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Some Results on Modal Axiomatization and Definability for Topological Spaces

Abstract. We consider two topological interpretations of the modal diamond—as the closure operator (C-semantics) and as the derived set operator (d-semantics). We call the logics arising from these interpretations C-logics and d-logics, respectively. We axiomatize a number of subclasses of the class of nodec spaces with respect to both semantics, and characterize exactly which of these classes are modally definable. It is demonstrated that the d-semantics is more expressive than the C-semantics. In particular, we show that the d-logics of the six classes of spaces considered in the paper are pairwise distinct, while the C-logics of some of them coincide.

Keywords: Modal logic, topological semantics, submaximal spaces.

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1. Introduction

In [24] McKinsey and Tarski suggested two novel topological interpretations of the modal diamond $\Diamond$: one as the closure operator C, and the other as the derived set operator d. They showed that for the former interpretation, the basic modal logic of all topological spaces is **S4**. In 1976 the second author showed that for the latter interpretation, the basic modal logic of all topological spaces is **wK4** (this result remained unpublished until [17]).

In order to distinguish between these two interpretations, the logics arising from interpreting $\Diamond$ as C we call *C-logics*. These are normal extensions of **S4**. And the logics arising from interpreting $\Diamond$ as d we call *d-logics*. These are normal extensions of **wK4**.

One of the main results of [24] states that **S4** is complete with respect to any metric separable dense-in-itself space. In particular, **S4** is complete with respect to the real line $\mathbb{R}$, the rational line $\mathbb{Q}$, or the Cantor space $\mathcal{C}$.

To mention a few other topological completeness results for C-logics, recall that a topological space X is *extremally disconnected* if the closure

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of every open subset of X is open; it is called *scattered* if every nonempty subspace of X contains an isolated point; X is called *weakly scattered* if the set of isolated points of X is dense in X.[1] We call X a *McKinsey space* if the set of dense subsets of X forms a filter. Also, X is called *irresolvable* if X is not the union of two disjoint dense subsets of X, and *hereditarily irresolvable (HI)* if every subspace of X is irresolvable. Then it is known that $\mathbf{S4.2} = \mathbf{S4} + \Diamond\Box p \to \Box\Diamond p$ is the logic of all extremally disconnected spaces [19], and that $\mathbf{S4.1} = \mathbf{S4} + \Box\Diamond p \to \Diamond\Box p$ is the logic of all McKinsey spaces, which coincides with the logic of all weakly scattered spaces [19, 9]. Moreover, $\mathbf{S4.Grz} = \mathbf{S4} + \Box(\Box(p \to \Box p) \to p) \to p$ is the logic of all HI spaces, and it coincides with the logic of all scattered spaces, the logic of all ordinal spaces, or the logic of any ordinal $\alpha \geq \omega^\omega$ [16, 2, 9].

As for the topological completeness results for d-logics, we recall that a space X satisfies the T_D *separation axiom* or is a T_D*-space* if every point in X is the intersection of an open and a closed subset of X. Equivalently, X is T_D iff $ddA \subseteq dA$ for each $A \subseteq X$.[2] Then it is known that $\mathbf{K4}$ is the logic of all T_D-spaces [17], $\mathbf{GL} = \mathbf{K} + \Box(\Box p \to p) \to \Box p$ is the logic of all scattered spaces, and it coincides with the logic of all ordinal spaces, or the logic of any ordinal $\alpha \geq \omega^\omega$ [16, 1, 10]. In addition, the logic of $\mathbb{Q}$ is equal to the logic of $\mathcal{C}$ and is $\mathbf{K4} + \Diamond\top$, and the logic of $\mathbb{R}^2$ is equal to the logic of $\mathbb{R}^n$ for any $n \geq 2$, and is $\mathbf{K4} + \Diamond\top + (\Diamond p \wedge \Diamond\neg p) \to \Diamond((p \wedge \Diamond\neg p) \vee (\neg p \wedge \Diamond p))$ [30].

Since in every topological space X we have that $CA = A \cup dA$ for each $A \subseteq X$, the derived set operator has more expressive power than the closure operator. Indeed, as follows from the completeness results mentioned above, d-logics can express such topological properties as being a T_D, scattered, or dense-in-itself space, which C-logics are not capable of expressing. Moreover, they distinguish between $\mathbb{Q}$ and $\mathbb{R}$, as well as between $\mathbb{R}$ and higher dimensional Euclidean spaces. None of this is distinguishable by C-logics.

The aim of this paper is to add to the abovementioned topological completeness results, as well as to give further indication of higher expressive power of d over C. In particular, we will consider the class of nodec spaces, its subclass of submaximal spaces, and such subclasses of submaximal spaces as the classes of door spaces, I-spaces, maximal spaces, and perfectly disconnected spaces. We axiomatize the C-logic of nodec spaces; show that the C-logic of submaximal spaces is its proper extension, axiomatize it, and prove that it coincides with the C-logics of door spaces and I-spaces. We also

[1] Weakly scattered spaces are sometimes called α-scattered (see, e.g., [28]).

[2] It is well known (see, e.g., [6]) that the T_D separation axiom is strictly in between the T_0 and T_1 separation axioms.

show that the C-logic of maximal spaces is a proper extension of the C-logic of submaximal spaces, axiomatize it, and prove that it coincides with the C-logic of perfectly disconnected spaces. On the other hand, we show that the d-logics of each of these six classes are distinct, and give an axiomatization of each one of them. These results are summed up in Table 1 at the end of the next section. We also answer the question of modal definability for each of the six classes with respect to both semantics.

Before plunging into the more technical part of the paper, let us say a few words about the general framework which this paper fits into. We view this work as a step in the direction set by McKinsey and Tarski [24]. There are at least two paths going in this direction. One of them views topological spaces as appropriate structures to provide models (for modal logics) that are richer than Kripke models. Another considers the modal language as a convenient formal language for expressing topologically meaningful properties and modal logic as a tool for reasoning about topological spaces. This paper is meant to follow the latter avenue. For a more detailed account of this and related fields of research see [3, 4, 7, 8, 19, 20, 27, 32] and references therein.

2. C-semantics and d-semantics for modal logic

2.1. C-semantics

We recall that **S4** is the least set of formulas of the basic modal language $\mathcal{L}$ containing the axioms (i) $\Box(p \to q) \to (\Box p \to \Box q)$, (ii) $\Box p \to p$, and (iii) $\Box p \to \Box\Box p$, and closed under modus ponens, substitution, and necessitation $(\varphi/\Box\varphi)$. A *topological model* is a pair $\langle X, \nu\rangle$, where X is a topological space and ν is a valuation, assigning to each propositional variable of $\mathcal{L}$ a subset of X. The connectives $\vee, \wedge$, and $\neg$ are interpreted in $\langle X, \nu\rangle$ as the set-theoretic union, intersection, and complement; and the modal operators $\Box$ and $\Diamond$ are interpreted as the *interior* and *closure* operators of X.

For a given topological model $\langle X, \nu\rangle$, $x \in X$, and φ a formula of $\mathcal{L}$, $x \models_C \varphi$ denotes that φ is satisfied in $x \in X$; we say that φ is *C-true* in $\langle X, \nu\rangle$ if $x \models_C \varphi$ for each $x \in X$; that φ is *C-valid in X* (notation $X \models_C \varphi$) if φ is C-true in $\langle X, \nu\rangle$ for each valuation ν; and that φ is *C-valid in a class* $\mathcal{K}$ of topological spaces ($\mathcal{K} \models_C \varphi$) if φ is valid in each member of $\mathcal{K}$.

For a class $\mathcal{K}$ of spaces, let $L_C(\mathcal{K})$ denote the set of formulas of $\mathcal{L}$ that are valid in $\mathcal{K}$. It is easy to verify that $L_C(\mathcal{K})$ is a normal extension of **S4**. We call $L_C(\mathcal{K})$ the *C-logic of* $\mathcal{K}$. A logic L is called a *C-logic* if L is the C-logic of some class of spaces. We will say that a class $\mathcal{K}$ of topological spaces is *C-definable* if there exists a set Γ of modal formulas such that for each space X:

$$X \in \mathcal{K} \qquad \text{iff} \qquad X \models_C \Gamma.$$

We recall that a space X is an *Alexandroff* space if the intersection of any family of open subsets of X is again open. Equivalently, X is Alexandroff iff every $x \in X$ has a least open neighborhood. It is well known that Alexandroff spaces correspond to **S4**-frames (see, e.g., [4]). We recall that a **S4**-frame is a pair $\mathfrak{F} = \langle X, R\rangle$, where $R \subseteq X^2$ is reflexive and transitive. For a given $\mathfrak{F}$, a subset A of X is called an *upset* of $\mathfrak{F}$ if $x \in A$ and xRy imply $y \in A$. Dually, A is called a *downset* if $x \in A$ and yRx imply $y \in A$. The topology on X is defined by declaring the upsets of $\mathfrak{F}$ to be open. Then the downsets of $\mathfrak{F}$ turn out to be closed, and it is routine to verify that the obtained space is Alexandroff, that a least neighborhood of $x \in X$ is $R(x) = \{y \in X : xRy\}$, that the closure of a set $A \subseteq X$ is

$$C_R(A) = R^{-1}(A) = \{x \in X : \exists y \in A \text{ with } xRy\},$$

and that the interior of A is

$$I_R(A) = (R^{-1}(A^c))^c = \{x \in X : (\forall y \in X)(xRy \to y \in A)\}.$$

Here and throughout the paper A^c denotes the complement of A in X.

For a topological space X, define the *specialization order* on X by setting xRy iff $x \in C(y)$. Then it is easy to check that the specialization order is reflexive and transitive,[3] and that the upsets of $\langle X, R\rangle$ are exactly the opens of X iff X is Alexandroff. These observations immediately imply that there is a 1-1 correspondence between Alexandroff spaces and **S4**-frames, and hence every Kripke complete normal extension of **S4** is a C-logic.

2.2. d-semantics

In order to emphasize distinction between the two topological interpretations, we introduce a new language $\mathcal{L}^*$ with modalities $\boxdot$ and $\Diamond\!\!\!\cdot$. We recall that **wK4** is the least set of formulas of $\mathcal{L}^*$ containing the axioms $\boxdot(p \to q) \to (\boxdot p \to \boxdot q)$ and $(p \wedge \boxdot p) \to \boxdot\boxdot p$, and closed under modus ponens, substitution, and necessitation $(\varphi/\boxdot\varphi)$.

For a topological space X and $A \subseteq X$, let $tA = (d(A^c))^c$. Then $x \in tA$ iff there exists a neighborhood U_x of x such that $U_x \subseteq A \cup \{x\}$. We call tA the set of *co-limit points* of A, and t the *co-derived set operator* of X.

[3]It is a partial order iff X is T_0.

Dually to $CA = A \cup dA$, we have that $IA = A \cap tA$ for every $A \subseteq X$. We interpret the modal operators $\boxdot$ and $\Diamonddot$ in a topological model $\langle X, \nu \rangle$ as the co-derived and derived set operators of X, respectively.

For a given topological model $\langle X, \nu \rangle$, $x \in X$, and φ a formula of $\mathcal{L}^*$, $x \models_d \varphi$ denotes that φ is satisfied in $x \in X$; we say that φ is *d-true* in $\langle X, \nu \rangle$ if $x \models_d \varphi$ for each $x \in X$; that φ is *d-valid in* X (notation $X \models_d \varphi$) if φ is d-true in $\langle X, \nu \rangle$ for each valuation ν; and that φ is *d-valid in a class* $\mathcal{K}$ of topological spaces (notation $\mathcal{K} \models_d \varphi$) if φ is valid in each member of $\mathcal{K}$.

For a class $\mathcal{K}$ of spaces, let $L_d(\mathcal{K})$ denote the set of formulas of $\mathcal{L}^*$ that are d-valid in $\mathcal{K}$. Similar to Section 2.1, we have that $L_d(\mathcal{K})$ is a normal extension of **wK4**. We call $L_d(\mathcal{K})$ the *d-logic of* $\mathcal{K}$. A logic L is said to be a *d-logic* if it is the d-logic of some class of topological spaces. We will say that a class $\mathcal{K}$ of topological spaces is *d-definable* if there exists a set Γ of formulas of $\mathcal{L}^*$ such that for each space X:

$$X \in \mathcal{K} \qquad \text{iff} \qquad X \models_d \Gamma.$$

There is a close correspondence between C-logics and d-logics. To see this, consider the translation of $\mathcal{L}$ into $\mathcal{L}^*$ that associates with every formula φ of $\mathcal{L}$ the formula φ^* of $\mathcal{L}^*$ obtained by replacing each subformula of φ of the form $\Box\psi$ by $\psi \wedge \boxdot\psi$.[4]

LEMMA 2.1. *Let $\mathcal{K}$ be a class of spaces, $X \in \mathcal{K}$, and φ be a formula of $\mathcal{L}$.*

1. *$X \models_C \varphi$ iff $X \models_d \varphi^*$.*
2. *$\mathcal{K} \models_C \varphi$ iff $\mathcal{K} \models_d \varphi^*$.*

PROOF. (1) By induction on the length of φ. If φ is a propositional letter or has one of the forms $\neg\psi$, $\psi \vee \chi$, or $\psi \wedge \chi$, then it obvious that $X \models_C \varphi$ iff $X \models_d \varphi^*$. The only nontrivial case is when $\varphi = \Box\psi$. But then $X \models_C \varphi$ iff $I(\nu(\psi)) = X$ iff $\nu(\psi) \cap t(\nu(\psi)) = X$ iff $X \models_d \psi \wedge \boxdot\psi$ iff $X \models_d \varphi^*$.

(2) follows from (1). ∎

For a normal extension L of **wK4**, let $T(L)$ denote $\{\varphi : \varphi^* \in L\}$. It is not difficult to verify that $T(L)$ is a normal extension of **S4**.

THEOREM 2.2. *If L is a d-logic, then $T(L)$ is a C-logic.*

PROOF. Suppose $L = L_d(\mathcal{K})$ for some $\mathcal{K}$. Then

$$\varphi \in T(L) \text{ iff } \varphi^* \in L \text{ iff } \mathcal{K} \models_d \varphi^* \text{ iff } \mathcal{K} \models_C \varphi \text{ iff } \varphi \in L_C(\mathcal{K}).$$ ∎

[4] In our notation, we aim to convey that $\Diamonddot$ should be read as the derived set operator, while $\Diamond$ is reserved to represent the closure.

We point out that $T(L)$ may be a C-logic without L being a d-logic. Thus, the converse of Theorem 2.2 is in general not true.

We recall that a **wK4**-frame is a pair $\mathfrak{F} = \langle X, R\rangle$, where $R \subseteq X^2$ is weakly transitive, that is $(\forall x, y, z \in X)(xRy \wedge yRz \wedge x \neq z \rightarrow xRz)$. We note that if $\mathfrak{F} = \langle X, R\rangle$ is a **wK4**-frame, then the reflexive closure $\overline{\mathfrak{F}} = \langle X, \overline{R}\rangle$ of $\mathfrak{F}$ is a **S4**-frame. Indeed, every **wK4**-frame is obtained from a **S4**-frame by deleting some reflexive arrows. However, if we view $\overline{\mathfrak{F}}$ as an Alexandroff space, then $d_{\overline{R}}(A)$ may not coincide with $R^{-1}(A)$. In fact, $d_{\overline{R}}(A) = \underline{R}^{-1}(A)$, where $\underline{R}$ is obtained from R by deleting *all* reflexive arrows, and so $d_{\overline{R}}(A) = R^{-1}(A)$ iff $R = \underline{R}$ (see [17, Proposition 14]). It follows that there is a 1-1 correspondence between (i) Alexandroff topologies, (ii) reflexive and transitive relations, and (iii) irreflexive and weakly transitive relations. In particular, since **GL**-frames are dually well-founded, and hence irreflexive, we have that every Kripke complete normal extension of **GL** is a d-logic. However, there exist Kripke complete normal extensions of **wK4** that are not d-logics. For example, every C-logic is such.

We recall [26] that $f : X \to Y$ is an *interior* map if f is continuous and open. We also recall [4, 19] that if $f : X \to Y$ is an onto interior map and φ is a formula of $\mathcal{L}$ such that $X \models_C \varphi$, then $Y \models_C \varphi$. An analogous characterization of the validity-preserving mappings in the d-semantics seems to bear a certain import. In the rest of this section we give such a characterization.

Definition 2.3. Suppose X, Y are topological spaces and $f : X \to Y$ is a map. We say that f is *pointwise discrete* if $f^{-1}(y)$ is a discrete subspace of X for each $y \in Y$. We call f a *d-map* if f is interior and pointwise discrete.

Theorem 2.4. *Let $f : X \to Y$ be a map. Then f is a d-map iff $f^{-1}(d_Y A) = d_X(f^{-1}A)$ for each $A \subseteq Y$.*

Proof. First suppose that $f^{-1}(d_Y A) = d_X(f^{-1}A)$ for each $A \subseteq Y$. Then $f^{-1}(C_Y A) = f^{-1}(A \cup d_Y A) = f^{-1}A \cup f^{-1}(d_Y A) = f^{-1}A \cup d_X(f^{-1}A) = C_X(f^{-1}A)$. So f is interior. Moreover, for each $y \in Y$ we have that $f^{-1}(y) \cap d_X(f^{-1}(y)) = f^{-1}(y) \cap f^{-1}(d_Y(y)) = f^{-1}(\{y\} \cap d_Y(y)) = f^{-1}\emptyset = \emptyset$. Thus, f is pointwise discrete, and so f is a d-map. Now suppose that f is a d-map. Then $x \notin d_X(f^{-1}A)$ implies there exists an open neighborhood U_x of x such that $U_x \cap (f^{-1}A - \{x\}) = \emptyset$. Let $U_{f(x)} = f(U_x)$. Since f is open, $U_{f(x)}$ is an open neighborhood of $f(x)$. Moreover, $U_{f(x)} \cap (A - \{f(x)\}) = \emptyset$. Therefore, $f(x) \notin d_Y A$, and so $x \notin f^{-1}(d_Y A)$. Conversely, if $x \notin f^{-1}(d_Y A)$, then $f(x) \notin d_Y A$, and so there exists an open neighborhood $U_{f(x)}$ of $f(x)$ such that $U_{f(x)} \cap (A - \{f(x)\}) = \emptyset$. As $x \in f^{-1}f(x)$ and $f^{-1}f(x)$ is discrete, there exists an open subset U of X such that $U \cap f^{-1}f(x) = \{x\}$. Let

$U_x = f^{-1}(U_{f(x)}) \cap U$. Since f is continuous, U_x is an open neighborhood of x. Moreover, $U_x \cap (f^{-1}A - \{x\}) = \emptyset$. Thus, $x \notin d_X(f^{-1}A)$. ■

COROLLARY 2.5. *If f is a d-map from X onto Y, then $L_d(X) \subseteq L_d(Y)$.*

PROOF. Suppose $\varphi \notin L_d(Y)$. Then there exists a valuation ν on Y such that $\nu(\varphi) \neq Y$. Define a valuation ν' on X by putting $\nu'(p) = f^{-1}(\nu(p))$ for each propositional letter p. Since f^{-1} commutes with the set-theoretic union, intersection, and complement, and since by Theorem 2.4 f^{-1} commutes with the derived set operator, we have that $\nu'(\varphi) = f^{-1}(\nu(\varphi)) \neq X$ as f is onto. Therefore, $\varphi \notin L_d(X)$. ■

Suppose $\mathfrak{F}$ is a **wK4**-frame and $\overline{\mathfrak{F}}$ is the reflexive closure of $\mathfrak{F}$. Then $\overline{\mathfrak{F}}$ is a **S4**-frame, and we view it as an Alexandroff space. Let $L(\mathfrak{F})$ denote the set of formulas of $\mathcal{L}^*$ valid in $\mathfrak{F}$. Then $L(\mathfrak{F})$ is a normal extension of **wK4**. In fact, $L(\mathfrak{F}) = L_d(\overline{\mathfrak{F}})$ whenever $\mathfrak{F}$ is irreflexive.

DEFINITION 2.6. Suppose X is a topological space, $\mathfrak{F} = \langle W, R\rangle$ is a **wK4**-frame, and $f : X \to W$ is a map. We call f *irreflexively discrete*, or *i-discrete* for short, if $f^{-1}(w)$ is a discrete subspace of X for each irreflexive $w \in W$. We also call f *reflexively dense*, or *r-dense* for short, if $f^{-1}(w) \subseteq d_X f^{-1}(w)$ for each reflexive $w \in W$. Finally, we call f a *d-morphism* if (i) $f : X \to \overline{\mathfrak{F}}$ is interior, (ii) f is i-discrete, and (iii) f is r-dense.

THEOREM 2.7. *Let X be a topological space, $\mathfrak{F} = \langle W, R\rangle$ be a* **wK4**-*frame, and $f : X \to W$ be a map. Then f is a d-morphism iff $f^{-1}(R^{-1}A) = d_X(f^{-1}A)$ for each $A \subseteq W$.*

PROOF. First suppose that $f^{-1}(R^{-1}A) = d_X(f^{-1}A)$ for each $A \subseteq W$. Then $f^{-1}(C_{\overline{\mathfrak{F}}}A) = f^{-1}([\overline{R}]^{-1}A) = f^{-1}(A \cup R^{-1}A) = f^{-1}A \cup f^{-1}(R^{-1}A) = f^{-1}A \cup d_X(f^{-1}A) = C_X(f^{-1}A)$. So f is interior. Moreover, for each irreflexive $w \in W$ we have $f^{-1}(w) \cap d_X(f^{-1}(w)) = f^{-1}(w) \cap f^{-1}(R^{-1}(w)) = f^{-1}(\{w\} \cap R^{-1}(w)) = f^{-1}\emptyset = \emptyset$. Therefore, f is i-discrete. Furthermore, for each reflexive $w \in W$ we have that $f^{-1}(w) \subseteq f^{-1}(R^{-1}(w)) = d_X(f^{-1}(w))$. Thus, f is r-dense, and so f is a d-morphism. Now suppose that f is a d-morphism. Then $x \notin d_X(f^{-1}A)$ implies there exists an open neighborhood U_x of x such that $U_x \cap (f^{-1}A - \{x\}) = \emptyset$. Since f is open, $f(U_x)$ is an upset of $\overline{\mathfrak{F}}$ containing $f(x)$ such that $f(U_x) \cap (A - \{f(x)\}) = \emptyset$. Therefore, $\overline{R}(f(x)) \cap (A - \{f(x)\}) = \emptyset$. This obviously implies that $R(f(x)) \cap A = \emptyset$ if $f(x)$ is irreflexive. However, if $f(x)$ is reflexive, then $f^{-1}f(x) \subseteq d_X(f^{-1}f(x))$, and so $f(x) \in A$ implies that $x \in f^{-1}f(x) \subseteq d_X(f^{-1}f(x)) \subseteq d_X(f^{-1}A)$, which is a contradiction. Therefore, $R(f(x)) \cap A = \emptyset$ in this case too. Thus, $f(x) \notin R^{-1}A$, and so $x \notin f^{-1}(R^{-1}A)$. Conversely, if $x \notin f^{-1}(R^{-1}A)$, then $f(x) \notin R^{-1}A$, and so $R(f(x)) \cap A = \emptyset$. If $f(x)$ is reflexive, then $R(f(x))$ is an upset of $\overline{\mathfrak{F}}$ containing $f(x)$. So, by continuity of f, $U_x = f^{-1}(R(f(x))$ is an open neighborhood of x with $U_x \cap f^{-1}A = \emptyset$, implying that $U_x \cap (f^{-1}A - \{x\}) = \emptyset$. And if $f(x)$ is irreflexive, then $\overline{R}(f(x))$ is an upset of $\overline{\mathfrak{F}}$ containing $f(x)$ such that $\overline{R}(f(x)) \cap (A - \{f(x)\}) = \emptyset$. Let $U_x = f^{-1}(\overline{R}(f(x))$. Again using continuity of f we obtain that U_x is an open neighborhood of x with $U_x \cap (f^{-1}A - f^{-1}f(x)) = \emptyset$. Since $f(x)$ is irreflexive, there exists an open subset U of X such that $U \cap f^{-1}f(x) = \{x\}$. So $U_x \cap U \cap (f^{-1}A - \{x\}) = \emptyset$. Therefore, in this case too, there exists an open neighborhood $O = U_x \cap U$ of x such that $O \cap (f^{-1}A - \{x\}) = \emptyset$. Thus, $x \notin d_X(f^{-1}A)$. ■

As an immediate consequence of Theorem 2.7 we obtain the following result first established by Shehtman [30, Sec. 5].

Corollary 2.8. *If $\mathfrak{F} = \langle W, R\rangle$ is finite, then $f : X \to W$ is a d-morphism iff $d_X f^{-1}(w) = f^{-1}(R^{-1}(w))$ for all $w \in W$.*

Another immediate consequence of Theorem 2.7 is the following result.

Corollary 2.9. *If f is a d-morphism from X onto $\mathfrak{F}$, then $L_d(X) \subseteq L(\mathfrak{F})$.*

Proof. Suppose $\varphi \notin L(\mathfrak{F})$. Then there exists a valuation ν on $\mathfrak{F}$ such that $\nu(\varphi) \neq W$. Define a valuation ν' on X by putting $\nu'(p) = f^{-1}(\nu(p))$ for each propositional letter p. Since f^{-1} commutes with the set-theoretic union, intersection, and complement, and since by Theorem 2.7 $f^{-1}(R^{-1}A) = d_X(f^{-1}A)$ for each $A \subseteq W$, we have that $\nu'(\varphi) = f^{-1}(\nu(\varphi)) \neq X$ as f is onto. Therefore, $\varphi \notin L_d(X)$. ■

Let $\mathcal{N}$ denote the class of nodec spaces; $\mathcal{S}$ the class of submaximal spaces; $\mathcal{D}$ the class of door spaces; $\mathcal{I}$ the class of I-spaces; $\mathcal{PD}$ the class of perfectly disconnected spaces; and $\mathcal{M}$ the class of maximal spaces. For precise definitions see Section 3. Then the axiomatization and definability results of this paper are summarized in Table 1 below.

3. Nodec spaces and their subclasses

3.1. Submaximal and nodec spaces

We recall that a topological space X is *submaximal* if every dense subset of X is open, and that X is *nodec* if every nowhere dense subset of X is closed. Different equivalent conditions for a space to be submaximal are given in

$\mathcal{K}$	$L_C(\mathcal{K})$	$L_d(\mathcal{K})$	C-def.	d-def.
$\mathcal{N}$	$\mathbf{S4} + \Box\Diamond\Box p \to (p \to \Box p)$	$\mathbf{K4} + \dot{\Diamond} p \to \Diamond\Box\Diamond p$	$+$	$+$
$\mathcal{S}$	$\mathbf{S4} + p \to \Box(\Diamond p \to p)$	$\mathbf{K4} + \boxdot(p \to \boxdot p)$	$+$	$+$
$\mathcal{D}$	”	$\mathbf{K4} + \dot{\Diamond}\dot{\Diamond} p \to \boxdot p$	$-$	$-$
$\mathcal{I}$	”	$\mathbf{K4} + \boxdot\boxdot\bot$	$-$	$+$
$\mathcal{PD}$	$\mathbf{S4.2} + p \to \Box(\Diamond p \to p)$	$\mathbf{K4} + \dot{\Diamond} p \to \boxdot p$	$+$	$+$
$\mathcal{M}$	”	$\mathbf{K4} + \dot{\Diamond} p \leftrightarrow \boxdot p$	$-$	$+$

Table 1. List of classes of spaces and the corresponding logics

[5, Theorem 1.2], and the ones for a space to be nodec in [13, Fact 1.14] and [25, Corollary to Proposition 4].[5] In particular, they imply that every submaximal space is nodec. The converse is not true: any trivial topology on a set with more than two elements is nodec, but not submaximal. This example shows that there exist nodec spaces that are not T_0. On the other hand, it is known (see, e.g., [9, Remark 2.6]) that every submaximal space is T_0.[6]

We point out that in Theorem 1.2 of [5], the conditions (d) and (f) require that the space X under consideration be T_1. We remove this restriction by adding an extra condition to both (d) and (f).

THEOREM 3.1. *The following conditions are equivalent:*

1. *X is submaximal.*
2. *$CA - A$ is closed for each $A \subseteq X$.*
3. *For each $A \subseteq X$, if $IA = \emptyset$, then A is closed and discrete.*
4. *$CA - A$ is closed and discrete for each $A \subseteq X$.*

PROOF. (1) $\Leftrightarrow$ (2) is the equivalence $(g) \Leftrightarrow (e)$ of [5, Theorem 1.2].

(2) $\Rightarrow$ (3) Suppose $IA = \emptyset$. Then

$$A^c = A^c \cup IA = A^c \cup (C(A^c))^c = (A \cap C(A^c))^c = (C(A^c) - (A^c))^c$$

is open since $C(A^c) - (A^c)$ is closed. So A is closed. Thus, $dA \subseteq A$. We show that $dA = \emptyset$. Let $x \in A$. Since $IA = \emptyset$, we also have that $I(A - \{x\}) = \emptyset$. Therefore, $A - \{x\}$ is closed, and so $\{x\} \cup A^c$ is open. But then there is an

[5] We point out that in [25] nodec spaces are called α-topologies.

[6] In fact, as follows from Corollary 3.5 below, every submaximal space is T_D.

open neighborhood $U_x = \{x\} \cup A^c$ of x such that $U_x \cap (A - \{x\}) = \emptyset$. Thus, $x \notin dA$. It follows that $dA = \emptyset$. Therefore, A is closed and discrete.

(3) $\Rightarrow$ (4) Since

$$I(CA - A) = ICA \cap I(A^c) = ICA \cap (CA)^c = \emptyset,$$

we have that $CA - A$ is closed and discrete.

(4) $\Rightarrow$ (2) Obvious. ■

To this end, we call closed and discrete sets simply *clods*. We recall that the *Hausdorff residue* $\rho(A)$ of a subset A of a space X is defined as $\rho(A) = A \cap C(CA - A)$.

Lemma 3.2. *For $A \subseteq X$ the following hold.*

1. *$CA - A$ is closed iff $\rho(A) = \emptyset$.*
2. *$CA - A$ is clod iff $d(dA - A) = \emptyset$.*

Proof. (1) If $CA - A$ is closed, then

$$\rho(A) = A \cap C(CA - A) = A \cap (CA - A) = \emptyset.$$

Conversely, if $\rho(A) = \emptyset$, then $C(CA - A) \subseteq A^c$. We also have that $C(CA - A) \subseteq CA$. Therefore, $C(CA - A) \subseteq CA \cap A^c = CA - A$. So $CA - A$ is closed.

(2) Since $CA - A = (A \cup dA) - A = dA - A$, we have that $CA - A$ is clod iff $dA - A$ is clod iff $d(dA - A) = \emptyset$. ■

Corollary 3.3. *The following two conditions are equivalent to the four conditions of Theorem 3.1:*

5. *$\rho(A) = \emptyset$ for each $A \subseteq X$.*
6. *$d(dA - A) = \emptyset$ for each $A \subseteq X$.*

Proof. It follows immediately from Theorem 3.1 and Lemma 3.2. ■

As an immediate consequence of Corollary 3.3 we obtain that the class of submaximal spaces is modally definable in both topological semantics.

Proposition 3.4. *For each space X we have:*

(i)	*X is submaximal*	*iff*	$X \models_C p \to \Box(\Diamond p \to p)$
(ii)	*X is submaximal*	*iff*	$X \models_d \boxdot(p \to \boxdot p)$

Thus, the class of submaximal spaces is both C- and d-definable.

PROOF. Suppose X is an arbitrary topological space and ν is a valuation on X. Denoting $\nu(p)$ by A and using Corollary 3.3, we obtain:

$$
\begin{aligned}
\text{(i)}\quad X \text{ is submaximal} &\Leftrightarrow (\forall A \subseteq X)[\rho(A) = \emptyset] \\
&\Leftrightarrow (\forall A \subseteq X)[A \cap C(CA - A) = \emptyset] \\
&\Leftrightarrow (\forall A \subseteq X)[A^c \cup (C(CA \cap A^c))^c = X] \\
&\Leftrightarrow (\forall A \subseteq X)[A^c \cup I((CA)^c \cup A) = X] \\
&\Leftrightarrow \nu(p \to \Box(\Diamond p \to p)) = X \\
&\Leftrightarrow X \models_C p \to \Box(\Diamond p \to p).
\end{aligned}
$$

$$
\begin{aligned}
\text{(ii)}\quad X \text{ is submaximal} &\Leftrightarrow (\forall A \subseteq X)[d(d(A^c) - (A^c)) = \emptyset] \\
&\Leftrightarrow (\forall A \subseteq X)[t(tA \cup A^c) = X] \\
&\Leftrightarrow \nu(\boxdot(p \to \boxdot p)) = X \\
&\Leftrightarrow X \models_d \boxdot(p \to \boxdot p).
\end{aligned}
$$

Thus, X is submaximal iff $X \models_C p \to \Box(\Diamond p \to p)$ iff $X \models_d \boxdot(p \to \boxdot p)$. ■

Another consequence of Corollary 3.3 worth mentioning is the following result.

COROLLARY 3.5.

1. *If X is submaximal, then X is HI.*

2. *If X is submaximal, then X is T_D.*

PROOF. (1) It follows from [9, Theorem 2.4] that X is HI iff $\rho(A) \subsetneq A$ for each nonempty subset A of X. Now if X is submaximal and A is a nonempty subset of X, then $\rho(A) = \emptyset \subsetneq A$. So X is HI.

(2) Every HI space is T_D. To see this, let $x \in X$. We need to show that x is isolated in $C(x)$. If not, then $C(x) = C(C(x) - \{x\})$. Therefore, $\{x\}$ and $C(x) - \{x\}$ are disjoint dense subsets of $C(x)$, implying that $C(x)$ is resolvable. Now apply (1).[7] ■

The converse of Corollary 3.5 is not true: already any ordinal $\alpha \geq \omega^2 + 1$ is not a submaximal space.

The next proposition, which was implicitly proved in [9, Proposition 3.4], will be instrumental in Section 5 for proving Theorem 5.4.

---

[7]We point out that if X is submaximal, then every point in X is in fact either open or closed. To see this, note that if $x \in X$ is not isolated, then $\{x\}^c$ is dense, so open, hence $\{x\}$ is closed.

PROPOSITION 3.6. *Every T_1 topology τ on X can be extended to a submaximal topology $\overline{\tau}$ on X in such a way that for each $U \in \tau$ we have $dU = \overline{d}U$, where $\overline{d}$ is the derived set operator of $(X, \overline{\tau})$.*

PROOF. Let (X, τ) be an arbitrary T_1 topological space. By [9, Lemma 3.3] there exists a filter $\mathcal{F}$ on X maximal among filters consisting of dense subsets of X. For the reader's convenience, we reproduce this result.

CLAIM 1. *Let X be a topological space. Then there is a filter $\mathcal{F}$ on X maximal among filters consisting of dense sets.*

PROOF. Let S be the set of all filters on X consisting of dense sets. Then S is nonempty since if A is dense, then $\mathcal{F}_A = \{B \in \wp(X) : A \subseteq B\}$ is a filter on X consisting of dense sets. Note that S is partially ordered by inclusion. We wish to apply Zorns lemma to S. To do this, let $\{\mathcal{F}_\alpha\}$ be a chain in S, where each $\mathcal{F}_\alpha$ is a filter consisting of dense sets. It is elementary to see that $\bigcup_\alpha \mathcal{F}_\alpha$ is a filter and consists of dense sets. This union is then an element of S. So by Zorns lemma, there is a maximal element $\mathcal{F}$ of S. ∎

Let $\overline{\tau}$ denote the topology generated by $\tau \cup \mathcal{F}$. Then a basis of $\overline{\tau}$ consists of all finite intersections of elements of $\tau \cup \mathcal{F}$. Since both τ and $\mathcal{F}$ are closed under finite intersections, this basis is $\{U \cap A : U \in \tau, A \in \mathcal{F}\}$. Let $\mathcal{D}(X, \tau)$ denote the set of dense subsets of (X, τ), and $\mathcal{D}(X, \overline{\tau})$ denote the set of dense subsets of $(X, \overline{\tau})$. We show that $\mathcal{D}(X, \overline{\tau}) = \mathcal{F}$. To prove the inclusion $\mathcal{F} \subseteq \mathcal{D}(X, \overline{\tau})$, let $B \in \mathcal{F}$. If $U \in \tau$ is nonempty and $A \in \mathcal{F}$, then $B \cap (U \cap A) = U \cap (A \cap B)$ is nonempty since $A \cap B \in \mathcal{F}$ and $\mathcal{F} \subseteq \mathcal{D}(X, \tau)$. Therefore, $B \in \mathcal{D}(X, \overline{\tau})$. For the converse inclusion, let $B \in \mathcal{D}(X, \overline{\tau})$. Then $B \in \mathcal{D}(X, \tau)$ because $\tau \subseteq \overline{\tau}$. Let $A \in \mathcal{F}$ and $U \in \tau$. Then $A \cap U \in \overline{\tau}$. Since B is dense in $(X, \overline{\tau})$, we have $B \cap (A \cap U) \neq \emptyset$. Thus, $A \cap B$ intersects each nonempty $U \in \tau$ nontrivially. Therefore, $A \cap B \in \mathcal{D}(X, \tau)$. This shows that the filter $\mathcal{F}'$ generated by $\mathcal{F} \cup \{B\}$ is a filter of dense sets. The maximality of $\mathcal{F}$ implies that $\mathcal{F}' = \mathcal{F}$, so $B \in \mathcal{F}$. Therefore, we proved that $\mathcal{D}(X, \overline{\tau}) = \mathcal{F}$. Now since $\mathcal{F} \subseteq \overline{\tau}$, it follows that $(X, \overline{\tau})$ is submaximal.

It is left to be shown that $dU = \overline{d}U$ for each $U \in \tau$. Let $U \in \tau$. Since $\tau \subseteq \overline{\tau}$, we have that $\overline{d}U \subseteq dU$. For the converse inclusion, let $x \in dU$ and let $V \cap A$ be a basic open neighborhood of x in $\overline{\tau}$, where $V \in \tau$ and $A \in \mathcal{F}$. Then V is an open neighborhood of x in τ, and so the set $V \cap (U - \{x\})$ is nonempty and open in τ as $x \in dU$ and τ is T_1. Since A is dense in τ, we obtain that $(V \cap (U - \{x\})) \cap A \neq \emptyset$. Therefore, $(V \cap A) \cap (U - \{x\}) \neq \emptyset$, implying that $x \in \overline{d}U$. ∎

We proceed with a characterization of nodec spaces.

THEOREM 3.7. *The following conditions are equivalent:*

1. *X is nodec.*
2. *Each nowhere dense subset of X is clod.*
3. *For each $A \subseteq X$, if $A \subseteq ICIA$, then A is open.*
4. *For each $A \subseteq X$, if $CICA \subseteq A$, then A is closed.*
5. *$dA \subseteq CICA$ for each $A \subseteq X$.*
6. *$CA = A \cup CICA$ for each $A \subseteq X$.*
7. *$IA = A \cap ICIA$ for each $A \subseteq X$.*

PROOF. For (1)⇔(2) see [13, Fact 1.14], and for (1)⇔(3) see [25, Corollary to Proposition 4].

(3)⇔(4) is obvious.

(2)⇒(5) Let $A \subseteq X$. Since

$$\begin{aligned} IC(A - ICA) &= IC(A \cap (ICA)^c) = IC(A \cap CI(A^c)) \subseteq I(CA \cap CI(A^c)) \\ &= ICA \cap ICI(A^c) = ICA \cap (CICA)^c = ICA - CICA = \emptyset \end{aligned}$$

we have that $A - ICA$ is nowhere dense. Therefore, $A - ICA$ is clod. Thus, $d(A - ICA) = \emptyset$, and as $dICA \subseteq CICA$ and $dA - dB \subseteq d(A - B)$, we have

$$dA - CICA \subseteq dA - dICA \subseteq d(A - ICA) = \emptyset$$

It follows that $dA \subseteq CICA$.

(5)⇒(6) As $A, CICA \subseteq CA$, we have that $A \cup CICA \subseteq CA$. Conversely, $CA = A \cup dA \subseteq A \cup CICA$. Therefore, $CA = A \cup CICA$.

(6)⇔(7) is obvious.

(6)⇒(1) If $N \subseteq X$ is nowhere dense, then $CN = N \cup CICN = N$. So N is closed. ∎

As an immediate consequence of Theorem 3.7 we obtain that the class of nodec spaces is modally definable in both topological semantics.

PROPOSITION 3.8. *For each space X we have:*

(i) *X is nodec* *iff* $X \models_C \Box\Diamond\Box p \to (p \to \Box p)$
(ii) *X is nodec* *iff* $X \models_d \diamond p \to \Diamond\Box\Diamond p$

Thus, the class of nodec spaces is both C- and d-definable.

PROOF. Suppose X is an arbitrary topological space and ν is a valuation on X. Denoting $\nu(p)$ by A and using Theorem 3.7, we obtain:

$$\begin{aligned}
\text{(i)}\quad X \text{ is nodec} &\Leftrightarrow (\forall A \subseteq X)[IA = A \cap ICIA] \\
&\Leftrightarrow (\forall A \subseteq X)[(ICIA \cap A)^c = (IA)^c] \\
&\Leftrightarrow (\forall A \subseteq X)[(ICIA \cap A)^c \cup IA = (IA)^c \cup IA] \\
&\Leftrightarrow (\forall A \subseteq X)[(ICIA)^c \cup A^c \cup IA = X] \\
&\Leftrightarrow \nu(\Box\Diamond\Box p \to (p \to \Box p)) = X \\
&\Leftrightarrow X \models_C \Box\Diamond\Box p \to (p \to \Box p).
\end{aligned}$$

$$\begin{aligned}
\text{(ii)}\quad X \text{ is nodec} &\Leftrightarrow (\forall A \subseteq X)[dA \subseteq CICA] \\
&\Leftrightarrow (\forall A \subseteq X)[(dA)^c \cup CICA = X] \\
&\Leftrightarrow \nu(\Diamonddot p \to \Diamond\Box\Diamond p) = X \\
&\Leftrightarrow X \models_d \Diamonddot p \to \Diamond\Box\Diamond p.
\end{aligned}$$

Thus, X is nodec iff $X \models_C \Box\Diamond\Box p \to (p \to \Box p)$ iff $X \models_d \Diamonddot p \to \Diamond\Box\Diamond p$. ■

3.2. Door spaces and I-spaces

We recall that a space X is *door* if every subset of X is either open or closed. It is obvious that every door space is submaximal. The converse however is not true: the spaces in [9, Proposition 3.4], where the original space is not a door space, are submaximal but not door. To give a characterization of door spaces, for a subset S of X and for a filter $\mathcal{F}$ on X, let $\tau_{S,\mathcal{F}} = \mathcal{F} \cup \{U \subseteq X : U \cap S = \emptyset\}$. It is easy to show that $\tau_{S,\mathcal{F}}$ is a topology on X.

THEOREM 3.9. *A space (X, τ) is a door space iff $\tau = \tau_{S,\mathcal{F}}$ and either S is a singleton or $\mathcal{F}$ is an ultrafilter.*

PROOF. See [31, Theorem 4.4] and [12, the paragraph after Proposition 1.1]. ■

The following lemma will be used in Section 5 for proving Theorem 5.5.

LEMMA 3.10. *If X is a door space, then one of the following holds:*

1. *$ddA = \emptyset$ for each $A \subseteq X$.*
2. *$dA \cap d(A^c) = \emptyset$ for each $A \subseteq X$.*

PROOF. Let X be a door space. By Theorem 3.9 the topology on X is $\tau_{S,\mathcal{F}}$, where S is a singleton or $\mathcal{F}$ is an ultrafilter. First suppose that $S = \{x\}$ for some $x \in X$. Then for each $A \subseteq X$ we have that $dA \subseteq \{x\}$ as each $y \neq x$ is an isolated point. Therefore, $ddA = \emptyset$ for each $A \subseteq X$. Now suppose that $\mathcal{F}$

is an ultrafilter and $A \subseteq X$. Then $A \in \mathcal{F}$ or $A^c \in \mathcal{F}$. If $A \in \mathcal{F}$, then for each $y \in X$, the set $U_y = \{y\} \cup A \in \mathcal{F}$ is an open neighborhood of y such that $U_y \cap (A^c - \{y\}) = \emptyset$. So $d(A^c) = \emptyset$. On the other hand, if $A^c \in \mathcal{F}$, then by a similar argument we obtain that $dA = \emptyset$. Therefore, $dA \cap d(A^c) = \emptyset$. ∎

Now we show that the class of door spaces is neither C- nor d-definable.

PROPOSITION 3.11. *The class of door spaces is neither C- nor d-definable.*

PROOF. The class of door spaces is not closed under topological sums: the Sierpinski space[8] is obviously door, however the sum of two Sierpinski spaces is not door. To conclude the proof it is sufficient to note that topological sums preserve modal validity in both C- and d-semantics. ∎

We recall that a space X is an *I-space* if $ddX = \emptyset$. As an immediate consequence of the definition, we obtain that the class of I-spaces is d-definable.

PROPOSITION 3.12. *For each space X we have:*

$$X \text{ is an I-space} \quad \text{iff} \quad X \models_d \boxdot\boxdot\bot.$$

Thus, the class of I-spaces is d-definable.

On the other hand, it will follow from Section 4 (see Proposition 4.5) that the class of I-spaces is not C-definable.

It is pointed out in [5] that for a space X the following three conditions are equivalent: (i) X is an I-space; (ii) X is nodec and (weakly) scattered; (iii) X is submaximal and (weakly) scattered. Examples of I-spaces that are not door are the ordinals $\alpha \in [\omega 2 + 1, \omega^2]$. For examples of door spaces that are not I-spaces, recall that a space X is *filtral* if the set $\tau - \{\emptyset\}$ of nonempty open subsets of X is a filter. Let X be an infinite filtral space, where $\tau - \{\emptyset\}$ is a free (non-principal) ultrafilter. Then X is a dense-in-itself door space [14], hence is not an I-space. We call X the *El′kin space*.

3.3. Maximal and perfectly disconnected spaces

We recall that a space X is *maximal* if every non-empty open subset of X is infinite and any strictly finer topology on X contains a finite open set. It is known (see, e.g., [21, Theorem 24]) that every maximal space is submaximal. Since maximal spaces are dense-in-itself and I-spaces are (weakly) scattered,

[8]Recall that the Sierpinski space is a two-point space where exactly one of the points is open.

the two classes have the empty intersection. The filtral spaces, where the filter $\tau - \{\emptyset\}$ is a principal ultrafilter, serve as examples of door spaces that are not maximal. For examples of maximal spaces that are not door, we note that it was shown in [21, Theorem 13] (see also [13, Theorem 1.2(b)]) that there exist Hausdorff maximal spaces. We point out that none of them can be door. Indeed, if X is a maximal door space, then as X is dense-in-itself, Theorem 3.9 implies that the topology on X is $\tau_{S,\mathcal{F}}$, where $S = X$ and $\mathcal{F}$ is an ultrafilter. Therefore, any two nonempty opens of X intersect, and so X cannot be Hausdorff.

As follows from Section 4 (see Proposition 4.8), the class of maximal spaces is not C-definable. On the other hand, it follows from Corollary 3.17 below that the class of maximal spaces is d-definable.

Closely related to the notion of maximality is the notion of perfectly disconnected spaces from [13]. We recall that a space X is *perfectly disconnected* if X is T_0 and disjoint subsets of X have no common limit points. Equivalently, X is perfectly disconnected iff X is T_0 and $dA \cap d(A^c) = \emptyset$ for each $A \subseteq X$. It is shown in [13, Theorem 2.2] that if X is dense-in-itself, then X is maximal iff X is perfectly disconnected. It follows that maximal spaces are perfectly disconnected. The Sierpinski space serves as an example of a perfectly disconnected space that is not maximal. Since the class of maximal spaces does not intersect with the class of I-spaces and since there exist maximal spaces that are not door, it follows that there exist perfectly disconnected spaces that are neither door nor I-spaces. The ordinal $\omega + 1$ serves as an example of a door space and an I-space that is not perfectly disconnected. The following is an immediate corollary of Lemma 3.10.

Corollary 3.13. *If X is a door space, then X is either an I-space or a perfectly disconnected space.*

Proposition 3.14. *If X is perfectly disconnected, then X is submaximal.*

Proof. We first show that X is a T_D-space. Suppose not. Then there exists $A \subseteq X$ such that $ddA \not\subseteq dA$. Therefore, there is $x \in ddA$ such that $x \notin dA$. The latter implies that there is an open neighborhood U_x of x such that $U_x \cap (A - \{x\}) = \emptyset$. Consequently, $U_x \subseteq A^c \cup \{x\}$. From $x \in ddA$ it follows that $U_x \cap (dA - \{x\}) \neq \emptyset$. So there is $y \neq x$ such that $y \in U_x \cap dA$. We show that $y \in dA \cap d(A^c)$. That $y \in dA$ follows from the selection of y. To show that $y \in d(A^c)$, we first show that $y \in d(x)$. Let V be an open neighborhood of y. It is sufficient to show that $x \in V$. We set $U = V \cap U_x$. Since $y \in dA$, we have that $U \cap (A - \{y\}) \neq \emptyset$. From $U \subseteq U_x \subseteq A^c \cup \{x\}$ it follows that $U \cap (A - \{y\}) = \{x\}$. Thus, $x \in U \subseteq V$, implying that $y \in d(x)$.

Now since X is T_0 and $y \in d(x)$, there must exist an open neighborhood V_x of x such that $y \notin V_x$. Let V be an open neighborhood of y. We set $O = V \cap U_x \cap V_x$. Since $x \in ddA$ and O is an open neighborhood of x, we have that $O \cap (dA - \{x\}) \neq \emptyset$. So there exists $z \neq x$ such that $z \in O \cap dA$. Since $z \neq x$ and $z \in O \subseteq V \cap U_x \cap V_x \subseteq U_x \subseteq A^c \cup \{x\}$, we have that $z \in A^c$. Clearly $z \neq y$ as $y \notin V_x$. Therefore, $z \in V \cap (A^c - \{y\})$. It follows that for any open neighborhood V of y we have $V \cap (A^c - \{y\}) \neq \emptyset$. This implies that $y \in d(A^c)$. Thus, $y \in dA \cap d(A^c)$, which is a contradiction because X was assumed to be perfectly disconnected. Therefore, $ddA \subseteq dA$ for each $A \subseteq X$, and so X is T_D.

To complete the proof, suppose X is not submaximal. Then by Corollary 3.3 there exists $A \subseteq X$ such that $d(dA - A) \neq \emptyset$. Since d preserves $\subseteq$ and X is T_D, we obtain that $d(dA - A) = d(dA \cap A^c) \subseteq ddA \subseteq dA$ and $d(dA - A) = d(dA \cap A^c) \subseteq d(A^c)$. Therefore, $\emptyset \neq d(dA - A) \subseteq dA \cap d(A^c)$, and so there exists $A \subseteq X$ such that $dA \cap d(A^c) \neq \emptyset$. Thus, X is not perfectly disconnected, which is a contradiction. ∎

Recall that $\mathcal{N}$ denotes the class of nodec spaces; $\mathcal{S}$ the class of submaximal spaces; $\mathcal{D}$ the class of door spaces; $\mathcal{I}$ the class of I-spaces; $\mathcal{PD}$ the class of perfectly disconnected spaces; and $\mathcal{M}$ the class of maximal spaces. Then it follows from the above that we have the following relationship between these classes of spaces (arrows denote strict set inclusion):

$$\begin{tikzcd}
 & \mathcal{D} \arrow[dr] & & & \\
\mathcal{M} \arrow[r] & \mathcal{PD} \arrow[r] & \mathcal{PD} \cup \mathcal{I} \arrow[r] & \mathcal{S} \arrow[r] & \mathcal{N} \\
 & \mathcal{I} \arrow[ur] & & &
\end{tikzcd}$$

We conclude this section by a characterization of perfectly disconnected spaces in terms of extremally disconnected spaces, which, in view of [13, Theorem 2.2], can be thought of as a generalization of the characterization of maximal spaces given in [22, 23].

Theorem 3.15. *A space X is perfectly disconnected iff X is submaximal and extremally disconnected.*

Proof. That perfectly disconnected spaces are submaximal follows from Proposition 3.14; that they are extremally disconnected follows from [15, Theorem 6.2.26]. Conversely, suppose that X is submaximal and extremally disconnected. Because X is submaximal, X is T_0. Moreover, for $A \subseteq X$, we have $d(A - IA) = d(A \cap (IA)^c) = d(A \cap C(A^c)) = d(C(A^c) - (A^c)) = d(d(A^c) - (A^c)) = \emptyset$ by Corollary 3.3. Therefore, $dA = d((A - IA) \cup IA) = d(A - IA) \cup dIA = dIA$. Similarly, $d(A^c - I(A^c)) = \emptyset$ and so $d(A^c) = dI(A^c)$. Thus, $dA \cap d(A^c) = dIA \cap dI(A^c) = \emptyset$ as X is extremally disconnected. It follows that X is perfectly disconnected. ∎

As an immediate consequence of Theorem 3.15 we obtain that the class of perfectly disconnected spaces is modally definable in both topological semantics.

Proposition 3.16. *The class of perfectly disconnected spaces is both C- and d-definable.*

Proof. Since both submaximal and extremally disconnected spaces are C-definable (see Proposition 3.4 and [19, Theorem 1.3.3]), it follows from Theorem 3.15 that the class of perfectly disconnected spaces is C-definable. Now it easily follows from Lemma 2.1 that every C-definable class is also d-definable. ∎

Corollary 3.17. *The class of maximal spaces is d-definable.*

Proof. By Proposition 3.16 the class of perfectly disconnected spaces is d-definable, and by [24, Pages 183–184] the class of dense-in-itself spaces is also d-definable. To finish the proof, we recall that X is maximal iff X is dense-in-itself perfectly disconnected [13, Theorem 2.2]. ∎

4. C-axiomatization

In this section we axiomatize the C-logics of the six classes of spaces described in Section 3. In particular, we show that

- $L_C(\mathcal{N}) = \mathbf{S4.Zem} = \mathbf{S4} + \Box\Diamond\Box p \to (p \to \Box p)$
- $L_C(\mathcal{S}) = L_C(\mathcal{I}) = L_C(\mathcal{D}) = \mathbf{S4} + p \to \Box(\Diamond p \to p)$
- $L_C(\mathcal{PD}) = L_C(\mathcal{M}) = \mathbf{S4.2} + p \to \Box(\Diamond p \to p)$

We already have the soundness part from the earlier C-definability results. As to the C-completeness of these logics, we establish it via their Kripke completeness. It is not difficult to notice that the three logics mentioned above are of depth ≤ 2. Thus, by Segerberg's theorem (see, e.g., [11, Theorem 8.85]), they have the finite model property, and hence are

Kripke complete. Therefore, using the correspondence between **S4**-frames and Alexandroff spaces, these logics are C-complete. In what follows, we take a deeper look at the Kripke semantics for these logics by examining the intersections of the six classes of spaces under consideration with the class of Alexandroff spaces. The result, being of an independent topological interest by itself, will enable us to prove all the equalities in the itemized list above.

Let X be an Alexandroff space and R be the specialization order on X. Then the opens of X are exactly the upsets of $\langle X, R\rangle$. We call $\langle X, R\rangle$ *rooted* if there exists $r \in X$ such that rRx for each $x \in X$. If this is the case, r is called a *root* of $\langle X, R\rangle$. We call $x \in X$ *maximal* if xRy implies $x = y$, and *quasi-maximal* if xRy implies yRx; similarly, $x \in X$ is called *minimal* if yRx implies $y = x$, and *quasi-minimal* if yRx implies xRy. Let $maxX$ and $qmaxX$ denote the sets of maximal and quasi-maximal points, and $minX$ and $qminX$ the sets of minimal and quasi-minimal points of X. If R is a partial order, it is obvious that $maxX = qmaxX$ and $minX = qminX$. We call $Y \subseteq X$ a *quasi-chain* if for every $x, y \in Y$ we have that xRy or yRx. If in addition xRy and $x \neq y$ imply $y\not R x$, then Y is called a *chain*. Again the two notions coincide if R is a partial order. A chain Y of X is said to be of *length* n if it consists of n elements. We say that $\langle X, R\rangle$ is of *depth* n if there exists a chain in X of length n and every other chain in X is of length $\leq n$.

Proposition 4.1. *Let X be an Alexandroff space with the specialization order R.*

1. *X is nodec iff $\langle X, R\rangle$ is of depth ≤ 2 and $qminX - qmaxX \subseteq minX$.*
2. *X is submaximal iff X is an I-space iff $\langle X, R\rangle$ is a partially ordered set of depth ≤ 2.*
3. *X is door iff $\langle X, R\rangle$ is a partially ordered set of depth ≤ 2 such that either $maxX - minX$ or $minX - maxX$ consists of at most one point.*

Proof. (1) Suppose X is nodec and there is a chain $Y \subseteq X$ of length > 2. Let $xRyRz$ be three distinct elements from Y. Then $z\not R y$ and $y\not R x$. So $z \notin R^{-1}(y)$, $qmaxX \cap R^{-1}(y) = \emptyset$, and so $\{y\}$ is nowhere dense. However, it is not a downset as $x \notin \{y\}$, and hence $\{y\}$ is not closed, contradicting to X being nodec. Therefore, $\langle X, R\rangle$ is of depth ≤ 2. Similarly, if there exist $x, y \in qminX - qmaxX$ such that xRy and yRx, then $\{y\}$ is nowhere dense but not closed, which is again a contradiction. Thus, $qminX - qmaxX \subseteq minX$. Conversely, suppose $\langle X, R\rangle$ is of depth ≤ 2 and $qminX - qmaxX \subseteq minX$. Then $N \subseteq X$ is nowhere dense iff $N \cap qmaxX = \emptyset$. Therefore, if N is nowhere dense, then $N \subseteq minX$, which implies that N is a downset, hence closed.

(2) If X is an I-space, then X is submaximal. If X is submaximal, then X is nodec, so (1) implies that $\langle X, R\rangle$ is of depth ≤ 2. Also, since submaximal spaces are T_0, $\langle X, R\rangle$ is a partially ordered set. Suppose $\langle X, R\rangle$ is a partially ordered set of depth ≤ 2. Since $maxX$ is the set of isolated points of X, we have that $ddX = d(X - maxX) = \emptyset$. Therefore, X is an I-space.

(3) Suppose X is door. Then X is submaximal and (2) implies that $\langle X, R\rangle$ is a partially ordered set of depth ≤ 2. If both $maxX - minX$ and $minX - maxX$ consist of at least two points, then either there exist $x \in maxX - minX$ and $y \in minX - maxX$ such that $y \not R x$ or all points in $minX - maxX$ are R-related to all points in $maxX - minX$. In either case, $\{x, y\}$ is neither an upset nor a downset. Hence, $\{x, y\}$ is neither open nor closed, which contradicts to X being door. Conversely, if $\langle X, R\rangle$ is a partially ordered set of depth ≤ 2 such that either $maxX - minX$ or $minX - maxX$ consists of at most one point, then as $X = maxX \cup minX$, it follows that X is door. ■

As an immediate consequence we obtain the following result.

Corollary 4.2. *If X is an Alexandroff space with the specialization order R such that $\langle X, R\rangle$ is rooted, then the following conditions are equivalent.*

1. *X is submaximal.*
2. *X is an I-space.*
3. *X is door.*
4. *$\langle X, R\rangle$ is a partially ordered set of depth ≤ 2.*

Proposition 4.3. *Let X be an Alexandroff space with the specialization order R.*

1. *X is perfectly disconnected iff $\langle X, R\rangle$ is a partially ordered set of depth ≤ 2 such that $(\forall x, y, z \in X)((xRy \wedge xRz) \to (\exists u \in X)(yRu \wedge zRu))$.*
2. *X is not maximal.*

Proof. (1) We recall that an Alexandroff space is extremally disconnected iff $(\forall x, y, z \in X)((xRy \wedge xRz) \to (\exists u \in X)(yRu \wedge zRu))$ [19, Theorem 1.3.3]. Now using Theorem 3.15 and Proposition 4.1 we obtain that X is perfectly disconnected iff X is submaximal and extremally disconnected iff

$\langle X, R\rangle$ is a partially ordered set of depth ≤ 2 such that $(\forall x, y, z \in X)((xRy \wedge xRz) \rightarrow (\exists u \in X)(yRu \wedge zRu))$.

(2) Suppose X is a maximal Alexandroff space. Then X is submaximal. So $\langle X, R\rangle$ is a partially ordered set of depth ≤ 2. Therefore, $maxX \neq \emptyset$. Thus, X has isolated points, contradicting to maximality of X. ■

Now we are in a position to show that the C-logics of submaximal spaces, door spaces, and I-spaces coincide.

THEOREM 4.4. $L_C(\mathcal{S}) = L_C(\mathcal{D}) = L_C(\mathcal{I}) = \mathbf{S4} + p \rightarrow \Box(\Diamond p \rightarrow p)$.

PROOF. Since $\mathcal{D}, \mathcal{I} \subseteq \mathcal{S}$, we have that $L_C(\mathcal{S}) \subseteq L_C(\mathcal{D}), L_C(\mathcal{I})$. By Proposition 3.4 we have $\mathbf{S4}+p \rightarrow \Box(\Diamond p \rightarrow p) \subseteq L_C(\mathcal{S}) \subseteq L_C(\mathcal{D}), L_C(\mathcal{I})$. Now since $\mathbf{S4} + p \rightarrow \Box(\Diamond p \rightarrow p)$ is complete with respect to all finite rooted partial orders of depth 2 (see, e.g., [29, 18, 11]), it follows from Corollary 4.2 that $\mathbf{S4} + p \rightarrow \Box(\Diamond p \rightarrow p) = L_C(\mathcal{D}) = L_C(\mathcal{I}) = L_C(\mathcal{S})$. ■

As an immediate consequence of Proposition 3.4, Theorem 4.4, and the fact that $\mathcal{D}, \mathcal{I}$ are strictly contained in $\mathcal{S}$, we obtain that neither the class of door spaces, nor the class of I-spaces is C-definable. Below we only state it for I-spaces since the analogous statement for door spaces we already mentioned in Proposition 3.11.

PROPOSITION 4.5. *The class of I-spaces is not C-definable.*

THEOREM 4.6. $L_C(\mathcal{N}) = \mathbf{S4.Zem}$.

PROOF. By Proposition 3.8 we have that $\mathbf{S4.Zem} \subseteq L_C(\mathcal{N})$. To show the converse, recall from [29, Theorem 7.5] that **S4.Zem** is complete with respect to all finite rooted **S4**-frames of depth 2 with a unique root. Since by Proposition 4.1(1) these are nodec spaces, we obtain $L_C(\mathcal{N}) \subseteq \mathbf{S4.Zem}$, thus the equality. ■

It follows that the C-logic of nodec spaces is **S4.Zem**, and that the C-logic of submaximal spaces is $\mathbf{S4} + p \rightarrow \Box(\Diamond p \rightarrow p)$, which is one of the five pre-tabular extensions of **S4** described in [18]. Moreover, $\mathbf{S4}+p \rightarrow \Box(\Diamond p \rightarrow p)$ coincides with the C-logics of door spaces and I-spaces, and is a proper normal extension of **S4.Zem**.

Next we show that the C-logics of perfectly disconnected and maximal spaces coincide and are equal to $\mathbf{S4.2} + p \rightarrow \Box(\Diamond p \rightarrow p)$. We point out that since the two element chain is the only frame among the rooted partially ordered frames of depth 2 that validate $\Diamond\Box p \rightarrow \Box\Diamond p$, and since $\mathbf{S4.2} + p \rightarrow \Box(\Diamond p \rightarrow p)$ is tabular, it is the logic of the two element chain.

THEOREM 4.7. $L_C(\mathcal{PD}) = L_C(\mathcal{M}) = \mathbf{S4.2} + p \to \Box(\Diamond p \to p)$

PROOF. As $\mathcal{M} \subseteq \mathcal{PD} \subseteq \mathcal{S}$, we have that $\mathbf{S4} + p \to \Box(\Diamond p \to p) = L_C(\mathcal{S}) \subseteq L_C(\mathcal{PD}) \subseteq L_C(\mathcal{M})$. Since $\Diamond\Box p \to \Box\Diamond p$ is valid in X iff X is extremally disconnected and since perfectly disconnected spaces are extremally disconnected, $\mathbf{S4.2} + p \to \Box(\Diamond p \to p) \subseteq L_C(\mathcal{PD})$. Moreover, as $\mathbf{S4.2} + p \to \Box(\Diamond p \to p)$ is the logic of the two element chain, i.e. the logic of the Sierpinski space, which is perfectly disconnected, then $\mathbf{S4.2} + p \to \Box(\Diamond p \to p) = L_C(\mathcal{PD}) \subseteq L_C(\mathcal{M})$.

To show that $L_C(\mathcal{M}) \subseteq \mathbf{S4.2} + p \to \Box(\Diamond p \to p)$, it is sufficient to show that the Sierpinski space is an interior image of any maximal space, and recall from [8, 19] that if Y is an interior image of X, then $L_C(X) \subseteq L_C(Y)$. Let X be a maximal space. To construct an interior map from X onto the Sierpinski space $S = \{u, v\}$, where $\{u\}$ is open and $\{v\}$ is closed, pick any $x \in X$ and set

$$f(y) = \begin{cases} v, & \text{if } y = x \\ u, & \text{otherwise} \end{cases}$$

It follows from the definition of f that it is a well-defined onto map. Since maximal spaces are T_1 and dense-in-itself, it is immediate that f is continuous and open. Therefore, $L_C(\mathcal{M}) = L_C(\mathcal{PD}) = \mathbf{S4.2} + p \to \Box(\Diamond p \to p)$. ■

As an immediate consequence of Proposition 3.16, Theorem 4.7, and the fact that $\mathcal{PD}$ strictly contains $\mathcal{M}$, we obtain the following result.

PROPOSITION 4.8. *The class of maximal spaces is not C-definable.*

REMARK 4.9. Since both **S4.Zem** and $\mathbf{S4} + p \to \Box(\Diamond p \to p)$ have the same superintuitionistic companion, viz. the superintuitionistic logic of all finite rooted partially ordered sets of depth 2, the superintuitionistic logics of nodec, submaximal, door, and I-spaces coincide, and can be axiomatized by adding the formula $q \vee (q \to (p \vee \neg p))$ to the intuitionistic propositional logic **Int**. The obtained logic is the least logic of the second slice of Hosoi, and is one of the three pre-tabular superintuitionistic logics.

Similarly, the superintuitionistic logics of perfectly disconnected and maximal spaces coincide with the superintuitionistic logic of the two element chain, and can be axiomatized by adding the formulas $q \vee (q \to (p \vee \neg p))$ and $(p \to q) \vee (q \to p)$ to **Int**. This logic is the greatest logic of the second slice of Hosoi, and can be alternatively axiomatized by adding the formula $(p \to q) \vee (q \to r) \vee (r \to s)$ to **Int**.

5. d-axiomatization

In this section we axiomatize the d-logics of the six classes of spaces considered in the paper. In fact, we show that

1. $L_d(\mathcal{N}) = \mathbf{wK4} + \dot{\Diamond} p \to \Diamond\Box\Diamond p$
2. $L_d(\mathcal{S}) = \mathbf{K4} + \boxdot(p \to \boxdot p)$
3. $L_d(\mathcal{D}) = \mathbf{K4} + \dot{\Diamond}\dot{\Diamond} p \to \boxdot p$
4. $L_d(\mathcal{I}) = \mathbf{K4} + \boxdot\boxdot\bot = \mathbf{GL} + \boxdot(p \to \boxdot p)$
5. $L_d(\mathcal{PD}) = \mathbf{K4} + \dot{\Diamond} p \to \boxdot p$
6. $L_d(\mathcal{M}) = \mathbf{K4} + \dot{\Diamond} p \leftrightarrow \boxdot p$

Here we recall that $\Box\varphi$ is an abbreviation of $\varphi \wedge \boxdot\varphi$, and that $\Diamond\varphi$ is an abbreviation of $\varphi \vee \dot{\Diamond}\varphi$.

Our strategy here is similar, but more involved than the one we employed in Section 4. We first determine Kripke frames of all six logics under consideration, and prove that each of these logics is Kripke complete. After that, we use the technique of d-morphisms developed in Section 2, to map appropriate topological spaces onto appropriate Kripke frames. This will allow us to establish the desired d-completeness of the logics under consideration.

Let $\mathfrak{F} = \langle W, R\rangle$ be a $\mathbf{wK4}$-frame. The notions of maximal, quasi-maximal, minimal, and quasi-minimal points, as well as the notion of the R-depth of $\mathfrak{F}$ are the same as in the case of $\mathbf{S4}$-frames. We say that $r \in W$ is a *root* of $\mathfrak{F}$ if rRw for each $r \neq w \in W$.

Proposition 5.1. *Let $\mathfrak{F} = \langle W, R\rangle$ be a rooted* $\mathbf{wK4}$*-frame with r a root.*

1. *$\mathfrak{F} \models \dot{\Diamond} p \to \Diamond\Box\Diamond p$ iff (i) $\mathfrak{F}$ is of depth ≤ 2, and (ii) if $r \notin qmax\mathfrak{F}$, then $w \not R r$ for $\forall w \in W$.*
2. *$\mathfrak{F} \models \boxdot(p \to \boxdot p)$ iff (i) $\overline{\mathfrak{F}}$ is a partially ordered set of depth ≤ 2, and (ii) if $r \notin max\mathfrak{F}$, then r is irreflexive.*
3. *$\mathfrak{F} \models \dot{\Diamond}\dot{\Diamond} p \to \boxdot p$ iff (i) $\overline{\mathfrak{F}}$ is a partially ordered set of depth ≤ 2, (ii) if $r \notin max\mathfrak{F}$, then r is irreflexive, and (iii) if $\overline{\mathfrak{F}}$ is not linearly ordered, then $\mathfrak{F}$ is irreflexive.*
4. *$\mathfrak{F} \models \boxdot\boxdot\bot$ iff (i) $\overline{\mathfrak{F}}$ is a partially ordered set of depth ≤ 2, and (ii) $\mathfrak{F}$ is irreflexive.*

5. *$\mathfrak{F} \models \diamond p \to \boxdot p$ iff (i) $\overline{\mathfrak{F}}$ is a linearly ordered set of depth ≤ 2, and (ii) if $r \notin max\mathfrak{F}$, then r is irreflexive.*

6. *$\mathfrak{F} \models \diamond p \leftrightarrow \boxdot p$ iff (i) $\overline{\mathfrak{F}}$ is a linearly ordered set of depth ≤ 2, (ii) if $r \notin max\mathfrak{F}$, then r is irreflexive, and (iii) if $r \in max\mathfrak{F}$, then r is reflexive.*

Proof. (1) First suppose that the depth of $\mathfrak{F}$ is > 2. Then there exist $w, v \in W$ such that $rRwRv$, $v\not{R}w$, and $w\not{R}r$. Therefore, by setting $\nu(p) = \{w\}$, we obtain that $r \models \diamond p$ and $r \not\models \Diamond\Box\Diamond p$. Now suppose that $r \notin qmax\mathfrak{F}$ and yet there exists $w \in W$ such that wRr (note that w may be r). Then again setting $\nu(p) = \{w\}$ gives us that $r \models \diamond p$ and $r \not\models \Diamond\Box\Diamond p$. Conversely, suppose that $\mathfrak{F}$ is of depth ≤ 2, and $r \notin qmax\mathfrak{F}$ implies $w\not{R}r$ for $\forall w \in W$. Then if $r \models \diamond p$, there exists $w \in W$ such that rRw and $w \models p$. Moreover, for any $v \in W$ we have that $w\overline{R}v$ implies $v\overline{R}w$. Therefore, $r \models \Diamond\Box\Diamond p$.

(2) If there exist $w, v \in W$ such that $rRwRv$, $v\not{R}w$, and $w\not{R}r$, then by setting $\nu(p) = \{w\}$, we obtain $r \not\models \boxdot(p \to \boxdot p)$. If there exist $w, v \in W$ such that wRv and vRw, then again setting $\nu(p) = \{w\}$ gives us $r \not\models \boxdot(p \to \boxdot p)$. Finally, if $r \notin max\mathfrak{F}$ and yet r is reflexive, then setting $\nu(p) = \{r\}$ gives us $r \not\models \boxdot(p \to \boxdot p)$. Conversely, suppose $\overline{\mathfrak{F}}$ is a partially ordered set of depth ≤ 2, and $r \notin max\mathfrak{F}$ implies that r is irreflexive. Then rRw implies $R(w) = \{w\}$ or $R(w) = \emptyset$. In either case we have that $w \models p$ implies $w \models \boxdot p$. So $r \models \boxdot(p \to \boxdot p)$.

(3) If either there exist $w, v \in W$ such that $rRwRv$, $v\not{R}w$, and $w\not{R}r$, or there exist $w, v \in W$ such that wRv and vRw, by setting $\nu(p) = \{v\}$, we obtain that $r \models \diamond\diamond p$ but $r \not\models \boxdot p$. If $r \notin max\mathfrak{F}$ and yet rRr, by setting $\nu(p) = \{r\}$, we obtain that $r \models \diamond\diamond p$ but $r \not\models \boxdot p$. Finally, if there exist $w, v \in W$ such that $w \neq v$, rRw, rRv, $w\not{R}v$, $v\not{R}w$, and one of the two points, say w, is reflexive, then by setting $\nu(p) = \{w\}$, we obtain that $r \models \diamond\diamond p$ but $r \not\models \boxdot p$. Conversely, if $\overline{\mathfrak{F}}$ is a partially ordered set of depth ≤ 2, $r \notin max\mathfrak{F}$ implies r is irreflexive, and $\overline{\mathfrak{F}}$ not linearly ordered implies that $\mathfrak{F}$ is irreflexive, then from $r \models \diamond\diamond p$ it follows that there exist $w, v \in W$ such that $rRwRv \models p$. Therefore, $w = v$, and so w is reflexive with $w \models p$. Thus, $W = \{r, w\}$ and $r \models \boxdot p$.

(4) $\mathfrak{F} \models \boxdot\boxdot\bot$ iff $(\forall w, v \in W)(wRv \to R(v){=}\emptyset)$ iff $\overline{\mathfrak{F}}$ is a partially ordered set of depth ≤ 2, and $\mathfrak{F}$ is irreflexive.

(5) $\mathfrak{F} \models \diamond p \to \boxdot p$ iff $(\forall w, v, u \in W)((wRv \wedge wRu) \to v{=}u)$ iff $\overline{\mathfrak{F}}$ is a linearly ordered set of depth ≤ 2, and if $r \notin max\mathfrak{F}$, then r is irreflexive.

(6) $\mathfrak{F} \models \diamond p \leftrightarrow \boxdot p$ iff $(\forall w, v, u \in W)((wRv \wedge wRu) \to v{=}u)$ and $(\forall w, v \in W)(wRv \to vRv)$ iff $\overline{\mathfrak{F}}$ is a linearly ordered set of depth ≤ 2, if $r \notin max\mathfrak{F}$, then r is irreflexive, and if $r \in max\mathfrak{F}$, then r is reflexive. ■

Corollary 5.2. *1.* $\mathbf{wK4} + \Diamonddot p \to \Diamond\Box\Diamond p$ *is the logic of finite rooted* $\mathbf{wK4}$*-frames* $\mathfrak{F}$ *of depth* ≤ 2 *such that* $r \notin qmax\mathfrak{F}$ *implies* $w\not{R}r$ *for* $\forall w \in W$*.*

2. $\mathbf{K4} + \boxdot(p \to \boxdot p)$ *is the logic of finite rooted* $\mathbf{K4}$*-frames* $\mathfrak{F}$ *such that* $\overline{\mathfrak{F}}$ *is a partially ordered set of depth* ≤ 2*, and if* $r \notin max\mathfrak{F}$*, then* r *is irreflexive.*

3. $\mathbf{K4} + \Diamonddot\Diamonddot p \to \boxdot p$ *is the logic of finite rooted* $\mathbf{K4}$*-frames* $\mathfrak{F}$ *such that* $\overline{\mathfrak{F}}$ *is a partially ordered set of depth* ≤ 2*, if* $r \notin max\mathfrak{F}$*, then* r *is irreflexive, and if* $\overline{\mathfrak{F}}$ *is not linearly ordered, then* $\mathfrak{F}$ *is irreflexive.*

4. $\mathbf{K4} + \boxdot\boxdot\bot = \mathbf{GL} + \boxdot(p \to \boxdot p)$ *and is the logic of finite rooted* $\mathbf{K4}$*-frames* $\mathfrak{F}$ *such that* $\overline{\mathfrak{F}}$ *is a partially ordered set of depth* ≤ 2*, and* $\mathfrak{F}$ *is irreflexive.*

5. $\mathbf{K4} + \Diamonddot p \to \boxdot p$ *is the logic of finite rooted* $\mathbf{K4}$*-frames* $\mathfrak{F}$ *such that* $\overline{\mathfrak{F}}$ *is a linearly ordered set of depth* ≤ 2*, and if* $r \notin max\mathfrak{F}$*, then* r *is irreflexive.*

6. $\mathbf{K4} + \Diamonddot p \leftrightarrow \boxdot p$ *is the logic of finite rooted* $\mathbf{K4}$*-frames* $\mathfrak{F}$ *such that* $\overline{\mathfrak{F}}$ *is a linearly ordered set of depth* ≤ 2*, if* $r \notin max\mathfrak{F}$*, then* r *is irreflexive, and if* $r \in max\mathfrak{F}$*, then* r *is reflexive.*

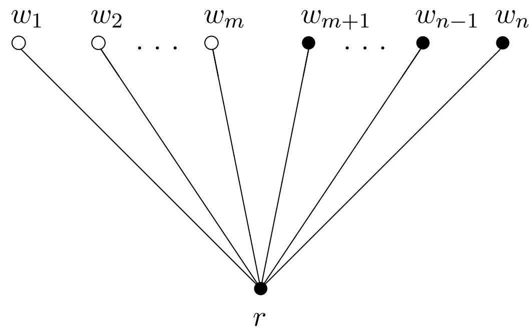
Proof. Since every logic under consideration except $\mathbf{wK4} + \Diamonddot p \to \Diamond\Box\Diamond p$ is a normal extension of $\mathbf{K4}$ of depth 2, it follows from [11, Theorem 8.85] that they all have the finite model property, hence are logics of their finite rooted frames. As $\mathbf{wK4} + \Diamonddot p \to \Diamond\Box\Diamond p$ is a normal extension of $\mathbf{wK4}$ of depth 2, its finite model property follows from an adaptation of [11, Theorem 8.85] to logics of finite depth over $\mathbf{wK4}$. Now apply Proposition 5.1. ∎

Theorem 5.3. $\mathbf{wK4} + \Diamonddot p \to \Diamond\Box\Diamond p = L_d(\mathcal{N})$*.*

Proof. We have $\mathbf{wK4} + \Diamonddot p \to \Diamond\Box\Diamond p \subseteq L_d(\mathcal{N})$ by Proposition 3.8. Conversely, we know from Corollary 5.2(1) that $\mathbf{wK4} + \Diamonddot p \to \Diamond\Box\Diamond p$ is complete with respect to finite rooted $\mathbf{wK4}$-frames $\mathfrak{F}$ of depth ≤ 2 such that $r \notin qmax\mathfrak{F}$ implies $w\not{R}r$ for $\forall w \in W$. Since each one of these is a p-morphic image of an irreflexive frame of the same kind (just substitute each reflexive point by the two-point irreflexive 'cluster'), $\mathbf{wK4} + \Diamonddot p \to \Diamond\Box\Diamond p$ is complete with respect to irreflexive rooted frames of depth ≤ 2 such that $r \notin qmax\mathfrak{F}$ implies $w\not{R}r$ for $\forall w \in W$. Since each one of these gives rise to a nodec space, we obtain that $L_d(\mathcal{N}) \subseteq \mathbf{wK4} + \Diamonddot p \to \Diamond\Box\Diamond p$, thus the equality. ∎

THEOREM 5.4. $\mathbf{K4} + \boxdot(p \to \boxdot p) = L_d(\mathcal{S})$.

PROOF. By Proposition 3.4 we obtain $\mathbf{K4} + \boxdot(p \to \boxdot p) \subseteq L_d(\mathcal{S})$. Conversely, we know from Corollary 5.2(2) that $\mathbf{K4} + \boxdot(p \to \boxdot p)$ is complete with respect to finite rooted $\mathbf{K4}$-frames $\mathfrak{F}$ such that $\overline{\mathfrak{F}}$ is a partially ordered set of depth ≤ 2, and if $r \notin max\mathfrak{F}$, then r is irreflexive. Among these frames, the irreflexive ones give rise to submaximal spaces. Suppose $\mathfrak{F}$ is not irreflexive. Then either $\mathfrak{F}$ consists of a single reflexive point, or $\mathfrak{F}$ is of the form:



Here and below we denote reflexive points by empty circles, irreflexive points by bullets, and we draw lines between points to represent the relation going from bottom up. For each of these $\mathfrak{F}$ we construct a submaximal space which can be d-morphically mapped onto $\mathfrak{F}$. If $\mathfrak{F} = \circ$, then we take any dense-in-itself submaximal space X and define $f : X \to \circ$ by sending every point of X to the reflexive root. Since X is dense-in-itself, it is obvious that f is an onto d-morphism. Suppose $\mathfrak{F}$ is of depth 2, r is the irreflexive root, $max\mathfrak{F} = \{w_1, \ldots, w_n\}$, and the first $m \leq n$ points of $max\mathfrak{F}$ are reflexive. Consider the real plane $\mathbb{R}^2$. For $1 \leq a \leq m$ let

$$D_a = \{(x, y) \in \mathbb{R}^2 - \{(0,0)\} \, : \, y = ax\}$$

denote the line with the slope a and without the origin $(0,0)$; also for $m+1 \leq b \leq n$ let

$$I_b = \left\{ \left(\frac{1}{k}, \frac{b}{k} \right) \, : \, k \in \mathbb{N}, \, k \geq 1 \right\}$$

denote the collection of pairs of positive rational numbers that lay on D_b. Note that all I_b have $(0,0)$ as their sole limit point. We let

$$S = \{(0,0)\} \cup \bigcup_{1 \leq a \leq m} D_a \cup \bigcup_{m+1 \leq b \leq n} I_b$$

and view S as a subspace of $\mathbb{R}^2$ with the subspace topology τ. By Proposition 3.6, τ can be extended to $\overline{\tau}$ so that $\overline{S} = (S, \overline{\tau})$ is a submaximal space with $dU = \overline{d}U$ for each $U \in \tau$. We define $f : S \to \mathfrak{F}$ by putting

$$\begin{array}{ll} f(D_a) = w_a & 1 \leq a \leq m \\ f(I_b) = w_b & m < b \leq n \\ f(0,0) = r & \end{array}$$

Claim 2. *f is an onto d-morphism.*

Proof. That f is onto is obvious. We prove that f is a d-morphism. Since $\mathfrak{F}$ is finite, by Corollary 2.8 it is sufficient to show that

$$(\forall w \in W)(\overline{d}f^{-1}(w) = f^{-1}(R^{-1}(w)))$$

First suppose that $w = r$ is the root. Then $\overline{d}f^{-1}(w) = \overline{d}(0,0) = \emptyset = f^{-1}(\emptyset) = f^{-1}(R^{-1}(w))$. Next suppose that $w = w_a$ for $1 \leq a \leq m$. Then $\overline{d}f^{-1}(w_a) = \overline{d}(D_a) = d(D_a)$ by Proposition 3.6 as D_a is open in S. Therefore, $\overline{d}f^{-1}(w_a) = d(D_a) = \{(0,0)\} \cup D_a$. On the other hand, $f^{-1}(R^{-1}(w_a)) = f^{-1}(\{r, w_a\}) = \{(0,0)\} \cup D_a$. Thus the equality. Finally, suppose that $w = w_b$ for $m < b \leq n$. Then $\overline{d}f^{-1}(w_b) = \overline{d}(I_b) = d(I_b)$ by Proposition 3.6 as I_b is open in S. Therefore, $\overline{d}f^{-1}(w_b) = d(I_b) = \{(0,0)\}$. On the other hand, $f^{-1}(R^{-1}(w_b)) = f^{-1}(\{r\}) = \{(0,0)\}$. Thus the equality. It follows that $\overline{d}f^{-1}(w) = f^{-1}(R^{-1}(w))$ for each $w \in W$, so f is a d-morphism. ■

As an immediate consequence of Claim 2 and Corollary 2.9 we obtain that $L_d(\mathcal{S}) \subseteq \mathbf{K4} + \boxdot(p \to \boxdot p)$. Therefore, $\mathbf{K4} + \boxdot(p \to \boxdot p) = L_d(\mathcal{S})$. ■

Theorem 5.5. $\mathbf{K4} + \Diamonddot\Diamonddot p \to \boxdot p = L_d(\mathcal{D})$.

Proof. Let X be a door space and ν be a valuation on X. If $\nu(p) = A$, then $\nu(\Diamonddot\Diamonddot p \to \boxdot p) = (ddA)^c \cup tA = (ddA)^c \cup (d(A^c))^c = (ddA \cap d(A^c))^c = X$ by Lemma 3.10 (recall that door spaces are submaximal, hence T_D, so $ddA \subseteq dA$ holds for each A). Therefore, $\Diamonddot\Diamonddot p \to \boxdot p$ is valid in every door space, and so $\mathbf{K4} + \Diamonddot\Diamonddot p \to \boxdot p \subseteq L_d(\mathcal{D})$. Conversely, we know from Corollary 5.2(3) that $\mathbf{K4} + \Diamonddot\Diamonddot p \to \boxdot p$ is complete with respect to finite rooted $\mathbf{K4}$-frames $\mathfrak{F}$ such that $\overline{\mathfrak{F}}$ is a partially ordered set of depth ≤ 2, $r \notin max\mathfrak{F}$ implies r is irreflexive, and $\overline{\mathfrak{F}}$ not linearly ordered implies that $\mathfrak{F}$ is irreflexive. Among these all irreflexive frames give rise to door spaces, and the only frames containing reflexive points are $\begin{smallmatrix}\circ\\ |\\ \bullet\end{smallmatrix}$ and $\circ$. We show that both $\begin{smallmatrix}\circ\\ |\\ \bullet\end{smallmatrix}$ and $\circ$ are d-morphic images of the El′kin space X. We recall that the topology on X is defined by letting nonempty opens to be elements of a free ultrafilter on X. We define $f : X \to \circ$ by sending every point of X to the reflexive root;

also, we fix any $x \in X$ and define $g : X \to \overset{\circ}{\bullet}$ by sending x to the irreflexive root and $X - \{x\}$ to the reflexive point. Since X is a dense-in-itself T_1-space, it is obvious that both f and g are onto d-morphisms. Therefore, every finite rooted frame of $\mathbf{K4} + \Diamonddot\Diamonddot p \to \boxdot p$ is a d-morphic image of a door space. It follows that $L_d(\mathcal{D}) \subseteq \mathbf{K4} + \Diamonddot\Diamonddot p \to \boxdot p$, thus the equality. ■

THEOREM 5.6. *$\mathbf{K4} + \boxdot\boxdot\bot = \mathbf{GL} + \boxdot(p \to \boxdot p) = L_d(\mathcal{I})$.*

PROOF. It follows from Corollary 5.2(4) that $\mathbf{K4} + \boxdot\boxdot\bot = \mathbf{GL} + \boxdot(p \to \boxdot p)$. Let X be an I-space and ν be a valuation on X. Then $\nu(\boxdot\boxdot\bot) = tt(\emptyset) = (ddX)^c = X$. Therefore, $\mathbf{K4} + \boxdot\boxdot\bot \subseteq L_d(\mathcal{I})$. Since every finite rooted frame $\mathfrak{F}$ of $\mathbf{GL} + \boxdot(p \to \boxdot p)$ is such that $\overline{\mathfrak{F}}$ is a partially ordered set of depth ≤ 2, and $\mathfrak{F}$ is irreflexive, then every $\mathfrak{F}$ gives rise to a submaximal scattered space, which is an I-space. Thus, $L_d(\mathcal{I}) \subseteq \mathbf{GL} + \boxdot(p \to \boxdot p)$. The equality follows. ■

THEOREM 5.7. *$\mathbf{K4} + \Diamonddot p \to \boxdot p = L_d(\mathcal{PD})$.*

PROOF. Let X be a perfectly disconnected space and ν be a valuation on X. If $\nu(p) = A$, then $\nu(\Diamonddot p \to \boxdot p) = (dA)^c \cup tA = (dA)^c \cup (d(A^c))^c = (dA \cap d(A^c))^c = X$. Therefore, $\Diamonddot p \to \boxdot p$ is valid in every perfectly disconnected space, and so $\mathbf{K4}+\Diamonddot p \to \boxdot p \subseteq L_d(\mathcal{PD})$. Conversely, we know from Corollary 5.2(5) that the only rooted frames of $\mathbf{K4} + \Diamonddot p \to \boxdot p$ are $\overset{\circ}{\bullet}$, $\overset{\bullet}{\bullet}$, $\circ$ and $\bullet$. Both $\overset{\bullet}{\bullet}$ and $\bullet$ give rise to perfectly disconnected spaces. Since both $\overset{\circ}{\bullet}$ and $\circ$ are d-morphic images of the El′kin space (see the proof of Theorem 5.5), and since the El′kin space is maximal, hence perfectly disconnected, it follows that both $\overset{\circ}{\bullet}$ and $\circ$ are d-morphic images of a perfectly disconnected space.[9] Therefore, every finite rooted frame of $\mathbf{K4} + \Diamonddot p \to \boxdot p$ is a d-morphic image of a perfectly disconnected space. It follows that $L_d(\mathcal{PD}) \subseteq \mathbf{K4} + \Diamonddot p \to \boxdot p$, thus the equality. ■

THEOREM 5.8. *$\mathbf{K4} + \Diamonddot p \leftrightarrow \boxdot p = L_d(\mathcal{M})$.*

PROOF. Let X be a maximal space. Then X is perfectly disconnected, and so $\Diamonddot p \to \boxdot p$ is valid in X by Theorem 5.7. Also, if ν is a valuation on X and $\nu(p) = A$, then $\nu(\boxdot p \to \Diamonddot p) = (tA)^c \cup dA = d(A^c) \cup dA = d(A^c \cup A) = dX = X$ as X is dense-in-itself. Therefore, $\Diamonddot p \leftrightarrow \boxdot p$ is valid in X, and so $\mathbf{K4} + \Diamonddot p \leftrightarrow \boxdot p \subseteq L_d(\mathcal{M})$. Conversely, we know from Corollary 5.2(6) that $\overset{\circ}{\bullet}$ and $\circ$ are the only rooted frames of $\mathbf{K4} + \Diamonddot p \leftrightarrow \boxdot p$. As follows from the

[9] In fact, the same argument as in the proof of Theorem 5.5 shows that both $\overset{\circ}{\bullet}$ and $\circ$ are d-morphic images of *any* maximal space.

proof of Theorem 5.7, both of them are d-morphic images of any maximal space. Therefore, $L_d(\mathcal{M}) \subseteq \mathbf{K4} + \Diamond p \leftrightarrow \Box p$, thus the equality. ∎

It follows that the above six d-logics are in the following relationship (arrows denote strict set inclusion):

$$\begin{tikzcd}
 & & & & L_d(\mathcal{I}) & \\
L_d(\mathcal{N}) \arrow[r] & L_d(\mathcal{S}) \arrow[r] & L_d(\mathcal{D}) \arrow[r] & L_d(\mathcal{I}) \cap L_d(\mathcal{PD}) \arrow[ur] \arrow[dr] & & \\
 & & & & L_d(\mathcal{PD}) \arrow[r] & L_d(\mathcal{M})
\end{tikzcd}$$

Therefore, even though the d-logics of the six classes of spaces considered in this paper are distinct, and so they express the relationship between these classes of spaces much more accurately than the corresponding C-logics, they still do not give a complete picture of the relationship between them. For example, $L_d(\mathcal{I})$, $L_d(\mathcal{PD})$, and $L_d(\mathcal{M})$ are all proper normal extensions of $L_d(\mathcal{D})$. However, the classes $\mathcal{D}, \mathcal{I}$; $\mathcal{D}, \mathcal{PD}$; and $\mathcal{D}, \mathcal{M}$ are pairwise incomparable.

We conclude the paper by mentioning that in order to capture complete picture of the relationship between the above six classes of spaces, it is worthwhile to consider different extensions of our basic modal language. For example, it appears to be very plausible that adding the universal modality or the nominals will give us powerful enough language to be able to express all the intrinsic connections between the classes of spaces considered in this paper.

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