ and let $W_\alpha = \{w_1, w_2, \dots\}$. Since h is an interior map, $U = h^{-1}(w)$ is an open dense subset of $[a, b]$. We divide U into countably many disjoint dense subsets $U_1, U_2, \dots$ and define $g : [a, b] \to \mathfrak{F} \Lsh \mathfrak{C}_\alpha$ by

$$g(x) = \begin{cases} h(x) & \text{if } x \notin U, \\ w_n & \text{if } x \in U_n \text{ for some } n < \omega. \end{cases}$$

It is straightforward to check that g is a well-defined onto map. To see that g is continuous, let V be a nonempty upset of $\mathfrak{F}\Lsh\mathfrak{C}_\alpha$. Then $V = W \cup W_\alpha$, where W is an upset of $\mathfrak{F}$. Clearly $g^{-1}(V) = h^{-1}(W \cup \{w\})$, and so $g^{-1}(V)$ is open in $[a, b]$. Therefore, g is continuous. It remains to be shown that g is open. Let I be a nonempty open interval of $[a, b]$. Since U is open and dense in $[a, b]$ and each U_n is dense in U, we have that I meets each U_n. Moreover, $h(I - U) = h(I) - \{w\}$ is an upset of $\mathfrak{F}$. Therefore, $g(I) = h(I - U) \cup W_\alpha$ is an upset of $\mathfrak{F}\Lsh\mathfrak{C}_\alpha$, and so g is open. Consequently, g is an onto interior map. □

Now we build an interior map from $\mathbf{R}$ onto $\mathfrak{G}$. We have that either $\mathfrak{F}_1 \models L$ or $\mathfrak{F}_1 \not\models L$.

Case 1: First suppose that $\mathfrak{F}_1 \models L$. We build an interior map from $(0, \infty)$ onto $\mathfrak{G}$. Since $(0, \infty)$ is homeomorphic to $\mathbf{R}$, the result follows. Note that if the number of the $\mathfrak{G}_i$ is finite, then $\mathfrak{G}$ is finite and connected. Therefore, by [2, Cor. 20], there exists an interior map from $\mathbf{R}$ onto $\mathfrak{G}$. Thus, without loss of generality we may assume that there are infinitely many $\mathfrak{G}_i$. By Claim 3(1), for each finite $\mathfrak{G}_i \natural \mathfrak{F}_{k_i}$ there exists an interior map g_i from the interval $[2i+1, 2i+2]$ onto $\mathfrak{G}_i \natural \mathfrak{F}_{k_i}$ such that $g_i^{-1}(r_i) \supseteq \{2i+1, 2i+2\}$. Define $f : (0, \infty) \to \mathfrak{G}$ by

$$f(x) = \begin{cases} m & \text{if } x \in (2i, 2i+1) \text{ for some } i < \omega, \\ g_i(x) & \text{if } x \in [2i+1, 2i+2] \text{ for some } i < \omega. \end{cases}$$

It is straightforward that f is a well-defined onto map. To see that f is open, it suffices to note that the restriction of f to the intervals $(2i, 2i+1)$ and $[2i+1, 2i+2]$ is open by the construction, f commutes with arbitrary unions, and the union of opens is open. The only nontrivial part to check is that f is continuous. Let $U \subseteq W$ be an upset of $\mathfrak{G}$ and $x \in f^{-1}(U)$. If there exists n such that $n < x < n+1$, then, by the construction, for sufficiently small ε, we have $(x - \varepsilon, x + \varepsilon) \subseteq f^{-1}(U)$. Suppose that $x = n$. Without loss of generality we may assume that $n = 2i + 1$. Then $f(x) = g_i(x)$ and as g_i is continuous, there exists a sufficiently small ε such that $[x, x + \varepsilon) \subseteq g_i^{-1}(U \cap [2i+1, 2i+2])$. We also know that $x = 2i + 1 \in g_i^{-1}(r_i)$. Therefore, $f(x) = r_i$. As U is an upset and $f(x) = r_i \in U$, we have $m \in U$. But then $(2i, 2i+1) \subseteq f^{-1}(U)$. Since we also have $[2i+1, 2i+1+\varepsilon) \subseteq f^{-1}(U)$, we obtain $(2i, 2i+1+\varepsilon) \subseteq f^{-1}(U)$. Thus, $x = 2i+1$ has an open neighborhood contained in $f^{-1}(U)$. Consequently, $f^{-1}(U)$ is open, and so $f : \mathbf{R} \twoheadrightarrow \mathfrak{G}$ is an interior map.

Case 2: Next suppose that $\mathfrak{F}_1 \not\models L$. By Claim 3(2), each frame $\mathfrak{G}_i\Lsh\mathfrak{C}_\alpha$ is an interior image of the interval $[2i+1, 2i+2]$. Let $g_i : [2i+1, 2i+2] \twoheadrightarrow \mathfrak{G}_i\Lsh\mathfrak{C}_\alpha$ denote the corresponding interior map, and let $f_i : (2i, 2i+1) \twoheadrightarrow \mathfrak{C}_\alpha$ denote the interior map from $(2i, 2i+1)$ onto the α-cluster $\mathfrak{C}_\alpha$. Define $f : (0, \infty) \to \mathfrak{G}$ by

$$f(x) = \begin{cases} f_i(x) & \text{if } x \in (2i, 2i+1) \text{ for some } i < \omega, \\ g_i(x) & \text{if } x \in [2i+1, 2i+2] \text{ for some } i < \omega. \end{cases}$$

It is straightforward to check that f is a well-defined onto interior map, which completes the proof of the theorem. □

Remark 4.9 As follows from our Main Theorem, for each connected logic L with the fmp, there is a subalgebra A of $\mathbf{R}^+$ such that $L = L(A)$. In fact, by an argument similar to [2, Thms. 15 and 21], it can be ensured that A is a subalgebra of $B(C^\omega(\mathbf{R}))$.

Remark 4.10 The original McKinsey-Tarski theorem is concerned with an arbitrary dense-in-itself metrizable separable space. It is natural to ask whether our Main Theorem can be proved with the same generality. Most of the ingredients of the proof can easily be seen to be generalizable to arbitrary connected dense-in-itself metrizable separable spaces using the ideas and constructions of [16] and [18]. The more difficult part is building an interior map onto the infinite path-connected frame $\mathfrak{G}$. Our construction easily generalizes to any Euclidean space, but it is not entirely clear how to generalize it to an arbitrary connected dense-in-itself metrizable separable space. We leave this as an open problem.

5 Logics over S4.1

Many logics over **S4** are connected. We will show in the next section that some of the most well-known extensions of **S4** are connected logics. In fact, there are continuum many connected logics, among them continuum many with the fmp and continuum many without the fmp. It is the goal of this section to show that each logic over **S4.1** is connected. For this we will require the machinery of Esakia duality between closure algebras and descriptive **S4**-frames.

Let $(B, \Diamond)$ be a closure algebra and (X, R) its dual descriptive **S4**-frame. By Esakia duality, homomorphic images of $(B, \Diamond)$ correspond to closed upsets of (X, R) and subalgebras of $(B, \Diamond)$ correspond to correct partitions of (X, R). Here we recall that if $\sim$ is an equivalence relation on X, then $[x] = \{y \in X : x \sim y\}$, $[U] = \bigcup\{[x] : x \in U\}$, and $\sim$ is a correct partition whenever (i) from $x \not\sim y$ it follows that there exists a clopen subset U of X such that $[U] = U$ and U separates x and y and (ii) $x \sim y$ and yRz imply there exists $u \in X$ such that xRu and $u \sim z$. We will also need that $(B, \Diamond)$ is subdirectly irreducible iff X is rooted and the minimal cluster of X is clopen [9].

Given a descriptive **S4**-frame (X, R) and a valuation ν sending the propositional letters to subsets of X, we call ν *admissible* if each $\nu(p)$ is a clopen subset of X. Let L be a logic over **S4**. We say that (X, R) *validates* L (notation: $(X, R) \models L$) if each theorem of L is true at every point of X under each admissible valuation.

For a family $\{(X_i, R_i) : i \in I\}$ of descriptive **S4**-frames, let (X, R) denote their disjoint union $\bigsqcup_{i \in I}(X_i, R_i)$. Note that if I is infinite, then X is no longer compact. However, X is clearly locally compact Hausdorff, hence X has the one-point compactification $\alpha X = X \cup \{\infty\}$. Let $\alpha R = R \cup \{(\infty, \infty)\}$. We claim that $(\alpha X, \alpha R)$ is a descriptive **S4**-frame. That αX is a Stone space and that αR is reflexive and transitive is obvious. Let $x \in \alpha X$. Then $x = \infty$ or there is $i \in I$ such that $x \in X_i$. If $x = \infty$, then $(\alpha R)[x] = \{x\}$, and if $x \in X_i$, then $(\alpha R)[x] = R_i[x]$ is closed. In either case $(\alpha R)[x]$ is a closed subset of αX. Next let U be a clopen subset of αX. By the definition of the topology on αX (see, e.g., [7, Thm. 3.5.11]), there exist $i_1, \ldots, i_n$ and clopen subsets $U_{i_1} \subseteq X_{i_1}, \ldots, U_{i_n} \subseteq X_{i_n}$ such that $U = \bigcup_{k=1}^{n} U_{i_k}$ or $U = \alpha X - (\bigcup_{k=1}^{n} U_{i_k})$. But then $(\alpha R)^{-1}(U)$ has again the same form, hence is a clopen subset of αX. Consequently, $(\alpha X, \alpha R)$ is a descriptive **S4**-frame.

Lemma 5.1 *Let L be a logic over **S4**, let $\{(X_i, R_i) : i \in I\}$ be a family of descriptive **S4**-frames such that $(X_i, R_i) \models L$ for each $i \in I$, and let $X = \bigsqcup_{i \in I}(X_i, R_i)$. Then $(\alpha X, \alpha R) \models L$.*

Proof Suppose $(\alpha X, \alpha R) \not\models \varphi$ for some theorem φ of L. This means that under some admissible valuation on $(\alpha X, \alpha R)$, the clopen corresponding to $\neg\varphi$ is nonempty. Since all nonempty clopens of αX meet X, we obtain that φ can be refuted on some $x \in X$. We have $x \in X_i$ for some $i \in I$. Therefore, φ is refuted on (X_i, R_i), which is impossible since $X_i \models L$. The obtained contradiction proves that $(\alpha X, \alpha R) \models L$. □

The next lemma is a straightforward generalization of the characterization of **S4.1**-frames (see, e.g., [6, Prop. 3.46]).

Lemma 5.2 *Let (X, R) be a descriptive **S4**-frame. Then $(X, R) \models$ **S4.1** iff for each $x \in X$ there exists a maximal $y \in X$ with xRy.*

Note that each point in a descriptive **S4**-frame sees a quasi-maximal point, by [10, Thm. 2.1]. Thus, the descriptive **S4.1**-frames can be characterized as those descriptive **S4**-frames in which every quasi-maximal point is maximal (or, equivalently, each maximal cluster is simple).

We now prove that each modal logic over **S4.1** is connected.

Theorem 5.3 *Let L be a logic over **S4.1**. Then L is connected.*

Proof Let $\{(B_i, \Diamond_i) : i \in I\}$ be the family of all non-isomorphic finitely generated subdirectly irreducible closure algebras validating L. It is well-known that L is the logic of $\{(B_i, \Diamond_i) : i \in I\}$. For each i let (X_i, R_i) be the dual descriptive **S4**-frame of $(B_i, \Diamond_i)$. Then $(X_i, R_i) \models L$. We let (X, R) denote the disjoint union of $\{(X_i, R_i) : i \in I\}$. By Lemma 5.1, $(\alpha X, \alpha R)$ is a descriptive **S4**-frame such that $(\alpha X, \alpha R) \models L$. For each $i \in I$, let m_i denote a maximal point of (X_i, R_i). It exists by Lemma 5.2 since $(X_i, R_i) \models$ **S4.1**. Define a partition $\sim$ on αX by identifying all m_i with ∞.

We show that $\sim$ is a correct partition. For this we first show that the set $A = \{m_i : i \in I\} \cup \{\infty\}$ is closed in αX. Let $x \notin A$. Then there is $i \in I$ such that $x \in X_i - \{m_i\}$. Since X_i is a Stone space, there exists a clopen subset U of X_i such that $x \in U$ and $m_i \notin U$. Clearly $U \cap A = \emptyset$ and U is open in X, and hence in αX. Therefore, $\alpha X - A$ is open and so A is closed in αX.

Now let $x, y \in \alpha X$ with $x \not\sim y$. Without loss of generality we may assume that $x \notin A$. Therefore, there is a clopen subset U of αX such that $x \in U$ and $U \cap A = \emptyset$.

Thus, $[U] = U$. If $y \in A$, then we have found a clopen subset U of αX such that $[U] = U$ and U separates x from y. Now suppose that $y \notin A$. Then there exist $i, j \in I$ such that $x \in X_i$ and $y \in X_j$. We can clearly separate x from y by a clopen subset U of X_i such that $m_i \notin U$. Since $U \subseteq X_i$ and $m_i \notin U$, we have $[U] = U$. Therefore, in this case too, we have found a clopen subset U of αX such that $[U] = U$ and U separates x from y. Since each m_i and ∞ are maximal points of αX, it is also obvious that from $x \sim y$ and yRz it follows that $y = z$, and so there is $u \in \alpha X$ ($u = x$) such that xRu and $u \sim z$. Consequently, $\sim$ is a correct partition, and so $(\alpha X/\!\sim, (\alpha R)_\sim)$ is a descriptive **S4**-frame such that $(\alpha X/\!\sim, (\alpha R)_\sim) \models L$ (where we recall that $[x](\alpha R)_\sim[y]$ iff there exist $x' \in [x]$ and $y' \in [y]$ with $x'(\alpha R)y'$).

Moreover, it follows from the definition of $\sim$ that $(\alpha X/\!\sim, (\alpha R)_\sim)$ is path-connected, and so the corresponding closure algebra $(B, \Diamond)$ is connected. It is also clear that each X_i is (isomorphic to) a closed upset of $(\alpha X/\!\sim, (\alpha R)_\sim)$. Therefore, each $(B_i, \Diamond_i)$ is a homomorphic image of $(B, \Diamond)$. Thus, L is the logic of the connected algebra $(B, \Diamond)$, hence is a connected logic. (In fact, $(B, \Diamond)$ is a subdirect product of the family $\{(B_i, \Diamond_i) : i \in I\}$). □

On the other hand, we give a simple example of a logic over **S4.2** which is not connected. Let L be the logic of the frame $\mathfrak{G}_1 \sqcup \mathfrak{C}_2$, where $\mathfrak{G}_1$ and $\mathfrak{C}_2$ are shown in Fig. 6. Then it is easy to see that $\mathfrak{G}_2$ is not an L-frame, where $\mathfrak{G}_2$ is also shown in Fig. 6. On the other hand, if L were connected, then by Lemma 4.6, $\mathfrak{G}_2$ would also be an L-frame. The obtained contradiction proves that L is not a connected logic.

Corollary 5.4 1. *Each logic over **S4.1** with the fmp is the logic of a subalgebra of* $\mathbf{R}^+$.
2. *Each logic over **S4.Grz** is connected.*
3. *Each logic over **S4.Grz** with the fmp is the logic of a subalgebra of* $\mathbf{R}^+$.
4. *There are continuum many connected logics over **S4**, continuum many with the fmp, and continuum many without the fmp.*

Proof (1) follows from Theorems 4.8 and 5.3; since **S4.1** is contained in **S4.Grz**, (2) follows from Theorem 5.3 and (3) follows from (1); finally, (4) follows from (2) since it is well-known that there are continuum many extensions of **S4.Grz**, continuum many with the fmp, and continuum many without the fmp. □

Open Problem 2: Which connected logics without the fmp can be obtained as the logics of subalgebras of $\mathbf{R}^+$?

6 Examples

Clearly **S4** is a connected logic because it is the logic of $\mathbf{R}^+$. In this section we list specific subalgebras of $\mathbf{R}^+$ that generate well-known normal extensions of **S4** such as **S5**, **S4.1**, **S4.2**, **S4.1.2**, **S4.Grz**, and **S4.Grz.2**. Each of these systems is an extension of **S4** by finitely many axioms in one variable, hence has the fmp, by [6, Thm. 11.58]. It follows from Theorem 4.8 that each of these logics is connected. In the following table we recall the syntactic and semantic characterizations of these systems.

Logic	Defining axioms	Generating frame class
S5	$\Diamond p \to \Box\Diamond p$	Finite clusters
S4.1	$\Box\Diamond p \to \Diamond\Box p$	All finite rooted frames in which each maximal cluster is simple
S4.2	$\Diamond\Box p \to \Box\Diamond p$	All finite rooted frames with a unique maximal cluster
S4.1.2	$\Diamond\Box p \leftrightarrow \Box\Diamond p$	All finite rooted frames with a unique maximal cluster which is simple
S4.Grz	$\Box(\Box(p \to \Box p) \to p) \to p$	All finite rooted partially ordered frames
S4.Grz.2	$\Box(\Box(p \to \Box p) \to p) \to p$ $\Diamond\Box p \to \Box\Diamond p$	All finite rooted partially ordered frames with a unique maximal point

6.1 **S4.Grz**

Let $B(\mathrm{Op}(\mathbf{R}))$ denote the Boolean subalgebra of $\mathbf{R}^+$ generated by the open subsets of $\mathbf{R}$. Since all open, and hence closed subsets of $\mathbf{R}$ are contained in $B(\mathrm{Op}(\mathbf{R}))$, it is clear that $B(\mathrm{Op}(\mathbf{R}))$ is closed under **cl**. Therefore, $B(\mathrm{Op}(\mathbf{R}))$ is a subalgebra of $\mathbf{R}^+$. By [2, Rem. 10], the logic of $B(\mathrm{Op}(\mathbf{R}))$ is **S4.Grz**.

6.2 **S4.Grz.2**

Let $\mathrm{OD}(\mathbf{R})$ denote the set of open dense subsets of $\mathbf{R}$ and let $B(\mathrm{OD}(\mathbf{R}))$ denote the Boolean subalgebra of $\mathbf{R}^+$ generated by $\mathrm{OD}(\mathbf{R})$. We show that $B(\mathrm{OD}(\mathbf{R}))$ is closed under **cl**, and so is a subalgebra of $\mathbf{R}^+$. Since $\mathrm{OD}(\mathbf{R})$ is closed with respect to finite unions and intersections, $\mathrm{OD}(\mathbf{R}) \cup \{\emptyset\}$ is a bounded sublattice of $\mathrm{Op}(\mathbf{R})$. Therefore, for each $A \in B(\mathrm{OD}(\mathbf{R}))$ we have $A = \bigcap_{i=1}^{n} (-U_i \cup V_i)$, where each $U_i, V_i \in \mathrm{OD}(\mathbf{R}) \cup \{\emptyset\}$.

Lemma 6.1 *For each $A \in B(\mathrm{OD}(\mathbf{R}))$ we have $\mathbf{cl}(A) = \mathbf{R}$ or $\mathbf{int}(A) = \emptyset$.*

Proof Let $A \in B(\mathrm{OD}(\mathbf{R}))$. Then $A = \bigcap_{i=1}^{n} (-U_i \cup V_i)$ for some $U_i, V_i \in \mathrm{OD}(\mathbf{R}) \cup \{\emptyset\}$. If each $V_i \neq \emptyset$, then each V_i is open dense, and hence so is $\bigcap_{i=1}^{n} V_i$. Therefore, $\mathbf{cl}(A) \supseteq \mathbf{cl}(\bigcap_{i=1}^{n} V_i) = \mathbf{R}$, and so $\mathbf{cl}(A) = \mathbf{R}$. On the other hand, if at least one $V_j = \emptyset$, then $\mathbf{int}(A) \subseteq \mathbf{int}(-U_j) = -\mathbf{cl}(U_j) = -\mathbf{R} = \emptyset$. □

Corollary 6.2 1. *$B(\mathrm{OD}(\mathbf{R}))$ is a subalgebra of $\mathbf{R}^+$.*
2. *$B(\mathrm{OD}(\mathbf{R}))$ is a **S4.Grz.2**-algebra.*

Proof (1) It is sufficient to show that if $A \in B(\mathrm{OD}(\mathbf{R}))$, then $\mathbf{cl}(A) \in B(\mathrm{OD}(\mathbf{R}))$. But this follows from Lemma 6.1.
(2) Since $B(\mathrm{OD}(\mathbf{R}))$ is a subalgebra of $B(\mathrm{Op}(\mathbf{R}))$ and $B(\mathrm{Op}(\mathbf{R}))$ is a **S4.Grz**-algebra, so is $B(\mathrm{OD}(\mathbf{R}))$. Also, by Lemma 6.1, for each $A \in B(\mathrm{OD}(\mathbf{R}))$

we have $\mathbf{cl}(A) = \mathbf{R}$ or $\mathbf{int}(A) = \emptyset$. Therefore, $\mathbf{cl}(\mathbf{int}(A)) \subseteq \mathbf{int}(\mathbf{cl}(A))$, and so $B(\mathrm{OD}(\mathbf{R}))$ is a **S4.2**-algebra. Consequently, $B(\mathrm{OD}(\mathbf{R}))$ is a **S4.Grz.2**-algebra. □

We show that each non-theorem of **S4.Grz.2** can be refuted on $B(\mathrm{OD}(\mathbf{R}))$. Consider $\mathfrak{F} \urcorner \mathfrak{C}_1$, where $\mathfrak{F} = (W, R)$ is a finite rooted partially ordered frame and $\mathfrak{C}_1 = (\{w_1\}, \{(w_1, w_1)\})$ is a one-point cluster. As follows from [16, Theorem 5.10] (see also [1, Lemma 4.5] for a simplified proof), there exists an interior map g from the Cantor space $\mathbf{C}$ onto $\mathfrak{F}$. Let $f : \mathbf{R} \to \mathfrak{F} \urcorner \mathfrak{C}_1$ be given by

$$f(x) = \begin{cases} g(x) & \text{if } x \in \mathbf{C}, \\ w_1 & \text{if } x \in \mathbf{R} - \mathbf{C}. \end{cases}$$

It is easily seen that f is an onto interior map. Let $\mathfrak{F} \urcorner \mathfrak{C}_1 = (W_1, R_1)$. For each $w \in W_1$, the set $R_1[w]$ is open and dense in $\mathfrak{F} \urcorner \mathfrak{C}_1$; for each $w \in W$, the set $R_1^{-1}[w]$ is closed and has empty interior; and $R_1^{-1}[w_1] = W_1$. Therefore, for each $w \in W_1$, we have $f^{-1}R_1[w], f^{-1}R_1^{-1}[w] \in B(\mathrm{OD}(\mathbf{R}))$. Thus, $f^{-1}(w) = f^{-1}R_1[w] \cap f^{-1}R_1^{-1}[w] \in B(\mathrm{OD}(\mathbf{R}))$. It follows that f^{-1} is a closure algebra homomorphism from $(\wp W_1, R_1^{-1})$ into $B(\mathrm{OD}(\mathbf{R}))$. Consequently, each formula refutable on $\mathfrak{F} \urcorner \mathfrak{C}_1$ is also refutable on $B(\mathrm{OD}(\mathbf{R}))$. Now since **S4.Grz.2** is complete with respect to the finite frames $\mathfrak{F} \urcorner \mathfrak{C}_1$, with $\mathfrak{F}$ a rooted partially ordered frame and $\mathfrak{C}_1$ a one-point cluster, we obtain that **S4.Grz.2** is the logic of $B(\mathrm{OD}(\mathbf{R}))$.

6.3 **S4.1.2**

We recall that a subset A of a topological space is *nowhere dense* if $\mathbf{int}(\mathbf{cl}(A)) = \emptyset$. Let $\mathrm{ND}(\mathbf{R})$ denote the set of nowhere dense subsets of $\mathbf{R}$ and let $B(\mathrm{ND}(\mathbf{R}))$ denote the Boolean subalgebra of $\mathbf{R}^+$ generated by $\mathrm{ND}(\mathbf{R})$. We show that $B(\mathrm{ND}(\mathbf{R}))$ is closed under **cl**, and so is a subalgebra of $\mathbf{R}^+$.

Lemma 6.3 $\mathrm{ND}(\mathbf{R})$ *is an ideal of* $\mathbf{R}^+$.

Proof Clearly if A is nowhere dense and $B \subseteq A$, then B is also nowhere dense. Let A and B be nowhere dense subsets of $\mathbf{R}$. Then $\mathbf{int}(\mathbf{cl}(A)) = \emptyset$. Therefore, $-\mathbf{cl}-\mathbf{cl}(A) = \emptyset$, so $\mathbf{cl} - \mathbf{cl}(A) = \mathbf{R}$, and so $-\mathbf{cl}(A) = \mathbf{int}(-A)$ is a dense subset of $\mathbf{R}$. Similarly $\mathbf{int}(-B)$ is a dense subset of $\mathbf{R}$.

If $A \cup B$ is not nowhere dense, then $\mathbf{int}(\mathbf{cl}(A \cup B)) \neq \emptyset$. As $\mathbf{int}(-A)$ is dense, we have

$$\mathbf{int}(-A) \cap \mathbf{int}(\mathbf{cl}(A \cup B)) \neq \emptyset$$

This implies

$$\mathbf{int}(\mathbf{int}(-A)) \cap \mathbf{int}(\mathbf{cl}(A \cup B)) \neq \emptyset$$

Therefore,

$$\mathbf{int}[\mathbf{int}(-A) \cap (-\mathbf{int}(-A) \cup \mathbf{cl}(B))] \neq \emptyset$$

Thus,

$$\mathbf{int}(\mathbf{int}(-A) \cap \mathbf{cl}(B)) \neq \emptyset$$

Also, as $\mathbf{int}(-B)$ is dense, we have

$$\mathbf{int}(-B) \cap \mathbf{int}(\mathbf{int}(-A) \cap \mathbf{cl}(B)) \neq \emptyset$$

Therefore,

$$\mathbf{int}(\mathbf{int}(-B)) \cap \mathbf{int}(\mathbf{int}(-A) \cap -\mathbf{int}(-B)) \neq \emptyset$$

Thus,

$$\mathbf{int}[\mathbf{int}(-B) \cap \mathbf{int}(-A) \cap -\mathbf{int}(-B)] \neq \emptyset$$

The obtained contradiction proves that $A \cup B$ is nowhere dense. Consequently, $\mathrm{ND}(\mathbf{R})$ is an ideal of $\mathbf{R}^+$. □

We call a subset A of a topological space X *interior dense* if it has a dense interior; that is, $\mathbf{cl}(\mathbf{int}(A)) = X$. We denote the collection of all interior dense subsets of X by $\mathrm{ID}(X)$. It is obvious that $A \in \mathrm{ID}(X)$ iff $-A \in \mathrm{ND}(X)$.

Lemma 6.4 $B(\mathrm{ND}(\mathbf{R})) = \mathrm{ND}(\mathbf{R}) \cup \mathrm{ID}(\mathbf{R})$.

Proof Clearly $\mathrm{ND}(\mathbf{R}) \cup \mathrm{ID}(\mathbf{R}) \subseteq B(\mathrm{ND}(\mathbf{R}))$. Therefore, it is sufficient to show that $\mathrm{ND}(\mathbf{R}) \cup \mathrm{ID}(\mathbf{R})$ is a Boolean subalgebra of $\mathbf{R}^+$. That $\mathrm{ND}(\mathbf{R}) \cup \mathrm{ID}(\mathbf{R})$ is closed under $-$ is obvious. We show that $\mathrm{ND}(\mathbf{R}) \cup \mathrm{ID}(\mathbf{R})$ is closed under $\cup$. By Lemma 6.3, $\mathrm{ND}(\mathbf{R})$ is an ideal of $\mathbf{R}^+$. Therefore, $\mathrm{ID}(\mathbf{R}) = \{-A : A \in \mathrm{ND}(\mathbf{R})\}$ is a filter of $\mathbf{R}^+$. Thus, for $A, B \in \mathrm{ND}(\mathbf{R}) \cup \mathrm{ID}(\mathbf{R})$, if both $A, B \in \mathrm{ND}(\mathbf{R})$, then $A \cup B \in \mathrm{ND}(\mathbf{R})$, and if at least one of A, B belongs to $\mathrm{ID}(\mathbf{R})$, then so does $A \cup B$. Consequently, $\mathrm{ND}(\mathbf{R}) \cup \mathrm{ID}(\mathbf{R})$ is a Boolean subalgebra of $\mathbf{R}^+$, and so $B(\mathrm{ND}(\mathbf{R})) = \mathrm{ND}(\mathbf{R}) \cup \mathrm{ID}(\mathbf{R})$. □

Lemma 6.5 $B(\mathrm{ND}(\mathbf{R}))$ *is a subalgebra of* $\mathbf{R}^+$.

Proof Let $A \in B(\mathrm{ND}(\mathbf{R}))$. By Lemma 6.4, either $A \in \mathrm{ND}(\mathbf{R})$ or $A \in \mathrm{ID}(\mathbf{R})$. If $A \in \mathrm{ND}(\mathbf{R})$, then

$$\mathbf{int}(\mathbf{cl}(\mathbf{cl}(A))) = \mathbf{int}(\mathbf{cl}(A)) = \emptyset$$

Therefore, $\mathbf{cl}(A) \in \mathrm{ND}(\mathbf{R})$. On the other hand, if $A \in \mathrm{ID}(\mathbf{R})$, then as $A \subseteq \mathbf{cl}(A)$ and $\mathrm{ID}(\mathbf{R})$ is a filter of $\mathbf{R}^+$, we have $\mathbf{cl}(A) \in \mathrm{ID}(\mathbf{R})$. In either case, $A \in B(\mathrm{ND}(\mathbf{R}))$ implies $\mathbf{cl}(A) \in B(\mathrm{ND}(\mathbf{R}))$, and so $B(\mathrm{ND}(\mathbf{R}))$ is a subalgebra of $\mathbf{R}^+$. □

Lemma 6.6 *$B(\mathrm{ND}(\mathbf{R}))$ is a* **S4.1.2**-*algebra.*

Proof Let $A \in B(\mathrm{ND}(\mathbf{R}))$. By Lemma 6.4, $A \in \mathrm{ND}(\mathbf{R})$ or $A \in \mathrm{ID}(\mathbf{R})$. If $A \in \mathrm{ND}(\mathbf{R})$, then $\mathbf{int}(\mathbf{cl}(A)) = \emptyset$. Therefore, $\mathbf{int}(A) = \emptyset$, so $\mathbf{cl}(\mathbf{int}(A)) = \emptyset$, and so $\mathbf{int}(\mathbf{cl}(A)) = \mathbf{cl}(\mathbf{int}(A))$. On the other hand, if $A \in \mathrm{ID}(\mathbf{R})$, then $\mathbf{cl}(\mathbf{int}(A)) = \mathbf{R}$. This implies $\mathbf{cl}(A) = \mathbf{R}$. Thus, $\mathbf{int}(\mathbf{cl}(A)) = \mathbf{R}$, and so $\mathbf{int}(\mathbf{cl}(A)) = \mathbf{cl}(\mathbf{int}(A))$. Consequently, $B(\mathrm{ND}(\mathbf{R}))$ is a **S4.1.2**-algebra. □

We show that **S4.1.2** is the logic of $B(\mathrm{ND}(\mathbf{R}))$. We recall that a finite rooted **S4.1.2**-frame is of the form $\mathfrak{F}^{\urcorner}\mathfrak{C}_1 = (W_1, R_1)$, where $\mathfrak{F} = (W, R)$ is a finite rooted **S4**-frame and $\mathfrak{C}_1 = (\{w_1\}, \{(w_1, w_1)\})$ is a one-point cluster. As follows from [16, Theorem 5.10] (see also [1, Lemma 4.5] for a simplified proof), there exists an interior map g from $\mathbf{C}$ onto $\mathfrak{F}$. We extend g to the map $f : \mathbf{R} \to \mathfrak{F}^{\urcorner}\mathfrak{C}_1$ by sending all points of $\mathbf{R} - \mathbf{C}$ to w_1. It is easy to see that f is onto and interior. Let $A \subseteq W_1$. Then $w_1 \in A$ or $w_1 \notin A$. In the first case, $w_1 \in -R_1^{-1}[-A]$, so $R^{-1} - R_1^{-1}[-A] = W_1$, and so A is interior dense. In the second case, $w_1 \notin R_1^{-1}[A]$, so $-R_1^{-1} - R^{-1}[A] = \emptyset$, and so A is nowhere dense. It follows that $f^{-1}[A]$ is in $B(\mathrm{ND}(\mathbf{R}))$. Therefore, f^{-1} is a closure algebra homomorphism from $(\mathfrak{F}^{\urcorner}\mathfrak{C}_1)^+$ into $B(\mathrm{ND}(\mathbf{R}))$. Thus, each formula refutable on $\mathfrak{F}^{\urcorner}\mathfrak{C}_1$ is also refutable on $B(\mathrm{ND}(\mathbf{R}))$, and so **S4.1.2** is the logic of $B(\mathrm{ND}(\mathbf{R}))$.

6.4 **S4.1**

Let X be a topological space and let

$$\mathfrak{A}_X = \{A \subseteq X : \mathbf{int}(\mathbf{cl}(A)) \subseteq \mathbf{cl}(\mathbf{int}(A))\}$$

We recall that the *boundary* of $A \subseteq X$ is defined as

$$\mathbf{fr}(A) = \mathbf{cl}(A) \cap \mathbf{cl}(-A)$$

and that A has *small boundary* if $\mathbf{int}(\mathbf{fr}(A)) = \emptyset$. Note that since $\mathbf{fr}(A)$ is closed, A has small boundary iff $\mathbf{fr}(A)$ is nowhere dense. We have:

$$\begin{aligned} A \text{ has small boundary } \quad & \text{iff } \ \mathbf{int}(\mathbf{fr}(A)) = \emptyset \\ & \text{iff } \ \mathbf{int}(\mathbf{cl}(A) \cap \mathbf{cl}(-A)) = \emptyset \\ & \text{iff } \ \mathbf{int}(\mathbf{cl}(A)) \cap \mathbf{int}(\mathbf{cl}(-A)) = \emptyset \\ & \text{iff } \ \mathbf{int}(\mathbf{cl}(A)) \cap -\mathbf{cl}(\mathbf{int}(A)) = \emptyset \\ & \text{iff } \ \mathbf{int}(\mathbf{cl}(A)) \subseteq \mathbf{cl}(\mathbf{int}(A)). \end{aligned}$$

Therefore, $\mathfrak{A}_X$ coincides with the set of subsets of X with small boundary.

Lemma 6.7 *(Esakia [11]) $\mathfrak{A}_X$ is a subalgebra of X^+.*

Proof Clearly if A has small boundary, then so does $-A$. Therefore, $A \in \mathfrak{A}_X$ implies $-A \in \mathfrak{A}_X$, and so $\mathfrak{A}_X$ is closed under complementation. To see that $\mathfrak{A}_X$ is closed under **cl**, let $A \in \mathfrak{A}_X$. Then $\mathbf{int}(\mathbf{fr}(A)) = \emptyset$. Therefore,

$$\begin{aligned}\mathbf{int}(\mathbf{fr}(\mathbf{cl}(A))) &= \mathbf{int}[\mathbf{cl}(\mathbf{cl}(A)) \cap \mathbf{cl}(-\mathbf{cl}(A))] \\ &\subseteq \mathbf{int}[\mathbf{cl}(A) \cap \mathbf{cl}(-A)] \\ &= \mathbf{int}(\mathbf{fr}(A)) \\ &= \emptyset\end{aligned}$$

Thus, $\mathbf{cl}(A) \in \mathfrak{A}_X$, and so $\mathfrak{A}_X$ is closed under **cl**. It is left to be shown that $\mathfrak{A}_X$ is closed under union. Let $A, B \in \mathfrak{A}_X$. Then $\mathbf{fr}(A)$ and $\mathbf{fr}(B)$ are nowhere dense. By Lemma 6.3, $\mathbf{fr}(A) \cup \mathbf{fr}(B)$ is also nowhere dense. Therefore, $\mathbf{int}(\mathbf{fr}(A) \cup \mathbf{fr}(B)) = \emptyset$. By [7, Thm. 1.3.2], $\mathbf{fr}(A \cup B) \subseteq \mathbf{fr}(A) \cup \mathbf{fr}(B)$. Thus, $\mathbf{int}(\mathbf{fr}(A \cup B)) = \emptyset$, so $A \cup B$ has small boundary, and so $A \cup B \in \mathfrak{A}_X$. Consequently, $\mathfrak{A}_X$ is a subalgebra of X^+. □

We show that **S4.1** is the logic of $\mathfrak{A}_{\mathbf{R}}$. It follows from the definition that $\mathfrak{A}_{\mathbf{R}}$ is an **S4.1**-algebra. In fact, $\mathfrak{A}_{\mathbf{R}}$ is the largest subalgebra of $\mathbf{R}^+$ that is an **S4.1**-algebra. Let $\mathfrak{F} = (W, R)$ be a finite rooted **S4.1**-frame. As follows from [16, Theorem 5.10] (see also [2, Corollary 14] for a simplified proof), there exists an interior map f from $\mathbf{R}$ onto $\mathfrak{F}$. Therefore, f^{-1} is a closure algebra homomorphism from $\mathfrak{F}^+$ into $\mathbf{R}^+$. We claim that $f^{-1}(A) \in \mathfrak{A}_{\mathbf{R}}$ for each $A \subseteq W$. Since $\mathfrak{F}$ is an **S4.1**-frame, we have $-R^{-1}-R^{-1}[A] \subseteq R^{-1}-R^{-1}[-A]$. Applying f^{-1} and using the fact that f^{-1} is a closure algebra homomorphism, we obtain $\mathbf{int}(\mathbf{cl}(f^{-1}(A))) \subseteq \mathbf{cl}(\mathbf{int}(f^{-1}(A)))$. Thus, $f^{-1}(A) \in \mathfrak{A}_{\mathbf{R}}$. It follows that each formula refutable on $\mathfrak{F}$ is also refutable on $\mathfrak{A}_{\mathbf{R}}$. Now as **S4.1** has the fmp, we obtain that **S4.1** is the logic of $\mathfrak{A}_{\mathbf{R}}$.

6.5 S5

Let $P = \{P_i : i \in \omega\}$ be a partition of $\mathbf{R}$ into countably many dense subsets; that is, $\bigcup P_i = \mathbf{R}$, $P_i \cap P_j = \emptyset$ whenever $i \neq j$, and $\mathbf{cl}(P_i) = \mathbf{R}$ for each i. Clearly such a partition exists. Let $B(P)$ be the Boolean subalgebra of $\mathbf{R}^+$ generated by P. Then each element of $B(P)$ is a finite union of P_i's or the complement of a finite union of P_i's. Since for each $A \neq \emptyset$ in $B(P)$ we have $\mathbf{cl}(A) = \mathbf{R}$, it is obvious that $B(P)$ is a subalgebra of $\mathbf{R}^+$. We show that **S5** is the logic of $B(P)$. Let $A \in B(P)$. If $A = \emptyset$, then $\mathbf{cl}(A) = \emptyset$, and so $\mathbf{int}(\mathbf{cl}(A)) = \emptyset = \mathbf{cl}(A)$. If $A \neq \emptyset$, then $\mathbf{cl}(A) = \mathbf{R}$, and so $\mathbf{int}(\mathbf{cl}(A)) = \mathbf{R} = \mathbf{cl}(A)$. In either case, $\mathbf{int}(\mathbf{cl}(A)) = \mathbf{cl}(A)$, and so $B(P)$ is an **S5**-algebra.

We show that each finite cluster $\mathfrak{C}_n = (W_n, R_n)$ is an onto interior image of $\mathbf{R}$ so that the preimages of subsets of $\mathfrak{C}_n$ are elements of $B(P)$. Define $h : \mathbf{R} \to \mathfrak{C}_n$ by

$$h(x) = \begin{cases} w_i & \text{if } i < n \text{ and } x \in P_i, \\ w_n & \text{if } i \geq n \text{ and } x \in P_i. \end{cases}$$

Then it is easy to see that h is onto, interior, and the h-preimages of subsets of $\mathfrak{C}_n$ belong to $B(P)$. Therefore, h^{-1} is a closure algebra homomorphism from $\mathfrak{C}_n^+$ into

$B(P)$. Thus, each formula refutable on $\mathfrak{C}_n$ is also refutable on $B(P)$. Since **S5** is complete with respect to the class of finite clusters, it follows that **S5** is the logic of $B(P)$.

6.6 S4.2

Now we can combine the approaches to **S4.1.2** and **S5** to obtain a subalgebra of $\mathbf{R}^+$ whose logic is **S4.2**. Let $\mathfrak{B}$ be the Boolean subalgebra of R^+ generated by $B(\mathrm{ND}(\mathbf{R})) \cup B(P)$. We show that **S4.2** is the logic of $\mathfrak{B}$.

Lemma 6.8 *For each $A \in \mathfrak{B}$ we have $\mathbf{cl}(A) = \mathbf{R}$ or $\mathbf{int}(A) = \emptyset$.*

Proof Let $A \in \mathfrak{B}$. Then $A = \bigcup_{i=1}^{n}(B_i \cap C_i)$, where $B_i \in B(\mathrm{ND}(\mathbf{R}))$ and C_i is a nonempty element of $B(P)$. By Lemma 6.4, $B(\mathrm{ND}(\mathbf{R})) = \mathrm{ND}(\mathbf{R}) \cup \mathrm{ID}(\mathbf{R})$. First suppose that $B_i \in \mathrm{ND}(\mathbf{R})$ for all $i = 1, \ldots, n$. By Lemma 6.3, $\mathrm{ND}(\mathbf{R})$ is an ideal of $\mathbf{R}^+$. Therefore, $B_i \cap C_i \in \mathrm{ND}(\mathbf{R})$, and so $A = \bigcup_{i=1}^{n}(B_i \cap C_i) \in \mathrm{ND}(\mathbf{R})$. Thus, $\mathbf{int}(\mathbf{cl}(A)) = \emptyset$, which implies that $\mathbf{int}(A) = \emptyset$. Now suppose that $B_i \in \mathrm{ID}(\mathbf{R})$ for some i. Then $\mathbf{cl}(\mathbf{int}(B_i)) = \mathbf{R}$, and so $\mathbf{int}(B_i)$ is dense. As $C_i \neq \emptyset$, we also have that C_i is dense. But then $\mathbf{int}(B_i) \cap C_i$ is also dense as $\mathbf{int}(B_i)$ is open and both $\mathbf{int}(B_i)$ and C_i are dense. Therefore, $\mathbf{cl}(A) \supseteq \mathbf{cl}(B_i \cap C_i) \supseteq \mathbf{cl}(\mathbf{int}(B_i) \cap C_i) = \mathbf{R}$. Thus, $\mathbf{cl}(A) = \mathbf{R}$. □

It is an immediate corollary to Lemma 6.8 that $\mathfrak{B}$ is a subalgebra of $\mathbf{R}^+$. Moreover, for each $A \in \mathfrak{B}$, if $\mathbf{int}(A) = \emptyset$, then $\mathbf{cl}(\mathbf{int}(A)) = \emptyset$, and so $\mathbf{cl}(\mathbf{int}(A)) \subseteq \mathbf{int}(\mathbf{cl}(A))$; and if $\mathbf{cl}(A) = \mathbf{R}$, then $\mathbf{int}(\mathbf{cl}(A)) = \mathbf{R}$, and again $\mathbf{cl}(\mathbf{int}(A)) \subseteq \mathbf{int}(\mathbf{cl}(A))$. Consequently, $\mathfrak{B}$ is an **S4.2**-algebra.

It is left to be shown that each non-theorem of **S4.2** can be refuted on $\mathfrak{B}$. Recall that a finite rooted **S4.2**-frame has the form $\mathfrak{F}\urcorner\mathfrak{C}_n$, where $\mathfrak{F}$ is a finite rooted **S4**-frame and $\mathfrak{C}_n$ is the n-cluster. Let $g : \mathbf{C} \to \mathfrak{F}$ and $h : \mathbf{R} \to \mathfrak{C}_n$ be the onto interior maps described above. Define $\alpha : \mathbf{R} \to \mathfrak{F}\urcorner\mathfrak{C}_n$ by

$$\alpha(x) = \begin{cases} g(x) & \text{if } x \in \mathbf{C}, \\ h(x) & \text{if } x \notin \mathbf{C}. \end{cases}$$

Then α is an onto interior map, and so α^{-1} is a closure algebra homomorphism from $(\mathfrak{F}\urcorner\mathfrak{C}_n)^+$ into $\mathfrak{B}$. Therefore, each formula refutable on $\mathfrak{F}\urcorner\mathfrak{C}_n$ is also refutable on $\mathfrak{B}$. Now since **S4.2** has the fmp, it follows that **S4.2** is the logic of $\mathfrak{B}$.

In the following table we list all the logics considered in this section together with the corresponding subalgebras of $\mathbf{R}^+$ that generate them.

Logic	Subalgebra of $\mathbf{R}^+$	Description
S4.Grz	$B(\mathrm{Op}(\mathbf{R}))$	Boolean combinations of open subsets of **R**
S4.Grz.2	$B(\mathrm{OD}(\mathbf{R}))$	Boolean combinations of open dense subsets of **R**
S4.1.2	$\mathrm{ND}(\mathbf{R}) \cup \mathrm{ID}(\mathbf{R})$	Nowhere dense and interior dense subsets of **R**
S4.1	$\mathfrak{A}_{\mathbf{R}}$	All subsets of **R** with small boundary
S5	$B(P)$	Boolean combinations of a partition P of **R** into countably many dense subsets of **R**
S4.2	$\mathfrak{B}$	Boolean combinations of nowhere dense subsets of **R** and a partition of **R** into countably many dense subsets of **R**

7 Logics of subalgebras of $\mathbf{Q}^+$ and $\mathbf{C}^+$

In this section we show that each normal extension of **S4** with the fmp is the logic of a subalgebra of $\mathbf{Q}^+$ as well as of a subalgebra of $\mathbf{C}^+$. The argument for subalgebras of $\mathbf{Q}^+$ is an easy consequence of the McKinsey-Tarski theorem [16, Theorem 5.10], while the one for subalgebras of $\mathbf{C}^+$ requires a little more work.

Theorem 7.1 *Let L be a normal extension of* **S4** *with the fmp. Then L is the logic of a subalgebra of* $\mathbf{Q}^+$.

Proof Let $\mathfrak{F}_1, \mathfrak{F}_2, \ldots$ be an enumeration of finite rooted L-frames. It follows from [16, Theorem 5.10] that each finite rooted **S4**-frame is an onto interior image of **Q**. Therefore, for each $i \in \omega$ there is an onto interior map $f_i : \mathbf{Q} \to \mathfrak{F}_i$. But then $\bigsqcup_{i\in\omega} f_i : \bigsqcup_{i\in\omega} \mathbf{Q} \to \bigsqcup_{i\in\omega} \mathfrak{F}_i$ is also an onto interior map. Clearly $\bigsqcup_{i\in\omega} \mathbf{Q}$ is a countable, dense-in-itself, metrizable space. Thus, by Sierpinski's theorem (see, e.g., [7, Exercise 6.2.A(d)]), $\bigsqcup_{i\in\omega} \mathbf{Q}$ is homeomorphic to **Q**. Therefore, there is an onto interior map $f : \mathbf{Q} \to \bigsqcup_{i\in\omega} \mathfrak{F}_i$. This implies that f^{-1} is a closure algebra homomorphism from $(\bigsqcup_{i\in\omega} \mathfrak{F}_i)^+$ into $\mathbf{Q}^+$. Let A be the image of f^{-1} in $\mathbf{Q}^+$. Then A is a subalgebra of $\mathbf{Q}^+$ isomorphic to $(\bigsqcup_{i\in\omega} \mathfrak{F}_i)^+$. Therefore, A is an L-algebra. Since each non-theorem of L is refuted on $\bigsqcup_{i\in\omega} \mathfrak{F}_i$, it is also refuted on A. Thus, L is the logic of A. □

In order to prove that L is the logic of a subalgebra of $\mathbf{C}^+$, we need the following

Lemma 7.2 *Let X be a countably infinite disjoint union of* **C**. *Then X is a noncompact locally compact Hausdorff space and* **C** *is homeomorphic to the one-point compactification of X.*

Proof Since X is a countably infinite disjoint union of **C**, it is obvious that X is a noncompact locally compact Hausdorff space. Therefore, by [7, Theorem 3.5.11], X has the one-point compactification αX. Clearly αX is a compact Hausdorff space. Since **C** is zero-dimensional, by [7, Theorem 6.2.13], X is zero-dimensional. This, by

[4, Corollary 3.16], implies that αX is also zero-dimensional, hence a Stone space. Clearly X is dense-in-itself. Therefore, so is αX. Also, as X has a countable basis, so does αX. Thus, αX is a dense-in-itself metrizable Stone space. By Brouwer's theorem (see, e.g., [7, Exercise 6.2.A(c)]), αX is homeomorphic to $\mathbf{C}$. □

Theorem 7.3 *Let L be a normal extension of* **S4** *with the fmp. Then L is the logic of a subalgebra of* $\mathbf{C}^+$.

Proof Let $\mathfrak{F}_1 = (W_1, R_1), \mathfrak{F}_2 = (W_2, R_2), \ldots$ be an enumeration of finite rooted L-frames and let $\mathfrak{F} = (W, R)$ be the disjoint union $\bigsqcup_{i\in\omega} \mathfrak{F}_i$. As follows from [16, Theorem 5.10] (see also [1, Lemma 4.5] for a simplified proof), each finite rooted **S4**-frame is an onto interior image of $\mathbf{C}$. Therefore, for each $i \in \omega$ there is an onto interior map $f_i : \mathbf{C} \to \mathfrak{F}_i$. Let X denote the disjoint union $\bigsqcup_{i\in\omega} \mathbf{C}$. Then $\bigsqcup_{i\in\omega} f_i : X \to \mathfrak{F}$ is also an onto interior map.

We view W as a topological space with the discrete topology. Then W is noncompact locally compact Hausdorff, and so has the one-point compactification $\alpha W = W \cup \{\infty\}$. Let $\alpha R = R \cup \{(\infty, \infty)\}$. By the argument preceding Lemma 5.1, $(\alpha W, \alpha R)$ is a descriptive **S4**-frame.

Let αX be the one-point compactification of X with Ω being the point at infinity. By Lemma 7.2, αX is homeomorphic to $\mathbf{C}$. We extend $\bigsqcup_{i\in\omega} f_i$ to a map $f : \alpha X \twoheadrightarrow \alpha W$ by sending Ω to ∞. Let $\mathfrak{A}$ be the closure algebra of clopen subsets of $(\alpha W, \alpha R)$. We claim that f^{-1} is a closure algebra homomorphism from $\mathfrak{A}$ into $[\alpha X]^+$. Since f is onto, we have f^{-1} is 1-1. We show that $f^{-1}((\alpha R)^{-1}[U]) = \mathbf{cl}_{\alpha X}(f^{-1}(U))$ for each clopen subset U of αW.

Let U be a clopen subset of αW. Then there exist $i_1, \ldots, i_n$ and $U_{i_1} \subseteq W_{i_1}, \ldots, U_{i_n} \subseteq W_{i_n}$ such that $U = \bigcup_{k=1}^{n} U_{i_k}$ or $U = \alpha W - (\bigcup_{k=1}^{n} U_{i_k})$. If $U = \bigcup_{k=1}^{n} U_{i_k}$, then

$$f^{-1}(\alpha R)^{-1}[U] = f^{-1}(\alpha R)^{-1}\left[\bigcup_{k=1}^{n} U_{i_k}\right] = \bigcup_{k=1}^{n} f^{-1}(\alpha R)^{-1}\left[U_{i_k}\right]$$

$$= \bigcup_{k=1}^{n} f_{i_k}^{-1} R_{i_k}^{-1}\left[U_{i_k}\right].$$

Since each f_{i_k} is an interior map from a copy of $\mathbf{C}$ onto W_{i_k} and each copy of $\mathbf{C}$ is a clopen subset of αX, we have $f_{i_k}^{-1} R_{i_k}^{-1}(U_{i_k}) = \mathbf{cl}_{\alpha X} f_{i_k}^{-1}(U_{i_k})$. Therefore,

$$f^{-1}(\alpha R)^{-1}[U] = \bigcup_{k=1}^{n} \mathbf{cl}_{\alpha X} f_{i_k}^{-1}(U_{i_k}) = \mathbf{cl}_{\alpha X} f^{-1}\left(\bigcup_{k=1}^{n} U_{i_k}\right) = \mathbf{cl}_{\alpha X} f^{-1}(U).$$

If $U = \alpha W - (\bigcup_{k=1}^{n} U_{i_k})$, then $U = (W - (\bigcup_{k=1}^{n} U_{i_k})) \cup \{\infty\}$. Let $V = W - (\bigcup_{k=1}^{n} U_{i_k})$. We show that $f^{-1}R^{-1}[V] = \mathbf{cl}_X f^{-1}(V)$. Indeed,

$$\begin{aligned}
f^{-1}R^{-1}[V] &= f^{-1}R^{-1}\left[W - \bigcup_{k=1}^{n} U_{i_k}\right] \\
&= f^{-1}R^{-1}\left[\bigcup_{j\neq i_k} W_j \cup \bigcup_{k=1}^{n}(W_{i_k} - U_{i_k})\right] \\
&= \bigcup_{j\neq i_k} f^{-1}R_j^{-1}\left[W_j\right] \cup \bigcup_{k=1}^{n} f_{i_k}^{-1}R_{i_k}^{-1}\left[W_{i_k} - U_{i_k}\right].
\end{aligned}$$

Note that $R_j^{-1}[W_j] = W_j$ and $f^{-1}[W_j]$ is a copy of $\mathbf{C}$. Therefore, $\bigcup_{j\neq i_k} f^{-1}R_j^{-1}[W_j] = \bigcup_{j\neq i_k} f^{-1}(W_j)$ is a union of all but finitely many copies of $\mathbf{C}$, hence is a clopen subset of X. Furthermore, by the same argument as in the previous case, $f_{i_k}^{-1}R_{i_k}^{-1}[W_{i_k} - U_{i_k}] = \mathbf{cl}_X f_{i_k}^{-1}(W_{i_k} - U_{i_k}) = \mathbf{cl}_X f^{-1}(W_{i_k} - U_{i_k})$. Thus,

$$\begin{aligned}
f^{-1}R^{-1}[V] &= \bigcup_{j\neq i_k} f^{-1}(W_j) \cup \bigcup_{k=1}^{n} \mathbf{cl}_X f^{-1}(W_{i_k} - U_{i_k}) \\
&= \mathbf{cl}_X f^{-1}\Big(\bigcup_{j\neq i_k} W_j \cup \bigcup_{k=1}^{n}(W_{i_k} - U_{i_k})\Big) \\
&= \mathbf{cl}_X f^{-1}(V).
\end{aligned}$$

Now,

$$\begin{aligned}
f^{-1}((\alpha R)^{-1}[U]) &= f^{-1}((\alpha R)^{-1}[V \cup \{\infty\}]) \\
&= f^{-1}(R^{-1}[V] \cup \{\infty\}) \\
&= f^{-1}R^{-1}[V] \cup f^{-1}(\{\infty\}) \\
&= \mathbf{cl}_X(f^{-1}(V)) \cup \{\Omega\} \\
&= \mathbf{cl}_{\alpha X}(f^{-1}(V)).
\end{aligned}$$

Consequently, f^{-1} is a closure algebra homomorphism from $\mathfrak{A}$ into $[\alpha X]^+$.

By Lemma 7.2, $\alpha X = \alpha(\bigsqcup_{i\in\omega} \mathbf{C})$ is homeomorphic to $\mathbf{C}$. Therefore, $\mathfrak{A}$ is isomorphic to a subalgebra $\mathfrak{B}$ of $\mathbf{C}^+$. Since each non-theorem of L can be refuted on one of the $\mathfrak{F}_i$'s, which are finite upsets (and downsets) of $(\alpha W, \alpha R)$, each non-theorem of L can be refuted on the descriptive **S4**-frame $(\alpha W, \alpha R)$ with an admissible valuation that sends propositional letters to finite (and thus clopen) subsets. This shows that each non-theorem of L can be refuted on $\mathfrak{A}$. By Lemma 5.1, $\mathfrak{A}$ is an L-algebra. Since $\mathfrak{A}$ is isomorphic to $\mathfrak{B}$, each non-theorem of L is refuted on an L-algebra $\mathfrak{B}$, and so L is the logic of $\mathfrak{B}$. Consequently, L is the logic of a subalgebra of $\mathbf{C}^+$. □

8 Intermediate logics

Let **CPC** denote the classical propositional calculus and **IPC** the intuitionistic propositional calculus. It is well-known that **IPC** is properly contained in **CPC** and that there are continuum many logics in between **IPC** and **CPC**, called *intermediate logics*. In the domain of intermediate logics we can obtain even sharper results. Recall that the algebraic semantics for intermediate logics is provided by *Heyting algebras*. In fact, there is a dual isomorphism between the lattice of intermediate logics and the lattice of non-degenerate varieties of Heyting algebras. We recall that a Heyting algebra A is a bounded distributive lattice with a binary operation $\to$ such that for all $a, b, c \in A$ we have $a \wedge c \leq b$ iff $c \leq a \to b$. Let A be a Heyting algebra and let $a \in A$. As usual, $\neg a$ abbreviates $a \to 0$. If $a \vee \neg a = 1$, then we say that a is *complemented*. It is always the case that 0 and 1 are complemented elements of A. We call A *connected* if 0, 1 are the only complemented elements of A. We also call an intermediate logic L *connected* if the corresponding variety $\mathcal{V}_L$ of Heyting algebras is generated by a connected Heyting algebra.

There is a close connection between intermediate logics and consistent normal extensions of **S4**. Each intermediate logic can be viewed as a fragment of a consistent normal extension of **S4**. There are different embeddings of the lattice of intermediate logics into the lattice of normal extensions of **S4**. The celebrated Blok-Esakia theorem states that the lattice of intermediate logics is isomorphic to the lattice of consistent normal extensions of **S4.Grz**. This together with the technique developed in Sects. 4 and 5 provide us with the following strengthening of Theorem 4.8 for intermediate logics.

Theorem 8.1 *Let L be an intermediate logic. Then L is connected. Moreover, if L has the fmp, then:*

(1) $L = L(\mathfrak{F})$ *for some path-connected partial order* $\mathfrak{F}$.
(2) $L = L(X)$ *for some connected space* X.
(3) $L = L(A)$ *for some Heyting subalgebra A of the Heyting algebra* $\mathrm{Op}(\mathbf{R})$ *of all open subsets of* $\mathbf{R}$.
(4) $L = L(A)$ *for some Heyting subalgebra A of* $\mathrm{Op}(\mathbf{Q})$.
(5) $L = L(A)$ *for some Heyting subalgebra A of* $\mathrm{Op}(\mathbf{C})$.

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