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THE PREDIFFERENTIAL OF A PATH FIBRATION

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Abstract. For a simply connected space B the Hirsch model of path fibration is constructed in terms of B. In particular this means the calculation of loop space cohomology. As an application, the Hirsch model of the fiber of any fibration over B is given.

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1. Introduction

Let k be a commutative ring and

$$F \to E \xrightarrow{p} B$$

be a Serre fibration. If $H^n(F,k)$, $n = 0,1,2,\dots$, are free k-modules, then the cohomological Hirsch model of the space E is the bigraded k-module

$$Y^{*,*} = C^*(B, H^*(F,k))$$

with the twisted differential defined by some (not uniquely determined) twisting cochain

$$a \in C^*(B, \operatorname{Hom}^*(H^*(F,k), H^*(F,k)), \quad \dim a = +1, \quad da = -aa.$$

There is a chain map of the Hirsch model to the double complex X^{**} (cohomologically equivalent to the cochain complex $C^*(E,k)$)

$$C^*(B, H^*(F,k))_a \to X^{**},$$

inducing an isomorphism of cohomology [5, 3]. If we assume B simply connected, then a is of the form

$$a = a^{2,-1} + a^{3,-2} + a^{4,-3} + \dots + a^{i,-i+1} + \cdots\,,$$
$$a^{n,-n+1} \in C^n(B, \operatorname{Hom}^{-(n-1)}(H^*(F,k), H^*(F,k))).$$

The condition

$$d_A a = -aa$$

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in our case is equivalent to the sequence of equations

$$\begin{gathered}\delta_B a^{2,-1} = 0,\\ \delta_B a^{3,-2} = -a^{2,-1}a^{2,-1},\\ \delta_B a^{4,-3} = -a^{2,-1}a^{3,-2} - a^{3,-2}a^{2,-1},\\ \delta_B a^{5,-4} = -a^{2,-1}a^{4,-3} - a^{3,-2}a^{3,-2} - a^{4,-3}a^{2,-1},\\ \vdots\\ \delta_B a^{q+1,-q} = -a^{2,-1}a^{q,-q+1} - \cdots - a^{q,-q+1}a^{2,-1}\end{gathered} \tag{1.1}$$

and so on.

The twisted differential on $Y^{*,*}$ is defined by the formula

$$d_a(c) = \delta_B(c) + a \smile c.$$

In details,

$$\begin{aligned} d_a(c^{p,q}) = \delta_B(c^{p,q}) &+ a_q^2 \smile c^{p,q} + a_q^3 \smile c^{p,q}\\ &+ a_q^4 \smile c^{p,q} + \cdots + a_q^{q+1} \smile c^{p,q}\end{aligned}$$

for $c^{p,q} \in C^p(B, H^q(F,k))$, where $a_q^n \in C^n(B, \mathrm{Hom}(H^q, H^{q-n+1}))$ is the component of $a^{n,-n+1}$ acting on H^q. Clearly, the components a_q^n determine $a^{n,-n+1}$ uniquely. For the path fibration

$$\Omega B \to PB \to B$$

PB is acyclic, so spectral sequence arguments immediately give $H^1(\Omega B) = H^2(B)$, but $H^2(\Omega B)$ already involves the differentials of spectral sequence.

The aim of this paper is to write inductively on q, $q = 1, 2, \ldots$, the cohomology $H^q(\Omega B)$ and a_q^*, the q-component of twisting cochain a, in terms of space B. The procedure is simple and the result is formulated as Theorem 2 in Section 4. Section 5 gives an application for the fiber of a fibration over B.

Adams cobar construction and Eilenberg–Moore spectral sequence are presently the most convenient tools in this area. We think an alternative approach presented here deserves some attention.

2. Preliminaries

In this section, for convenience, we give the notions of the Hirsch model and the predifferential of a fibration as in [1, 2].

We shall consider a Serre fibration

$$F \to E \to B \tag{2.1}$$

with B simply connected.

Let k be a ring and G be a module over k and assume that all k-modules $H^i(F,G)$ are free. Consider $A = C^*(B, \mathrm{Hom}_k^*(H^*(F,G), H^*(F,G)))$ and $Y = C^*(B, H^*(F,G))$. It is clear that A is a bigraded algebra since $\mathrm{Hom}^*(H^*, H^*)$ is a graded ring, with the product being the composition. Furthermore, Y is a left module over A: there is a pairing $A \otimes Y \to Y$, where the product is defined by the $\smile$-product of B and the obvious pairing of coefficients: $\mathrm{Hom}^*(H^*, H^*) \otimes H^* \to H^*$.

Consider the filtration $F^p(A)$ of the bigraded algebra A by first degree and let $a \in A$ be such that dim $a = +1$, $a \in F^2(A)$ (i.e., $a = a^{2,-1} + a^{3,-2} + a^{4,-3} + \cdots + a^{i,-i+1} + \cdots$) and $d_A a = -aa$. Such elements are called twisting elements of the algebra A. The set of all twisting elements is denoted by $T(A)$. It turns out that $d_a y = d_Y y + ay$ is a differential in Y (i.e., $d_a d_a = 0$). Denote the complex (Y, d_a) by Y_a.

Let

$$g = 1 - u^{1,-1} - u^{2,-2} - u^{3,-3} - \cdots ,$$
$$u^{n,-n} \in A^{n,-n} = C^n(B, \mathrm{Hom}_k^{-n}(H^*(F,G), H^*(F,G))).$$

The set of such g's is a group with multiplication as in the algebra A. It is called the group of units: $G(A) = \{g \mid g = 1 - u,\ \ u \in F^1 A,\ \ \dim u = 0\}$. Clearly,

$$(1-u)^{-1} = 1 + u + uu + uuu + uuuu + \cdots + u^i + \cdots .$$

The group $G(A)$ acts on the set $T(A)$ via the formula

$$g * a = gag^{-1} - dgg^{-1}.$$

It is easy to verify that the map $f_g : Y \to Y$ defined by $f_g(y) = gy$ is an isomorphism of complexes $Y_a \to Y_{g*a}$.

Definition 1. We define the set of predifferentials of fibrations with base B and fiber cohomology $H^*(F,G)$ as the set of orbits of the set $T(A)$ under the action of group $G(A)$, $D(A) = T(A)/G(A) = D(B, H^*(F,G))$.

The set $D(B, H^*)$ is a contravariant functor on the category of topological spaces B, and if $f : B_1 \to B$ induces an isomorphism of homology, then $D(f)$ is one-to-one ([1, 2]).

The basic fact we will use below is

Theorem 1 ([1, 2])**.** *Fibration* (2.1) *defines uniquely an element* $d(E) \in D(B, H^*(F,G))$*, called the predifferential of fibration, such that:*

1) *for every* $a \in d(E)$ *the twisted complex* $Y_a = C^*(B, H^*(F,G))_a$ *is a model of the cochain complex* $C^*(E,G)$ (*i.e., there is a chain map* $Y_a \to X^{**} \equiv C^*(E,G)$ *inducing an isomorphism of homology; we call* Y_a *the Hirsch model of the fibration*)*;*

2) *for every two elements* $a, b \in d(E)$ *such that* $b = g * a$, $g \in G(A)$, *the diagram*

$$\begin{tikzcd} Y_b \arrow[r] & X^{**} \\ Y_a \arrow[u, "f_g"] \arrow[ur] & \end{tikzcd}$$

commutes up to a chain homotopy;

3) $d(E)$ *is functorial: if* $\phi : B_1 \to B$ *is a map and* $F \to E_1 \to B_1$ *is the induced fibration, then* $d(E_1) = D(\phi)[d(E)]$.

(The first assertion of the theorem is due to G. Hirsch [5] and E. H. Brown [3].)

Note that the Hirsch model is not multiplicative: it does not carry a multiplicative structure.

Remark 1. This theorem is also true for the general case where cohomologies of fibers are not free k-modules and the base is not simply connected. In this case

$$Y = C^*(B, R^*H^*(F,G)), \quad A = C^*(B, \mathrm{Hom}^* * (R^*H^*(F,G), R^*H^*(F,G)))$$

with $R^*H^q(F,G)$ being a free resolution of the module $H^q(F,G)$. The differential is that of local systems [1, 2].

The first filtration of $Y_a = C^*(B, H^*(F,G))_a$ defined by

$$F^p = F^p(Y_a) = \sum_{i \geq p} C^i(B, H^*(F,G)),$$

is decreasing:

$$Y_a = F^0 \supset F^1 \supset \cdots \supset F^{p-1} \supset F^p \supset F^{p+1} \supset \cdots .$$

For the corresponding spectral sequence one has

$$E_2^{p,q} = H^p(B, H^q(F,G)) \Longrightarrow H^{p+q}(Y_a). \tag{2.2}$$

This spectral sequence coincides with the Serre spectral sequence of fibration (2.1). So the questions posed as regards the Serre spectral sequence are to be redirected to the spectral sequence (2.2). All differentials of (2.2) are easy to write in terms of the cochain a ([2]). Even partial knowledge of a gives some information about differentials of (2.2). For example, the pair $a^{2,-1} + a^{3,-2}$, the beginning of a, determines the differentials d_2 and d_3 of the spectral sequence. Moreover, the Hirsch model is more handy than the fibration.

3. More about the Hirsch Model

Let us pay more attention to Y_a and let us forget for convenience that it comes from fibration, i.e., consider the following algebraic case: let U^* be a free graded k-module with $U^i = 0$, $i < 0$, $U^0 = k$, and $Y = C^*(B, U^*)$, $A = C^*(B, \mathrm{Hom}(U^*, U^*))$.

If $a \in T(A)$, then we have Y_a. Consider the natural imbedding

$$U^q \subset Y_a \tag{3.1}$$

as $(0,q)$-cocycles in the differential module $Y = C^*(B, U^*)$, which reflects the fact that in the spectral sequence (2.2) $U^q = E_2^{0,q}$.

The filtration of complex Y_a by second degree (we call it the vertical filtration) is increasing: by definition, $F^q = F^q(Y_a) = \sum\limits_{0 \leq i \leq q} C^*(B, U^i)$; it is a subcomplex in Y_a and

$$F^0 = C^*(B,k) \subset F^1 \subset \cdots \subset F^{q-1} \subset F^q \subset F^{q+1} \subset \cdots .$$

For each u in U^{q+1} the $(q+2)$-dimensional element of F^q :

$$\begin{array}{llllll} a^2_{q+1}(u) & & & & & \\ & +a^3_{q+1}(u) & & & & \\ & & +a^4_{q+1}(u) & & & \\ & & & \ddots & & \\ & & & & +a^{q+1}_{q+1}(u) & \\ & & & & & +a^{q+2}_{q+1}(u) \end{array} \tag{3.2}$$

is $a(u)$ and if by (3.1) we regard u as an element of $C^0(B, U^{q+1}) \subset Y_a$, it is au and from the formula $d_{Y_a}(u) = d_Y(u) + au$ it follows that

$$d_{Y_a}(u) = d_Y(u) + au = 0 + a(u) = a(u).$$

Hence it is a cocycle in Y_a and belongs to F^q.

Let

$$\begin{array}{llllll} x^2 & & & & & \\ & +x^3 & & & & \\ & & +x^4 & & & \\ & & & \ddots & & \\ & & & & +x^{q+1} & \\ & & & & & +x^{q+2} \end{array}$$

be another $(n+2)$-cocycle of F^q homological to (3.2) in F^q. Hence there is a $(q+1)$- cochain $b(u)$ in F^q, say,

$$\begin{array}{llllll} b^1(u) & & & & & \\ & +b^2(u) & & & & \\ & & +b^3(u) & & & \\ & & & \ddots & & \\ & & & & +b^q(u) & \\ & & & & & +b^{q+1}(u) \end{array}$$

such that $d_{Y_a} b(u) = x - a(u)$.

Lemma 1. *Assume that u is one of the basis elements of U^{q+1} and define $g^i_{q+1} \in C^i(B, \mathrm{Hom}(U^{q+1}, U^{q+1-i}))$ on the basis elements of U^{q+1} by $g^i_{q+1}(u_k) = 0$, $u_k \neq u$, and $g^i_{q+1}(u) = b^i$. Then, denoting $g_{q+1} = g^1_{q+1} + g^2_{q+1} + \cdots + g^{q+1}_{q+1}$, one has:*

i) *$\overline{a} \equiv (1 - g_{q+1}) * a$ coincides with a for U^j, $j < q+1$, and for basis elements u_t of U^{q+1} others than u;*

ii) *$\overline{a}(u) \equiv ((1 - g_{q+1}) * a)(u) = \sum_i x^i = x$.*

Proof. The proof is a straightforward application of the formula

$$(1-g) * a = (1-g)a(1-g)^{-1} - d(1-g).(1-g)^{-1}$$

recalling that in our case where $g = g_{q+1}$, one has $gg = 0$, and hence

$$(1-g)^{-1} = 1 + g, \quad dg.g = 0, \quad gag = 0, \quad ga(u) = 0.$$

For example,

$$[(1-g)*a](u) = [(1-g)a(1+g)+dg](u) = (a-ga+ag+dg)(u)$$
$$= a(u)+ag(u)+dg(u) = a(u)+d_a(g(u)) = a(u)+d_a b(u) = x.$$

The lemma is proved. □

It is evident that

$$H^i(Y_a) = H^i(F^q), \quad i \le q. \tag{3.3}$$

Define

$$f : U^{q+1} \to H^{q+2}(F^q) \tag{3.4}$$

as the composition

$$U^{q+1} \to Z^{q+2}(F^q) \to H^{q+2}(F^q)$$

where the cycles are with respect to the differential d_a, U^{q+1} is considered as imbedded in Y_a as above in (3.1) and the first map is d_a.

For the first spectral sequence of Y_a we have

$$E_2^{p,q} = H^p(B,U^q) \Longrightarrow H^{p+q}(Y_a). \tag{3.5}$$

Lemma 2. *If Y_a is acyclic then for $q>0$ one has:*

i) $U^{q+1} = H^{q+2}(F^q)$. *This isomorphism is* (3.4) (*induced by the differential d_a on the subgroup $U^{q+1} \subset Y_a$ imbedded as* $(0,q+1)$*-cocycles in the differential module $Y_a = C^*(B,U^{q+1})$*);

ii) $H^i(F^q) = 0$ *if* $i \le q+1$.

Proof. For $i \le q$, assertion ii) follows from (3.3) and $H^*(Y_a)=0$; for $i=q+1$, it follows from simply connectedness of B by virtue of spectral sequence arguments. As for assertion i), in our case the map f in (3.4) is a monomorphism: if $f(u)=0$ then $d_a(u) = d_a(v)$, $\;dim\, v = q+1$, $\;v \in F^q$; and we see that all differentials of the spectral sequence (3.5) vanish on the element $u \in E_2^{0,q+1}$; but from $E_\infty^{0,q+1} = 0$ follows $u=0$. On the other hand, f is an epimorphism: if not, then for some $p \ge 2$ it would be $E_\infty^{p,q-p+2} \ne 0$, which contradicts the fact that $H^{q+2}(Y_a) = 0$. □

The main fact is the uniqueness of acyclic models over the same space given in the lemma below.

Lemma 3. *Let $Y = C^*(B,U^*)$ and $Z = C^*(B,V^*)$, $A = C^*(B, hom^*(U^*,U^*))$, $A_1 = C^*(B, hom^*(V^*,V^*))$. Let $a \in T(A)$, $b \in T(A_1)$ be such that Y_a and Z_b are acyclic complexes. Then there exist isomorphisms*

$$U^i \to V^i, \quad i = 1,2,\dots, \tag{3.6}$$

and elements $a' \sim a$, $b' \sim b$ such that the map $Y \to Z$ induced by (3.6) *is an isomorphism of complexes $Y_{a'} \to Z_{b'}$; hence U^i-s and the predifferential a are uniquely defined by the space B.*

Proof. We are going to prove that the following inductive assertion is valid.

(P_q) : there exist isomorphisms $U^i \to V^i$, $i \leq q$, and a and b coincide in the range $i \leq q$ via these isomorphisms (then, of course, the induced map $F^q(Y_a) \to F^q(Z_b)$ is an isomorphism of complexes).

The assertion (P_0) is true: $F^0 Y_a = C^*(B,k) = F^0 Z_b$.

Assume that (P_q) is valid. Then we can write $F^q(Y_a) = F^q(Z_b)$. By Lemma 2 we have two isomorphisms

$$U^{q+1} \to H^{q+2}(F^q) \leftarrow V^{q+1} \tag{3.7}$$

induced by the differentials in $F^{q+1}(Y_a)$ and $F^{q+1}(Z_b)$, respectively. This defines an isomorphism $U^{q+1} \to V^{q+1}$. Choose a basis for $H^{q+2}(F^q(Y_a) = F^q(Z_b))$, say, $h_1, h_2, \ldots, h_l, \ldots$ and choose cocycles in each h_l:

$$\sum_{i+j=q+2} x_l^{ij} \in h_l.$$

We have by isomorphism (3.7) that h_l is in U^{q+1} and $d_a(h_l)$ is homological to $\sum_i x_l^{ij}$, by Lemma 1 we can transform a to $\overline{a}$ assuming $u = h_l$:

$$\overline{a}(h_l) = \sum_i x_l^{ij}$$

In a similar way by Lemma 1 we can transform b to $\overline{b}$ assuming $u = h_l$:

$$\overline{b}(h_l) = \sum_i x_l^{ij}.$$

Recall that for F^q the elements a and b are not changed and the new $\overline{a}$ and $\overline{b}$ coincide for the range $q+1$, too, via isomorphism (3.7) (and still Y_a and Z_b are acyclic). Hence P_{q+1} is valid. □

The equivalent formulation of this lemma is

Lemma 4. *Let $Y = C^*(B, U^*)$, $a \in T(A)$, $A = C^*(B, \mathrm{Hom}(U^*, U^*))$ and let Y_a be acyclic, then the predifferential $d(Y_a) \in D(B, U^*)$ with $a \in d(Y_a)$ is defined uniquely in the sense that if Z_b is another acyclic complex with $Z = C^*(B, V^*)$, $b \in T(A_1)$, $A_1 = C^*(B, \mathrm{Hom}(V^*, V^*))$, then there are isomorphisms*

$$U^i \to V^i, \quad i = 0, 1, \ldots,$$

such that, after identifying the algebras A and A_1 via these isomorphisms the predifferentials $d(Y_a)$ and $d(Z_b)$ coincide, i.e., a and b become equivalent twisting elements: $b = g * a$.

4. The Main Theorem

Lemma 2 gives a clue how to construct acyclic Y_a for given B. It runs as follows.

Let B be a simply connected space and k be a field. Let $U(B)^0 = k$, then $F^0 Y_a = C^*(B,k)$. It consists of one row and the differential is the coboundary operator δ of B. One has

$$H^i(F^0 Y_a) = 0, \quad i < 2. \tag{4.1}$$

Let $U(B)^1$ be $H^2(F^0Y_a)$. Choose a basis in $H^2(F^0Y_a)$, say $\{u_\lambda\}$, and for every u_λ choose a cocycle $l(u_\lambda)$ in it, $l(u_\lambda) \in u_\lambda$. Identify these basis elements as basis elements of U^1 and define a cochain $a_1^2 \in C^2(B, \mathrm{Hom}(U(B)^1, U(B)^0))$ for a simplex $\sigma^2 \in B$ by

$$[a_1^2(\sigma^2)](u_\lambda) = l(u_\lambda)(\sigma^2).$$

The constructed $a_1^2 \in C(B, \mathrm{Hom}(U(B)^1, U(B)^0)))$ is a cocycle,

$$\delta_B a_1^2 = 0 \tag{4.2}$$

(the first equation of (1.1)): from

$$[(\delta_B a_1^2)(\sigma^3)](u_\lambda) = \sum(-1)^i [a_1^2(\sigma_i^3)](u_\lambda) = \sum(-1)^i l(u_\lambda)(\sigma_i^3) = 0$$

follows (4.2). This constructs the twisted complex $F^1Y_a = C^*(B, U(B)^i)$, $i = 0, 1$, with two rows. The differential for the new row is

$$dc^{p,1} = \delta c^{p,1} + a_1^2 c^{p,1}.$$

One has $F^0Y_a \subset F^1Y_a$ and $H^i(F^1Y_a) = 0$, $i < 3$.

Let $U(B)^2$ be $H^3(F^1Y_a)$, choose a basis in $H^3(F^1Y_a)$, say $\{u_\lambda\}$, and for every u_λ choose a cocycle $l(u_\lambda)$ in it, $l(u_\lambda) \in u_\lambda$. Let its components be $l^{2,1}(u_\lambda)$ and $l^{3,0}(u_\lambda)$. Identify these basis elements as basis elements of U^2 and let cochain $a_2^2 \in C^2(B, \mathrm{Hom}(U^2, U^1))$ be defined for a simplex $\sigma^2 \in B$ by $[a_2^3(\sigma^2)](u_\lambda) = l^{2,1}(u_\lambda)(\sigma^2)$ and define the cochain $a_2^3 \in C^3(B, \mathrm{Hom}(U^2, U^0))$ for a simplex $\sigma^3 \in B$ by

$$[a_2^2(\sigma^3)](u_\lambda) = l^{3,0}(u_\lambda)(\sigma^3).$$

Evidently, the equation $d_a l(u_\lambda) = 0$ is equivalent to the following two equations

$$\delta l^{2,1}(u_\lambda) = 0,$$
$$\delta l^{3,0}(u_\lambda) + a_1^2 l^{2,1}(u_\lambda) = 0.$$

From the first equation it follows that

$$\delta a_2^2 = 0$$

and from the second one that

$$\delta a_2^3 = -a_1^2 a_2^2.$$

This constructs the twisted complex $F^2Y_a = C^*(B, U(B)^i)$, $i = 0, 1, 2$, with three rows. One has $F^1Y_a \subset F^2Y_a$. The differential for the new row is

$$dc^{p,2} = \delta c^{p,2} + a_1^2 c^{p,2} + a_2^3 c^{p,2}.$$

It is obvious that

$$H^i(F^2Y_a) = 0, \quad i < 4.$$

Suppose now that F^qY_a with $q+1$ rows has already been constructed and let us construct $F^{q+1}Y_a$ as follows.

Let $U(B)^{q+1}$ be $H^{q+2}(F^qY_a)$. Choose a basis in $H^{q+2}(F^qY_a)$, say $\{u_\lambda\}$, and for every u_λ choose a cocycle $l(u_\lambda)$ in it, $l(u_\lambda) \in u_\lambda$; let its components be

$$\begin{matrix} l^{2,q}(u_\lambda) & & & & \\ & l^{3,q-1}(u_\lambda) & & & \\ & & \ddots & & \\ & & & l^{q+1,1}(u_\lambda) & \\ & & & & l^{q+2,0}(u_\lambda). \end{matrix}$$

This means that the equations

$$\begin{gathered} \delta l^{2,q}(u_\lambda) = 0, \\ \delta l^{3,q-1}(u_\lambda) + a^2_{q-1} l^{2,q}(u_\lambda) = 0, \\ \delta l^{4,q}(u_\lambda) + a^2_{q-2} l^{3,q-1}(u_\lambda) + a^3_{q-2} l^{2,q}(u_\lambda) = 0, \\ \vdots \\ \delta l^{q+1,1,}(u_\lambda) + a^2_2 l^{q,2}(u_\lambda) + a^3_3 l^{q-1,3}(u_\lambda) + \cdots + a^q_q l^{2,q}(u_\lambda) = 0, \\ \delta l^{q+2,0,}(u_\lambda) + a^2_1 l^{q+1,1}(u_\lambda) + a^3_2 l^{q,2}(u_\lambda) + \cdots + a^{q+1}_q l^{2,q}(u_\lambda) = 0 \end{gathered}$$

are valid. From the first equation it follows that

$$\delta a^{2,-1}_{q+1} = 0;$$

from the second equation that

$$\delta a^{3,-2}_{q+1} = -a^{2,-1}_q a^{2,-1}_{q+1};$$

from the third equation that

$$\delta a^{4,-3}_{q+1} = -a^{2,-1}_{q-1} a^{3,-2}_{q+1} - a^{3,-2}_q a^{2,-1}_{q+1}$$

and so on. From the last equation it follows that

$$\delta a^{q+2,-q-1}_{q+1} = -a^{2,-1}_1 a^{q+1,-q}_{q+1} - \cdots - a^{q,-q+1}_q a^{2,-1}_{q+1}.$$

This constructs the twisted complex $F^{q+1}Y_a = C^*(B, U(B)^i)$, $i = 0, 1, \ldots, q+1$, with $q+2$ rows. It is clear that

$$F^qY_a \subset F^{q+1}Y_a$$

and

$$H^i(F^{q+1}Y_a) = 0, \quad i < q+3.$$

The differential for the new row is

$$dc^{p,q+1} = \delta c^{p,q+1} + a^{2,-1}_{q+1} c^{p,q+1} + a^{3,-2}_{q+1} c^{p,q+1} + \cdots + a^{q+2,-q-1}_{q+1} c^{p,q+1}.$$

This ends the inductive construction of Y^a and a^*_*.

Denote the constructed U, a, Y by $U(B)$, $a(B)$, $Y(B)$, respectively.

Now the first assertion of Theorem 1 in Section 2 and Lemma 3 in Section 3 immediately imply

Theorem 2. *Let*

$$\Omega B \to PB \to B$$

be the path fibration of a simply connected space B and let $Y_{a(B)} = C^*(B, U(B)^*))_{a(B)}$ be some acyclic twisted complex constructed above. Then $H^i(\Omega B, k) = U(B)^i$ and $d(PB) = d(Y_a)$.

5. The Hirsch Model of Fiber

Let $F \to E \xrightarrow{p} B$ be a Serre fibration and $\Omega B \to PB \to B$ the path fibration of the basis. If

$$\Omega B \to F' \to E \tag{5.1}$$

is the fibration induced from the path fibration by $E \to B$, then F' is homotopy equivalent to F. Hence the Hirsch model of fibration (5.1) is the Hirsch model of F. So, by the theorem of the preceding section, the twisting cochain of the model for F' is $p^*(a(B)) \in A_1 = C^*(E, \operatorname{Hom}(U(B)^*, U(B)^*))$, where

$$a(B) \in A = C^*(B, \operatorname{Hom}(U(B)^*, U(B)^*))$$

is constructed in the preceding section. So one has

Theorem 3. *Let F be the fiber of a Serre fibration*

$$F \to E \xrightarrow{p} B,$$

with $\pi_1(B) = 0$ and $(U(B), a(B))$, $a(B) \in A = C^*(B, \operatorname{Hom}(U(B)^*, U(B)^*))$, be the pair constructed in the preceding section; then the twisted product

$$C^*(E, U^*(B))_{p^*(a(B))},$$

where

$$p^* : A = C^*(B, \operatorname{Hom}(U(B)^*, U(B)^*) \to A_1 = C^*(E, \operatorname{Hom}(U(B)^*, U(B)^*))$$

is the induced homomorphism, is the Hirsch model of the fiber F.

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