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From Simplex to Cube

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ABSTRACT. **Interplay of simplex-cube contained in Serre map of cube to simplex is formalized. Tower of cubical CW-complexes between simplex and cube is constructed and a preparing result for stable obstruction theory is given.**

Key words: simplex, cube, CW-complex, fibration, cross section, obstruction.

1. Introduction. The problem of existence of cross sections in stable ranges was solved by M.Mahowald in [1]. The aim of the present paper is to prepare tools for new approach to the problem. The main technical innovations is construction of auxiliary CW-complexes which are close to simplex and cube. The Serre map of cube to simplex [2] was used in [3-6]. The formalism developed there is given here in more convenient form.

2. Cubes $\omega(0,1,\ldots,(n+1))$ and $\beta(0,1,\ldots,(n+1))$. For the standard $(n+1)$-simplex let $\omega(0,1,\ldots,(n+1))$ be a n-cube defined by the formulas

$$d_i^1\omega(0,1,2,\ldots,(n+1)) = \omega(0,1,2,\ldots,\hat{i},\ldots,(n+1)), \quad 1 \le i \le n,$$

$$d_i^0\omega(0,1,2,\ldots,(n+1)) = \omega(0,1,2,\ldots,i)\omega(i,\ldots,(n+1)), \quad 1 \le i \le n.$$

(Left side of the second formula is a cartesian product of two cubes.)

Let $\beta(0,1,\ldots,(n+1))$ be $(n+1)$-cube defined by the formulas

$$d_i^1\beta(0,1,2,\ldots,(n+1)) = \beta(0,1,2,\ldots,\widehat{(i-1)},\ldots,(n+1)), \quad 1 \le i \le n+1.$$

$$d_i^0\beta(0,1,2,\ldots,(n+1)) = \omega(0,1,2,\ldots,(i-1))\omega((i-1),\ldots,(n+1)), \quad 1 \le i \le n+1.$$

From formulas above it follows that an arbitrary face of the cube $\omega(0,1,\ldots,(n+1))$ is the word

$$\omega(a_{i_0},\ldots,a_{i_1})\omega(a_{i_1},\ldots,a_{i_2})\omega(a_{i_2},\ldots,a_{i_3})\cdots\omega(a_{i_{k-1}},\ldots,a_{i_k}),$$

where

$$(a_0,a_1,a_2,\ldots,a_q)\subset(0,1,\ldots,(n+1)),$$

$$a_{i_0} = a_0 = 0 \quad \text{and} \quad a_{i_k} = a_q = n+1$$

and arbitrary face of cube $\beta(0,1,\ldots,(n+1))$ is the word

$$\beta(\ldots,a_{i_1})\omega(a_{i_1},\ldots,a_{i_2})\omega(a_{i_2},\ldots,a_{i_3})\cdots\omega(a_{i_{k-1}},\ldots,a_{i_k}),$$

where

$$(a_0,a_1,a_2,\ldots,a_q)\subset(0,1,\ldots,(n+1)), \quad a_{i_k} = a_q = n+1$$

and in β some a's precedent to a_{i_1} may be omitted. Note that in the case of β the first multiplier is always with β and all others with ω. Multiplier with ω has as argument not less than 2 vertexes and multiplier with β has as argument not less than 1 vertex.

There is a map

$$\phi: \beta(0,1,2,\ldots,i)\to(0,1,2,\ldots,i),$$

which is surjective and homeomorphic on the interiors. It is defined inductively. Being $\beta(0)$ a point, the map $\beta(0)\to(0)$ is defined. Assume the map is defined for $i\le k$. Define it for $k+1$ as follows.

We define the map of boundaries

$$\partial\beta(0,1,2,\dots,k+1)\to\partial(0,1,2,\dots,k+1)$$

as a map defined for d_i^1-faces by

$$\phi:\beta(0,1,2,\dots,\hat{i},\dots,(k+1))\to(0,1,2,\dots,\hat{i},\dots,(k+1)),$$

for d_i^0-faces by composition of projektion

$$\beta(0,1,2,\dots,i\text{-}1)\omega((i\text{-}1),\dots,(k+1))\to\beta(0,1,2,\dots,i\text{-}1)$$

and map

$$\phi:\beta(0,1,2,\dots,(i\text{-}1))\to(0,1,2,\dots,(i\text{-}1)).$$

The map is defined correctly. This defines factor space

$$\partial\beta(0,1,2,\dots,(k+1))/\sim=\partial(0,1,2,\dots,(k+1))$$

and hence the factor space

$$\beta(0,1,2,\dots,(k+1))/\sim=(0,1,2,\dots,(k+1)).$$

3. Complex $\beta_n(0, 1, 2,\dots, s)$. Fix the integer n and consider in $\beta(0, 1, 2,\dots, s)$ an equivalence relation by assuming that in all faces of it the multipliers $\omega(a_0,a_1,a_2,\dots,a_k)$ with dimension less than n and bigger than 0 is contracted to point $\omega(a_0,a_k)$ (i.e. if $1<k<n+1$).

We will have the space $\beta_n(0, 1, 2,\dots, s)$ as a cubical CW-complex. It is done as follows. Consider the following faces of $\beta(0, 1, 2,\dots, s)$ as pre-cells of $\beta_n(0, 1, 2,\dots, s)$:

$$\beta(\dots,a_1)\omega(a_1,\dots,a_2)\dots\omega(a_{k-1},\dots,a_k),$$

where $(a_0<a_1<a_2<\dots<a_k)$, $a_k=s$, is a subset of $(0,1,2,3,\dots,s)$ and: i) dimension of ω's are 0 or $>n$-1; ii) two or more 0-dimensional ω's are not standing together; iii) ω's may not be.

These cubes we are going to have as all cells of $\beta_n(0, 1, 2,\dots, s)$ and we must give a procedure how to build up from them cubical CW-complex $\beta_n(0, 1, 2,\dots, s)$. We need a following remark.

If

$$\beta(\dots,a_1)\omega(a_1,\dots,a_2)\dots\omega(a_{k-1},\dots,s)$$

is an arbitrary face of cube $\beta(0, 1, 2,\dots, s)$ then reducing all the $\omega(a_i,\dots,a_r)$'s to $\omega(a_i,a_r)$ if $\dim\omega(a_i,\dots,a_r)<n$, we become the face to which we can project given face; the new face may have neighboring 0-dimensional ω's, say $\omega(a_i,a_p)$ and $\omega(a_p,a_k)$ which we replace by one $\omega:\omega(a_i,a_k)$. Such way we become one of the standard cells of the kind described above. So for each face of $\beta(0, 1, 2,\dots, s)$ we have named the standard cell (may be of dimension less) and the continuous map of given face to this cell.

Below without confusion if in pre-cell last ω is of dimension 0 we can drop it. We see that if $p<n$ then we have only a cell of kind $\beta(a_0,\dots,a_p)$. We can this cell identify with the simplex $(a_0,\dots,a_p)$. So n-skeleton of standard simplex $(0, 1, 2,\dots, s)$ is a part of n-skeleton of $\beta_n(0, 1, 2,\dots, s)$. Other n-cells are of kind:

$$\beta(a_0)\omega(a_0,\dots,a_{n+1}) \text{ and } \beta(a_0)\omega(a_0,a_1)\omega(a_1,\dots,a_{n+2}).$$

Both of them, say τ is attached to vertex (a_0) with the attaching map $\partial\tau\to a_0$. The n-skeleton $\beta_n(0, 1, 2,\dots, s)^{(n)}$ is constructed. Assume that $(n+k)$-skeleton is already constructed. Consider arbitrary $(n+k+1)$-pre-cell σ. The remark made above defines the map for each of the $(n+k)$-faces in already constructed $(n+k)$-skeleton; they define the attaching map of its boundary $\partial\sigma$ correctly.

The CW-complex $\beta_n(0, 1, 2, \ldots, s)$ is constructed.

The remark shows too that there is a continuous map

$$\varphi : \beta(0,1,2,\ldots,s) \to \beta_n(0,1,2,\ldots,s).$$

On the other hand, there is an obvious map

$$p: \beta_n(0,1,2,\ldots,s) \to (0,1,2,\ldots,s)$$

and the triangle

$$\begin{array}{ccc} \beta(0,1,2,\ldots,s) & \to & \beta_n(0,1,2,\ldots,s) \\ & \searrow & \downarrow \\ & & (0,1,2,\ldots,s) \end{array}$$

is commutative. Here p is identical on the n-skeleton of simplex $(0,1,2,\ldots,s)$.

Now if $(a_0,a_1,a_2,\ldots,a_r) \subset (0,\ldots,s)$ is subsimplex then $\beta_n(a_0,a_1,a_2,\ldots,a_r) \subset \beta_n(0,\ldots,s)$. Of course this is not true for $\beta(0,\ldots,i)$.

4. Complex $\beta_{n,n\text{-}1}(0,\ldots,s)$. The complex $\beta_{n,n\text{-}1}(0,\ldots,s)$ will be constructed from complex $\beta_n(0,\ldots,s)$ by restoring some degenerated faces of its cells.

First two preliminary remarks. Let natural $r>0$ be fixed and consider all faces of cube $\beta_n(0,\ldots,s)$ which as the last positive dimensional multiplier has ω of dimension t. We say that such a face is of kind p, $p>0$, if immediately before last positive dimensional ω stands exactly $(p\text{-}1)$ multipliers ω of dimension 0. Other faces we say that are of kind 0. The second remark is this. Let σ^k be k-dimensional face of $\beta_n(0,\ldots,s)$ which has last multiplier of dimension $\geq$ than t. Then its face which has as last multiplier ω of dimension $t\text{-}1$ is determined uniquely, let it be $\tau^{k\text{-}1}$ and in the pair $(\sigma^k, \tau^{k\text{-}1})$ both components define uniquely each other.

Let $t=n$. We see that $\beta_n(0,\ldots,s)$ has cells only of kind 0, 1 and 2.

Construction of $\beta_{n,n\text{-}1}(0,\ldots,s)$ proceeds as follows. Let $(n+1)$-skeleton $\beta_n(0,\ldots,s)^{(n+1)}$ be part of $\beta_{n,n\text{-}1}(0,\ldots,s)^{(n+1)}$ and proceed constructing $\beta_{n,n\text{-}1}(0,\ldots,s)^{(n+1)}$ by attaching to it new cells of dimension $(n+1)$, $(n+2)$, $(n+3)$ and so on. This procedure will use restoration of some degenerated faces of already constructed cells. To begin with let $\sigma^{n+2} \in \beta_n(0,\ldots,s)$ be cell of kind 0 and let $(\sigma^{n+2}, \tau^{n+1})$ be corresponding pair. Then τ^{n+1} has as pre-cell the face of $\beta(0,\ldots,s)$ which ends with multiplier ω of dimension $(n\text{-}1)$ and being a degenerate face in $\beta_n(0,\ldots,s)$ it is mapped in $\beta_n(0,\ldots,s)^{(n)}$. Considering this map only on the boundary, τ^{n+1} becomes a new $(n+1)$-cell. After this attach the first component of the pair with new obviously defined attaching map. This we do for all $(n+2)$-dimensional cells of kind 0. Considering $(n+3)$-dimensional cells $\sigma^{n+3} \in \beta_n(0,\ldots,s)$ we are forced to restore some of its $(n+2)$-dimensional cells which turns out to be of kind 1. Considering pairs $(\sigma^{n+2}, \tau^{n+1})$ with first component of kind 1 of complex $\beta_n(0,\ldots,s)$ or one of restored cells we attach the second component and after this the first component. The $\sigma^{n+3} \in \beta_n(0,\ldots,s)$ of kind 0 will be attached with new attaching map. And so on in this manner. $\beta_{n,n\text{-}1}(0,\ldots,s)$ is constructed.

The projection $\beta_{n,n\text{-}1}(0,\ldots,s) \to \beta_n(0,1,2,\ldots,s)$ is defined in obvious way indicated by the procedure of construction of complexe $\beta_{n,n\text{-}1}(0,\ldots,s)$. It is identical on $\beta_n(0,\ldots,s)^{(n+1)}$. If $(a_0,a_1,a_2,\ldots,a_r) \subset (0,\ldots,s)$ is a subsimplex then $\beta_{n,n\text{-}1}(a_0,a_1,a_2,\ldots,a_r) \subset \beta_{n,n\text{-}1}(0,\ldots,s)$.

5. Complexes $\beta_{n,n\text{-}1,\ldots,n\text{-}k}(a_0,a_1,a_2,\ldots,a_s)$. For given n constructed above tower of CW-complexes

$$(0,\ldots,s) \leftarrow \beta_n(0,1,2,\ldots,s) \leftarrow \beta_{n,n\text{-}1}(0,\ldots,s)$$

is a beginning of tower consisting from $n+1$ CW-complexes

$$(0,\dots,s) \leftarrow \beta_n(0,1,2,\dots,s) \leftarrow \beta_{n,n-1}(0,\dots,s) \leftarrow \beta_{n,n-1,n-2}(0,\dots,s) \leftarrow \dots \leftarrow \beta_{n,n-1,n-2,\dots,2,1}(0,\dots,s).$$

Denote

$$L_{n-k} = \beta_{n,n-1,\dots,n-k}(0,\dots,s)$$

then we will have above tower as

$$L_{n+1} = (0,\dots,s) \leftarrow L_n \leftarrow L_{n-1} \leftarrow L_{n-2} \leftarrow \dots \leftarrow L_1$$

with projection $\psi_k : L_k \to L_{k+1}$. Let $K^{(i)}$ denote i-skeleton of complex K. The above tower has as well properties: i) $(0,\dots,s)^{(n)} \subset L_n^{(n)}$, and the projection ψ_n is identical on it, $L_n^{(n+1)} \subset L_{n-1}^{(n-1)}$ and the projection ψ_{n-1} is identical on it; $L_{n-1}^{(n+2)} \subset L_{n-2}^{(n-2)}$ and the projection ψ_{n-2} is identical on it and so on $L_{k+1}^{(2n-k)} \subset L_k^{(2n-k)}$, $k=n$, n-1, n-2, ...,2, 1, and the projection ψ_k is identical on it. ii) as pre-cells are used the faces of cube $\beta(0,1,2,\dots,s)$ and in cells of L_{n-k} does not occur multiplier of positive dimension less than n-k. iii) In cells of L_{n-k} may occur only once (n-k)-dimensional ω as multiplier. iv) the tower is functorial.

Each new complex of tower is constructed from precedent complex in the similar way as $\beta_{n,n-1}(0,\dots,s)$ was constructed from $\beta_n(0,\dots,s)$: L_k is constructed by attaching to (2n-k)-skeleton of L_{k+1} cells of dimension 2n-k+1, 2n-k+2 and so on. The projections are defined in obvious way indicated by the procedure of construction of complexes.

Tower is functorial.

6. Cross sections. Let for the standard simplex $(0, 1, 2,\dots, s)$ be given Serre fibration

$$F \to E \to (0,\dots,s)$$

Then for $k=n$, n-1, n-2,... , 2, 1, tower of precedent section defines induced fibration

$$F \to E \to L_k$$

Theorem. *If w is a cross section of the fibration over the n-skeleton of simplex $(0, 1, 2,\dots, s)$ then for induced fibration there is a cross section w_k over (2n-k+1)-skeleton of L_k, k=n ,n-1, n-2, ...,2, 1, such that w_n coincides with w on n-skeleton of simplex $(0, 1, 2,\dots, s)$ and w_k coincides with w_{k+1} on the (2n-k)-skeleton of complex L_{k+1}.*

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სიმპლექსიდან კუბისკენ

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