# MORAVA $K$-THEORY RINGS OF THE EXTENSIONS OF $C_{2}$ BY THE PRODUCTS OF CYCLIC 2-GROUPS. 

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#### Abstract

In 19 Schuster proved that mod 2 Morava $K$-theory $K(s)^{*}(B G)$ is evenly generated for all groups $G$ of order 32 . There exist 51 non-isomorphic groups of order 32. In [12], these groups are numbered by $1, \cdots, 51$. For the groups $G_{38}, \cdots, G_{41}$, that fit in the title, the explicit ring structure is determined in [5]. In particular, $K(s)^{*}(B G)$ is the quotient of a polynomial ring in 6 variables over $K(s)^{*}(p t)$ by an ideal generated by explicit polynomials. In this article we present some calculations using the same arguments in combination with a theorem of [2] on good groups in the sense of Hopkins-Kuhn-Ravenel. In particular, we consider the groups $G_{36}, G_{37}$, each isomorphic to a semidirect product $\left(C_{4} \times C_{2} \times C_{2}\right) \rtimes C_{2}$, the group $G_{34} \cong\left(C_{4} \times C_{4}\right) \rtimes C_{2}$ and its non-split version $G_{35}$. For these groups the action of $C_{2}$ is diagonal, i.e., simpler than for the groups $G_{38}, \cdots, G_{41}$, however the rings $K(s)^{*}(B G)$ have the same complexity.


## 1. Introduction

In this paper we apply the transfer methods to the study of Morava K-theory rings of finite groups.

Among many obvious motivations for studying Morava K-theory rings of finite groups let us mention that elliptic cohomology rings of classifying spaces of finite groups provide especially suitable testbed for studying possible features of tentative elliptic objects, as demonstrated by works of Baker, Thomas, Devoto and others.

For various examples of finite groups the Morava K-theory ring has been shown in the works of N. Yagita to be generated by Chern classes. M. Hopkins, N. Kuhn and D. Ravenel introduced the notion of good groups in the sense that Morava Ktheory of $B G$ is generated by transferred Euler classes of complex representations of subgroups of $G$. This clearly indicates that the part of the relations which can be derived from the properties of the transfer should play decisive role in determining the whole ring structure. Note however not all groups are good by [17.

In [24], [25], 26], 27] the multiplicative structure has only been determined modulo certain indeterminacy. In [19, [20, 21], [22, [23] some artificial generators are suggested in order to obtain explicit relation.

In the works by McClure and Snaith, Hunton and others cohomology groups of homotopy orbit spaces have been constructed, however it is especially interesting to express the ring structure purely in terms of transferred characteristic classes. Thus we are led to consider for finite coverings the interaction of transfers and

[^0]Chern classes along the lines taken in 6]. Initial results for some examples given in [7], 3], 4] and [5] are the first ones to describe multiplicative structure completely in terms of Euler classes and transferred Euler classes

## 2. Preliminaries

Let $K(s)^{*}(B G)$ be the $s$-th Morava $K$-theory of $B G$ at prime $p=2$. Note that by [15], the coefficient ring $K(s)^{*}(p t)$ is the Laurent polynomial ring in one variable $\mathbb{F}_{2}\left[v_{s}, v_{s}^{-1}\right]$, where $\mathbb{F}_{2}$ is the field of 2 elements and $\operatorname{deg}\left(v_{s}\right)=-2\left(2^{s}-1\right)$. So the coefficient ring is a graded field in the sense that all its graded modules are free, therefore Morava $K$-theories enjoy the Kunneth isomorphism. In particular, we have for the cyclic 2-groups

$$
K(s)^{*}\left(B C_{2^{n}} \times B C_{2^{m}}\right)=\mathbb{F}_{2}\left[v_{s}, v_{s}^{-1}\right][u, v] /\left(u^{2^{n s}}, v^{2^{m s}}\right)
$$

where $x$ and $y$ are Euler classes of canonical complex linear representations.
Recall from [13] the following definition.
a) For a finite group $G$, an element $x \in K(s)^{*}(B G)$ is good if it is a transferred Euler class of a complex subrepresentation of $G$, i.e., a class of the form $\operatorname{Tr}^{*}(e(\rho))$, where $\rho$ is a complex representation of a subgroup $H<G, e(\rho) \in K(s)^{*}(B H)$ is its Euler class (i.e., its top Chern class, this being defined since $K(s)^{*}$ is a complex oriented theory), and $\operatorname{Tr}: B G \rightarrow B H$ is the stable transfer map.
(b) $G$ is good if $K(s)^{*}(B G)$ is spanned by good elements as a $K(s)^{*}$ module.

The good groups in the sense, that $K(s)^{\text {odd }}=0$ (this fact is in principle weaker than being good in the sense of Hopkins-Kuhn-Ravenel) also play a role in the literature [19, 20, 21, 26].

In [17], Kriz proved a theorem about the Serre spectral sequence [13]

$$
E_{2}=H^{*}\left(B C_{p}, K(s)^{*}(B H)\right) \Rightarrow K(s)^{*}(B G)
$$

associated to a group extension $1 \rightarrow H \rightarrow G \rightarrow C_{p} \rightarrow 1$ for a $p$-group $G$ and normal subgroup $H \triangleleft G$ with $K(s)^{\text {odd }}(B H)=0$. Namely in 17 Kriz proved that $K(s)^{\text {odd }}(B G)=0$ if and only if the integral Morava $K$-theory $\tilde{K}(s)^{*}(B H)$ is a permutation module for the action of $G / H \cong C_{p}$. Kriz in [17] and Yagita in [26] proved that an extension of a cyclic $p$-group by elementary abelian $p$-group satisfies the even-dimensionality conjecture. For primes greater than 3 , groups of $p$-rank 2 were shown to have even Morava $K$ - theory by Tezuka-Yagita 24 and Yagita [25]. Furthermore, Yagita also proved that these groups are generated by transfered Euler classes and are thus good in the sense of Hopkins-Kuhn-Ravenel. Hunton [14] used "unitary-like embeddings" to show that if $K(s)^{*}(B H)$ is concentrated in even degrees, then so is $K(s)^{*}\left(B H_{2} C_{p}\right)$. An independent proof of the same fact in the sense HKR was given in [13];

One can consider the good groups in the stronger sense, i.e., $K(s)^{*}(B G)$ is generated by good elements as a $K(s)^{*}$-algebra and all generating relations are consequences of formal properties of the characteristic classes and transfer.

There exist 51 non-isomorphic groups of order 32 . Among these groups, the groups with numbers $3,14,16,31,34,39$ and 41 can be presented as a semidirect product $\left(C_{4} \times C_{4}\right) \rtimes C_{2}$, and the groups $G=G_{36}, G_{37}, G_{38}$ - as $\left(C_{4} \times C_{2} \times C_{2}\right) \rtimes C_{2}$.

We consider the examples $G_{34}, \cdots, G_{37}$ to apply the following theorem of [2] on good groups in the sense HKR, which in our case reads as
Theorem 2.1. Let $H_{i}$ and $G_{i}$ be finite 2-group, $i=1,2$, such that $H_{i}$ is good and $G_{i}$ fits into an extension $1 \rightarrow H_{i} \rightarrow G_{i} \rightarrow C_{2} \rightarrow 1$.

Let $G$ fits into an extension of the form $1 \rightarrow H \rightarrow G \rightarrow C_{2} \rightarrow 1$, with diagonal conjugation action of $C_{2}$ on $H=H_{1} \times H_{2}$. Denote by
$T r^{*}=T r_{\varrho}^{*}: K(s)^{*}(B H) \rightarrow K(s)^{*}(B G)$, the transfer homomorphism, associated to 2 -covering $\varrho=\varrho(H, G): B H \rightarrow B G$,
$T r_{i}^{*}=T r_{\varrho_{i}}^{*}: K(s)^{*}\left(B H_{i}\right) \rightarrow K(s)^{*}\left(B G_{i}\right)$, the transfer homomorphism, associated to 2-covering $\varrho_{i}=\varrho\left(H_{i}, G_{i}\right): B H_{i} \rightarrow B G_{i}, i=1,2$,
$\rho_{i}: B G \rightarrow B G_{i}$, the map, induced by projection $H \rightarrow H_{i}$ on $i$-th factor, and let $\rho^{*}$ be the restriction of

$$
\left(\rho_{1}, \rho_{2}\right)^{*}: K(s)^{*}\left(B G_{1} \times B G_{2}\right) \rightarrow K(s)^{*}(B G)
$$

on $K(s)^{*}\left(B G_{1}\right) / I m T r_{1}^{*} \otimes K(s)^{*}\left(B G_{2}\right) / I m T r_{2}^{*}$. Then
i) If $G_{i}$ are good so is $G$.
ii) $K(s)^{*}(B G)$ is spanned, as a $K(s)^{*}(p t)$-module, by elements of $I m T r^{*}$ and $I m \rho^{*}$

For our examples the action of $G / H \cong C_{2}$ is diagonal, i.e., is simpler than for the groups $G_{38}, \cdots, G_{41}$, 5 , but as we shall see the ring structures have the same complexity.

To prove that Theorems 3.1 and 4.1 below, indeed give the ring structure we should
(i) check the relations stated, and then to establish two facts:
(ii) the classes defined generate, and
(iii) the list of relations is complete.

For the first step (i) our tool is the formal properties of the transfer homomorphism.

Let $H \triangleleft G$ be of index 2. Consider the double covering $\varrho: B H \rightarrow B G$. Let

$$
\operatorname{Tr}^{*}=\operatorname{Tr}_{\varrho}^{*}=\operatorname{Tr}^{*}(H, G): K(s)^{*}(B H) \rightarrow K(s)^{*}(B G)
$$

be the associated transfer homomorphism induced by the stable transfer map [1], [16, 10].

For all formal properties enjoyed by the transfer, Frobenius reciprocity and the double coset formula, the best general reference is 22 .

We will need the following transfer formula from [6], which does not work for Morava $K$-theory at $s=1$. Therefore we restrict to $s>1$. Also, we set $v_{s}=1$ throughout the paper.

Let $\xi \rightarrow B H$ be a complex line bundle and $\xi_{\varrho}=\operatorname{Ind}_{H}^{G}(\xi)$ be its Atiyah transfer. Then

$$
\begin{equation*}
c_{1}\left(\xi_{\varrho}\right)=c_{1}(\psi)+\sum_{i=1}^{s-1} c_{1}(\psi)^{2^{s}-2^{i}} c_{2}\left(\xi_{\varrho}\right)^{2^{i-1}}+\operatorname{Tr}^{*}\left(c_{1}(\xi)\right), \tag{1}
\end{equation*}
$$

where $\psi \rightarrow B G$ is the pullback of the canonical line bundle over $B \mathbb{Z} / 2$ along the map $B G \rightarrow B \mathbb{Z} / 2$ classifying $\varrho$.

For the Chern classes $u, v$ and $T r^{*}=\operatorname{Tr}^{*}(H, G)$ the formula (11) implies

$$
\begin{equation*}
\operatorname{Tr}^{*}(u)=c+x_{1}+\sum_{i=1}^{s-1} c^{2^{s}-2^{i}} x_{2}^{2^{i-1}} \tag{2}
\end{equation*}
$$

and

$$
\begin{equation*}
\operatorname{Tr}^{*}(v)=c+y_{1}+\sum_{i=1}^{s-1} c^{2^{s}-2^{i}} y_{2}^{2^{i-1}} \tag{3}
\end{equation*}
$$

One has the following approximation formula for the formal group law in Morava $K$-theory ([7], Lemma 2.2 ii)).

$$
\begin{equation*}
F(x, y)=x+y+\Phi(x, y)^{2^{s-1}} \tag{4}
\end{equation*}
$$

where $\Phi(x, y)=x y+(x y)^{2^{s-1}}(x+y)$ modulo $(x y)^{2^{s-1}}(x+y)^{2^{s-1}}$.
We also will need the following Lemma [5]
Lemma 2.2. The tensor square of a complex plane vector bundle $\zeta$ has the following total Chern class $C\left(\zeta^{\otimes 2}\right)=\left(1+c_{1}^{2}(\operatorname{det} \zeta)\right)\left(1+c_{1}^{2^{s}}(\zeta)+c_{2}^{2^{s}}(\zeta)\right)$.

For the step (ii) our principal tool is the Serre spectral sequence (see [8, 9], [13] for details)

$$
\begin{equation*}
\left\{E_{2}^{*, *}(B G)\right\}=H^{*}\left(B C_{2}, K(s)^{*}(B H)\right) \Rightarrow K(s)^{*}(B G) \tag{5}
\end{equation*}
$$

associated to a group extension $1 \rightarrow H \xrightarrow{i} G \xrightarrow{\pi} C_{2} \rightarrow 1$.
Here $H^{*}\left(B C_{2}, K(s)^{*}(B H)\right)$ denotes the ordinary cohomology of $C_{2}$ with coefficients in the $\mathbb{F}_{2}\left[C_{2}\right]$-module $K(s)^{*}(B H)$, where the action of $C_{2}$ is induced by conjugation in $G$.

As a $C_{2}$-module $K(s)(B H)$ is a direct sum of free $F$ and trivial $T$ modules.
Here

$$
H^{i}\left(B C_{2}, F\right)= \begin{cases}{[F]^{C_{2}}} & \text { for } i=0 \\ 0 & \text { for } i>0\end{cases}
$$

and

$$
H^{*}\left(B C_{2}, T\right)=H^{*}\left(B C_{2}\right) \otimes T
$$

Recall $E_{2}^{0, *}$ is isomorphic to $\left[K(s)^{*}(B H)\right]^{C_{2}}$ via $\varrho^{*}: K(s)^{*}(B G) \rightarrow K(s)^{*}(B H)$. Also, via $\pi^{*}$, the spectral sequence $E_{2}^{*, *}(B G)$ is a module over the Atiyah-Hirzebruch spectral sequence $E_{2}^{*, *}\left(B C_{2}\right)$ that converges to $K(s)^{*}\left(B C_{2}\right)$.

The Frobenius reciprocity of the transfer says that the composition $\varrho^{*} T r^{*}$ is the trace map

$$
\begin{equation*}
\varrho^{*} T r^{*}=1+t, \text { where } t \text { is the generator } t \in G / H \cong C_{p} \tag{6}
\end{equation*}
$$

Clearly, the trace map $\varrho^{*} T r^{*}$ always maps the good elements of $K(s)^{*}(B H)$ onto $[F]^{C_{2}}$. Hence the group $G$ is good iff $T$ is also covered by $\varrho^{*}$-images of good elements.

In general if to operate roughly, even for $s=2$, this requires a serious computational effort, see [22] p.78. One can to simplify this task by suggesting the special bases for particular $C_{2}$-modules $K(s)^{*}(B H)$. This simple but convenient idea first was realized in [5].

Finally one can test (iii) by calculating the rank $\chi_{s}(G)=\frac{1}{2} 16^{s}-\frac{1}{2} 4^{s}+8^{s}$ from the relations. The answer is calculated in [19] using the Euler characteristic formula of Hopkins-Kuhn-Ravenel.

## 3. Groups $G_{36}, G_{37}$

The Morava ring for $G_{38}$, a semidirect product $\cong\left(C_{4} \times C_{2} \times C_{2}\right) \rtimes C_{2}$ was computed in [5]. The relations for the groups $G_{36}, G_{37}$ can be treated similarly (see Section 5). The fact that transferred Chern classes generate (see Subsection 5.2) uses rather different arguments.

Let us recall the presentations

$$
\begin{aligned}
G_{36} & =\left\langle\mathbf{a}, \mathbf{b}, \mathbf{c} \mid \mathbf{a}^{4}=\mathbf{b}^{4}=\mathbf{c}^{2}=[\mathbf{b}, \mathbf{c}]=1, \mathbf{a}^{-1} \mathbf{b} \mathbf{a}=\mathbf{b}^{-1}, \mathbf{c a c}=\mathbf{a}^{-1}\right\rangle \\
G_{37} & =\left\langle\mathbf{a}, \mathbf{b}, \mathbf{c} \mid \mathbf{a}^{4}=\mathbf{c}^{2}=\mathbf{d}^{2}=[\mathbf{b}, \mathbf{c}]=1, \mathbf{d}=[\mathbf{a}, \mathbf{c}], \mathbf{b}^{2}=\mathbf{a}^{2}, \mathbf{b} \mathbf{a b}^{-1}=\mathbf{a}^{-1}\right\rangle .
\end{aligned}
$$

Let $H \cong C_{4} \times C_{2} \times C_{2}$ be the maximal abelian subgroup in $G: H=\left\langle\mathbf{b}, \mathbf{c}, \mathbf{a}^{2}\right\rangle$ for $G=G_{36} ; H=\langle\mathbf{b}, \mathbf{c}, \mathbf{d}\rangle$ for $G=G_{37}$. Let $\lambda, \mu$ and $\nu$ be the complex line bundles over $B H$, the pullbacks of the canonical complex line bundles along the projections onto the first, second and third factor of $H$ respectively, that is,
for $H \triangleleft G_{36}$,

$$
\lambda(\mathbf{b})=i, \nu\left(\mathbf{a}^{2}\right)=\mu(\mathbf{c})=-1, \lambda\left(\mathbf{a}^{2}\right)=\lambda(\mathbf{c})=\nu(\mathbf{b})=\nu(\mathbf{c})=\mu(\mathbf{b})=\mu\left(\mathbf{a}^{2}\right)=1,
$$

and for $H \triangleleft G_{37}$,

$$
\lambda(\mathbf{b})=i, \mu(\mathbf{c})=\nu(\mathbf{d})=-1, \lambda(\mathbf{c})=\lambda(\mathbf{d})=\mu(\mathbf{b})=\mu(\mathbf{d})=\nu(\mathbf{b})=\nu(\mathbf{c})=1
$$

The quotient of $G$ by the center $Z \cong C_{2}^{2}$ is isomorphic to $C_{2}^{3}$. The projections of $G$ on three factors of $C_{2}^{3}$ induce line bundles $\alpha, \beta$ and $\gamma$ over $B G$ :

$$
\alpha(b)=\beta(c)=\gamma(a)=-1, \quad \alpha(a)=\alpha(c)=\beta(a)=\beta(b)=\gamma(b)=\gamma(c)=1
$$

Let us denote Chern classes by

$$
\begin{gathered}
x_{i}=c_{i}\left(\operatorname{Ind}_{H}^{G}(\nu)\right) ; \quad y_{i}=c_{i}\left(\operatorname{Ind}_{H}^{G}(\lambda)\right) ; \\
a=\left\{\begin{array}{l}
c_{1}(\alpha), \text { for } G=G_{36}, \\
c_{1}(\alpha \gamma), \text { for } G=G_{37},
\end{array}\right.
\end{gathered}
$$

and for both cases

$$
b=c_{1}(\beta), \quad c=c_{1}(\gamma)
$$

Let $T r^{*}: K(s)^{*}(B H) \rightarrow K(s)^{*}(B G)$ be the transfer homomorphism [1] associated to the double covering $\rho: B H \rightarrow B G$ and let

$$
T=T r^{*}(u v), \text { where } u=c_{1}(\nu) ; v=c_{1}(\lambda)
$$

One our main result is the following

Theorem 3.1. Let $G$ be one of the groups $G_{36}, G_{37}$. Then

$$
\begin{aligned}
& \text { i) } K(s)^{*}(B G) \cong K(s)^{*}\left[a, b, c, x_{2}, y_{2}, T\right] / R \text {, where the ideal } R \text { is generated by } \\
& a^{2^{s}}, b^{2^{s}}, c^{2^{s}} \text {, } \\
& c\left(c+x_{1}+\sum_{i=1}^{s-1} c^{2^{s}-2^{i}} x_{2}^{2^{i-1}}\right), \quad c\left(c+y_{1}+\sum_{i=1}^{s-1} c^{2^{s}-2^{i}} y_{2}^{2^{i-1}}\right), \\
& a\left(a+y_{1}+\sum_{i=1}^{s-1} a^{2^{s}-2^{i}} y_{2}^{2^{i-1}}\right), \quad b\left(b+x_{1}+\sum_{i=1}^{s-1} b^{2^{s}-2^{i}} x_{2}^{2^{i-1}}\right) \text {, } \\
& \left(c+y_{1}+\sum_{i=1}^{s-1} c^{2^{s}-2^{i}} y_{2}^{2^{i-1}}\right)\left(b+x_{1}+\sum_{i=1}^{s-1} b^{2^{s}-2^{i}} x_{2}^{2^{i-1}}\right)+b^{2^{s}-1} T \text {, } \\
& \left(c+x_{1}+v_{s} \sum_{i=1}^{s-1} c^{2^{s}-2^{i}} x_{2}^{2^{i-1}}\right)\left(a+y_{1}+\sum_{i=1}^{s-1} a^{2^{s}-2^{i}} y_{2}^{2^{i-1}}\right)+a^{2^{s}-1} T \text {, } \\
& T^{2}+T x_{1} y_{1}+x_{2} y_{1}\left(c+y_{1}+\sum_{i=1}^{s-1} c^{2^{s}-2^{i}} y_{2}^{2^{i-1}}\right)+x_{1} y_{2}\left(c+x_{1}+\sum_{i=1}^{s-1} c^{2^{s}-2^{i}} x_{2}^{2^{i-1}}\right) \text {, } \\
& T\left(b+x_{1}+\sum_{i=1}^{s-1} b^{2^{s}-2^{i}} x_{2}^{2^{i-1}}\right)+b^{2^{s}-1} x_{2}\left(c+y_{1}\right) \text {, } \\
& T\left(a+y_{1}+\sum_{i=1}^{s-1} a^{2^{s}-2^{i}} y_{2}^{2^{i-1}}\right)+a^{2^{s}-1} y_{2}\left(c+x_{1}\right), c T \text {, and } \\
& x_{2}^{2^{s}}+\left\{\begin{array}{lc}
c^{2}+b c & \text { for } G=G_{36}, \\
b c & G=G_{37},
\end{array}\right. \\
& y_{2}^{2^{s}}+\left\{\begin{array}{lc}
a^{2}+a c & \text { for } G=G_{36}, \\
a^{2}+a c+c^{2} & G=G_{37},
\end{array}\right.
\end{aligned}
$$

where

$$
\begin{gathered}
x_{1}=\left(x_{2}+x_{1} x_{2}^{2^{s-1}}\right)^{2^{s-1}}+\left\{\begin{array}{lc}
b & \text { for } G=G_{36} \\
b+c+(b c)^{2^{s-1}} & G=G_{37}
\end{array}\right. \\
y_{1}=\left(y_{2}+y_{1} y_{2}^{2^{s-1}}\right)^{2^{s-1}}+\left\{\begin{array}{lc}
c & \text { for } G=G_{36} \\
0 & G=G_{37}
\end{array}\right.
\end{gathered}
$$

ii) Some other relations are

$$
a^{2} c=a c^{2}, \quad b^{2} c=b c^{2}, \quad x_{1}^{2^{s}}=b^{2^{s-1}} c^{2^{s-1}}, \quad y_{1}^{2^{s}}=a^{2^{s-1}} c^{2^{s-1}}
$$

## 4. Group $G_{34}$ And its non-Split version $G_{35}$

Presentations of $G_{34}$ and $G_{35}$ are as follows:

$$
\begin{aligned}
& G_{34}=\left\langle\mathbf{a}, \mathbf{b}, \mathbf{c} \mid \mathbf{a}^{4}=\mathbf{b}^{4}=\mathbf{c}^{2}=[\mathbf{a}, \mathbf{b}]=1, \mathbf{c a c}=\mathbf{a}^{-1}, \mathbf{c b c}=\mathbf{b}^{-1}\right\rangle \\
& G_{35}=\left\langle\mathbf{a}, \mathbf{b}, \mathbf{c} \mid \mathbf{a}^{4}=\mathbf{b}^{4}=[\mathbf{a}, \mathbf{b}]=1, \mathbf{c}^{2}=\mathbf{a}^{2}, \mathbf{c a c}^{-1}=\mathbf{a}^{-1}, \mathbf{c b c}^{-1}=\mathbf{b}^{-1}\right\rangle
\end{aligned}
$$

Let $G$ be either $G_{34}$ or $G_{35}$ and let $H=\langle\mathbf{a}, \mathbf{b}\rangle \cong C_{4} \times C_{4}$ be the maximal abelian subgroup in $G$. Let $\lambda, \nu \rightarrow B H$ denote the pullbacks of the canonical complex line bundles along the projections onto the first and second factor of $H$ respectively:

$$
\lambda(\mathbf{a})=\nu(\mathbf{b})=i, \quad \lambda(\mathbf{b})=\lambda(c)=\nu(\mathbf{a})=\nu(\mathbf{c})=1 .
$$

The quotient of $G$ by the center $Z \cong C_{2}^{2}$ is isomorphic to $C_{2}^{3}$. The projections of $G$ on the three factors induce three line bundles $\alpha, \beta$ and $\gamma$ over $B G$ :

$$
\alpha(a)=\beta(b)=\gamma(c)=-1, \quad \alpha(b)=\alpha(c)=\beta(a)=\beta(c)=\gamma(a)=\gamma(b)=1
$$

For both cases let us denote Chern classes by

$$
\begin{gathered}
x_{i}=c_{i}\left(\operatorname{Ind}_{H}^{G}(\lambda)\right) ; \quad y_{i}=c_{i}\left(\operatorname{Ind}_{H}^{G}(\nu)\right) \\
a=c_{1}(\alpha), \quad b=c_{1}(\beta), \quad c=c_{1}(\gamma)
\end{gathered}
$$

Let $T r^{*}: K(s)^{*}(B H) \rightarrow K(s)^{*}(B G)$ be the transfer homomorphism [1] associated to the double covering $\rho: B H \rightarrow B G$ and let

$$
T=\operatorname{Tr}^{*}(u v), \text { where } u=c_{1}(\lambda) ; \quad v=c_{1}(\nu)
$$

Theorem 4.1. Let $G$ be one of the groups $G_{34}, G_{35}$. Then
i) $K(s)^{*}(B G) \cong K(s)^{*}\left[a, b, c, x_{2}, y_{2}, T\right] / R$, where the ideal $R$ is generated by $a^{2^{s}}, b^{2^{s}}, c^{2^{s}}$,
$c\left(c+x_{1}+\sum_{i=1}^{s-1} c^{2^{s}-2^{i}} x_{2}^{2^{i-1}}\right), \quad c\left(c+y_{1}+\sum_{i=1}^{s-1} c^{2^{s}-2^{i}} y_{2}^{2^{i-1}}\right)$,
$a\left(a+y_{1}+\sum_{i=1}^{s-1} a^{2^{s}-2^{i}} y_{2}^{2^{i-1}}\right), \quad b\left(b+x_{1}+\sum_{i=1}^{s-1} b^{2^{s}-2^{i}} x_{2}^{2^{i-1}}\right)$,
$\left(c+x_{1}+\sum_{i=1}^{s-1} c^{2^{s}-2^{i}} x_{2}^{2^{i-1}}\right)\left(b+y_{1}+\sum_{i=1}^{s-1} b^{2^{s}-2^{i}} y_{2}^{2^{i-1}}\right)+b^{2^{s}-1} T$,
$\left(c+y_{1}+\sum_{i=1}^{s-1} c^{2^{s}-2^{i}} y_{2}^{2^{i-1}}\right)\left(a+x_{1}+\sum_{i=1}^{s-1} a^{2^{s}-2^{i}} x_{2}^{2^{i-1}}\right)+a^{2^{s}-1} T$,
$T^{2}+T x_{1} y_{1}+x_{2} y_{1}\left(c+y_{1}+\sum_{i=1}^{s-1} c^{2^{s}-2^{i}} y_{2}^{2^{i-1}}\right)+x_{1} y_{2}\left(c+x_{1}+\sum_{i=1}^{s-1} c^{2^{s}-2^{i}} x_{2}^{2^{i-1}}\right)$,
$T\left(a+x_{1}+\sum_{i=1}^{s-1} a^{2^{s}-2^{i}} x_{2}^{2^{i-1}}\right)+a^{2^{s}-1} x_{2}\left(c+y_{1}\right)$,
$T\left(b+y_{1}+\sum_{i=1}^{s-1} b^{2^{s}-2^{i}} y_{2}^{2^{i-1}}\right)+b^{2^{s}-1} y_{2}\left(c+x_{1}\right), c T$,
$x_{2}^{2^{s}}+b^{2}+b c, \quad y_{2}^{2^{s}}+a^{2}+a c$,
where

$$
x_{1}=\left(x_{2}+x_{1} x_{2}^{2^{s-1}}\right)^{2^{s-1}}+\left\{\begin{array}{cc}
c & \text { for } G=G_{34} \\
0 & G=G_{35}
\end{array}\right.
$$

and

$$
y_{1}=c+\left(y_{2}+y_{1} y_{2}^{2^{s-1}}\right)^{2^{s-1}}, \text { for both cases }
$$

ii) Some other relations are
$a^{2} c=a c^{2}, \quad b^{2} c=b c^{2}, \quad x_{1}^{2^{s}}=b^{2^{s-1}} c^{2^{s-1}}, \quad y_{1}^{2^{s}}=a^{2^{s-1}} c^{2^{s-1}}$.

## 5. Proof of Theorem 3.1

Bundle relations. Let us give some relations of bundles over $B G$ we will need. We omit the proofs since they are completely standard and easily follow from the definitions and from Frobenius reciprocity of the transfer in complex $K$-theory.

Let $\rho: B H \rightarrow B G$ be the double covering $\rho=\rho(H, G)$ and let $\rho_{!} \lambda=\operatorname{Ind}_{H}^{G}(\lambda)$ and $\rho_{!} \nu=\operatorname{Ind}_{H}^{G}(\nu)$ in both cases. Both groups have the same character table. The only difference is in the determinants of $\rho_{!} \lambda$ and $\rho!\nu$, while restrictions, products, and two lemmas below are the same. Namely

$$
\begin{array}{ll}
\operatorname{det}\left(\rho_{!} \lambda\right)=\gamma, \quad \operatorname{det}\left(\rho_{!} \nu\right)=\beta, & \text { for } G=G_{36} \\
\operatorname{det}(\rho!\lambda)=1, \quad \operatorname{det}(\rho!\nu)=\beta \gamma, & \text { for } G=G_{37}
\end{array}
$$

and for both cases is easily checked the following

Restrictions
i) $\rho^{*} \alpha=\lambda^{2}, \quad \rho^{*} \beta=\mu, \quad \rho^{*} \gamma=1 ;$
i) $\rho^{*}(\rho!\lambda)=\lambda+\lambda^{3}, \quad \rho^{*}(\rho!\nu)=\nu+\nu \mu$;

Product relations
iii) $\alpha \rho_{!} \lambda=\rho_{!} \lambda, \quad \gamma \rho_{!} \lambda=\rho_{!} \lambda$;
iv) $\beta \rho_{!} \nu=\rho_{!} \nu, \quad \gamma \rho_{!} \nu=\rho_{!} \nu$;
v) $\left(\rho_{!} \nu\right)^{2}=1+\beta+\gamma+\beta \gamma$;
vi) $\left(\rho_{!} \lambda\right)^{2}=1+\alpha+\gamma+\alpha \gamma$.

The first relation of iii) suggests that $\rho_{!} \lambda$ should be also a transfer line bundle over the total space of a 2 -covering corresponding to $\alpha$. Namely we have

Lemma 5.1. Let $G=G_{36}$. There exists a subgroup $H^{\prime} \subset G_{36}$ and a line bundle $\lambda^{\prime} \rightarrow B H^{\prime}$ such that $\operatorname{Ind}_{H}^{G}(\lambda)=\operatorname{Ind} H_{H^{\prime}}^{G}\left(\lambda^{\prime}\right)$. The bundle $\alpha$ restricts to trivial bundle over $B H^{\prime}$.

Proof. One can choose the group $H^{\prime}=\left\langle\mathbf{b}^{2}, \mathbf{c}, \mathbf{a}\right\rangle$ and the line bundle $\lambda^{\prime}$ represented by $\lambda^{\prime}(\mathbf{a})=1, \lambda^{\prime}\left(\mathbf{b}^{2}\right)=-1, \lambda^{\prime}(\mathbf{c})=1$.

We will need also the following three easy facts.
Lemma 5.2. Let $G=G_{36}$. There exists $K^{\prime} \subset G$ and a line bundle $\nu^{\prime} \rightarrow B K^{\prime}$ such that $\operatorname{Ind}_{H}^{G}(\nu)=\operatorname{Ind}_{K^{\prime}}^{G}\left(\nu^{\prime}\right)$. The bundle $\beta$ restricts to trivial bundle over $B K^{\prime}$.
Proof. Put $K^{\prime}=\langle b, a\rangle$ and $\nu^{\prime}(b)=1, \nu^{\prime}(a)=i$.

Lemma 5.3. Let $G=G_{37}$. There exists $H^{\prime \prime} \subset G_{37}$ and a line bundle $\lambda^{\prime \prime} \rightarrow B H^{\prime \prime}$ such that $\operatorname{Ind}_{H}^{G}(\lambda)=\operatorname{Ind}_{H^{\prime \prime}}^{G}\left(\lambda^{\prime \prime}\right)$. The bundle $\alpha \gamma$ restricts to trivial bundle over $B H^{\prime \prime}$.

Proof. Let $H^{\prime \prime}=\left\langle\mathbf{b}^{2}, \mathbf{c}, \mathbf{a b}\right\rangle$ and $\lambda^{\prime \prime}$ be represented by $\lambda^{\prime \prime}(\mathbf{a b})=i, \lambda^{\prime \prime}\left(\mathbf{b}^{\mathbf{2}}\right)=-1$, $\lambda^{\prime \prime}(\mathbf{c})=1$.

Lemma 5.4. Let $G=G_{37}$. There exists $K^{\prime \prime} \subset G$ and a line bundle $\nu^{\prime \prime} \rightarrow B K^{\prime \prime}$ such that $\operatorname{Ind}_{H}^{G}(\nu)=\operatorname{Ind}_{K^{\prime \prime}}^{G}\left(\nu^{\prime \prime}\right)$. The bundle $\beta$ restricts to trivial bundle over $B K^{\prime \prime}$.
Proof. Let $K^{\prime \prime}=\langle\mathbf{b}, \mathbf{d}, \mathbf{a}\rangle$ and $\nu^{\prime \prime}(\mathbf{a})=\nu^{\prime \prime}(\mathbf{b})=1, \nu^{\prime \prime}(\mathbf{d})=-1$.
5.1. Relations of Theorem 3.1. Clearly the relations

$$
a^{2^{s}}=b^{2^{s}}=c^{2^{s}}=0
$$

are immediate consequences of the bundle relations $\alpha^{2}=\beta^{2}=\gamma^{2}=1$ for all cases.
The 4th and 5 th relations follow from (2) and (3) respectively.
For the 6 th relation

$$
a\left(a+y_{1}+\sum_{i=1}^{s-1} a^{2^{s}-2^{i}} y_{2}^{2^{i-1}}\right)=0
$$

consider the transfer $T r^{*}$ of the double covering $\rho: B H^{\prime} \rightarrow B G$ in case $G_{36}$ and apply formula (1) and Lemma 5.1. Then it follows that the second factor of the relation is the transfer of $c_{1}\left(\lambda^{\prime}\right)$, i.e.,

$$
\operatorname{Tr}^{\prime *}\left(c_{1}\left(\lambda^{\prime}\right)\right)=\left(a+y_{1}+\sum_{i=1}^{s-1} a^{2^{s}-2^{i}} y_{2}^{2^{i-1}}\right)
$$

Therefore

$$
a \operatorname{Tr}^{\prime *}\left(c_{1}\left(\lambda^{\prime}\right)\right)=\operatorname{Tr}^{\prime *}\left(\rho^{*}(a) c_{1}\left(\lambda^{\prime}\right)\right)=\operatorname{Tr}^{\prime *}\left(0 \cdot c_{1}\left(\lambda^{\prime}\right)\right)=0
$$

Similarly, for $G_{37}$ apply formula (11) and Lemma 5.3
In the same spirit, apply formula (1), Lemma 5.2 and Lemma 5.4 for the 7 th relation

$$
b\left(b+x_{1}+\sum_{i=1}^{s-1} b^{2^{s}-2^{i}} x_{2}^{2^{i-1}}\right)=0
$$

Now note that the 4 th and 6 th relations imply $a^{2} c=a c^{2}$, the first relations of Theorem 3.1 ii). For this multiply the 4 th relation by $a$ and the 6 th relation by $c$. The sum of these terms equals $a^{2} c+a c^{2}$ up to an invertible factor. Similarly the 5 th and 7 th relation imply $b^{2} c=b c^{2}$, the second relation of Theorem 3.1 ii). Note also

$$
\begin{equation*}
a^{i} c^{j}=0, \quad b^{i} c^{j}=0, i+j>2^{s} \tag{7}
\end{equation*}
$$

The first one is the consequence of $a^{2} c=a c^{2}$ and $a^{2^{s}}=c^{2^{s}}=0$. Similarly the second one follows from $b^{2} c=b c^{2}$ and $b^{2^{s}}=c^{2^{s}}=0$.

For the decompositions of $x_{2}^{2^{s}}, y_{2}^{2^{s}}$, (also for the formulas for $x_{1}^{2^{s}}$ and $y_{1}^{2^{s}}$ of Theorem 3.1 ii) we need the material of Section 3. Namely we have to apply Lemma 2.2 to all induced representations given in Section 3 and take into account that their determinants can written in terms of the bundles $\alpha, \beta, \gamma$. For example for $G=G_{36}$

$$
x_{2}^{2^{s}}=c^{2}+b c, x_{1}^{2^{s}}=b^{2^{s-1}} c^{2^{s-1}}
$$

and

$$
y_{2}^{2^{s}}=a^{2}+a c, \quad y_{1}^{2^{s}}=a^{2^{s-1}} c^{2^{s-1}}
$$

are the consequences of product relations v) and vi). Let us prove first two relations. Equate Chern classes in the bundle relation of v). Then taking into account (7) we get from the equation for the first Chern classes

$$
x_{1}^{2^{s}}=b+c+b+c+b^{2^{s-1}} c^{2^{s-1}}=a^{2^{s-1}} c^{2^{s-1}}
$$

For the decomposition of $x_{2}^{2^{s}}$ apply the equation for the second Chern classes and take into account (7):

$$
\begin{aligned}
x_{2}^{2^{s}} & =c_{2}\left(\rho_{!} \nu^{2}\right)+c_{1}\left(\operatorname{det} \rho_{!} \nu\right)^{2} \\
& =c_{2}(1+\beta+\gamma+\beta \gamma)+c_{1}(\beta)^{2}=b c+b F(b, c)+c F(b, c)+b^{2} \\
& =b c+c^{2}
\end{aligned}
$$

The 8th and 9th relations.

$$
\left(c+y_{1}+\sum_{i=1}^{s-1} c^{2^{s}-2^{i}} y_{2}^{2^{i-1}}\right)\left(b+x_{1}+\sum_{i=1}^{s-1} b^{2^{s}-2^{i}} x_{2}^{2^{i-1}}\right)=T b^{2^{s}-1}
$$

Proof. Let $G=G_{36}$. Consider the diagram


Then the left hand side of our relation is equal to

$$
\begin{array}{ll}
\operatorname{Tr}_{\gamma}^{*}(v)\left(b+x_{1}+\sum_{i=1}^{s-1} b^{2^{s}-2^{i}} x_{2}^{2^{i-1}}\right) & \text { by transfer formula (1) } \\
=\operatorname{Tr}_{\gamma}^{*}(v) \operatorname{Tr}_{\beta}^{*}\left(c_{1}\left(\nu^{\prime}\right)\right) & \text { by Lemma } 5.2 \\
=\operatorname{Tr}_{\gamma}^{*}\left(v \cdot \rho_{\gamma}^{*} \operatorname{Tr}_{\beta}^{*}\left(c_{1}\left(\nu^{\prime}\right)\right)\right) & \text { by Frobenius reciprocity of the transfer } \\
=\operatorname{Tr}_{\gamma}^{*}\left(v \cdot \operatorname{Tr}_{\mu}^{*}\left(\rho_{\mu}^{*}(u)\right)\right) & \text { by the double coset formula and Lemma } 5.2 \\
=\operatorname{Tr}_{\gamma}^{*}\left(\operatorname{Tr}_{\mu}^{*}\left(\rho_{\mu}^{*}(u v)\right)\right) & \text { by Frobenius reciprocity } \\
=\operatorname{Tr}_{\gamma}^{*}\left(u v \cdot \operatorname{Tr}_{\mu}^{*}(1)\right)=\operatorname{Tr}_{\gamma}^{*}\left(u v \cdot c_{1}^{2^{s}-1}(\mu)\right) & \text { by the formula for } \operatorname{Tr}^{*}(1) \\
=\operatorname{Tb}^{2^{s}-1} . &
\end{array}
$$

Similarly

$$
\left(c+x_{1}+\sum_{i=1}^{s-1} c^{2^{s}-2^{i}} x_{2}^{2^{i-1}}\right)\left(a+y_{1}+\sum_{i=1}^{s-1} a^{2^{s}-2^{i}} y_{2}^{2^{i-1}}\right)=T a^{2^{s}-1}
$$

Proof. Consider the diagram


With this notation the left hand side of the above relation is equal to

$$
\begin{array}{ll}
\operatorname{Tr}_{\gamma}^{*}(u) \operatorname{Tr}_{\alpha}^{*}\left(c_{1}\left(\lambda^{\prime}\right)\right) & \text { by Lemma } 5.1 \text { and formula (1) } \\
=\operatorname{Tr}_{\gamma}^{*}\left(u \cdot \rho_{\gamma}^{*} \operatorname{Tr}_{\alpha}^{*}\left(c_{1}\left(\lambda^{\prime}\right)\right)\right) & \\
=\operatorname{Tr}_{\gamma}^{*}\left(u \cdot \operatorname{Tr}_{\lambda^{2}}^{*}\left(\rho_{\lambda^{2}}^{*}(v)\right)\right) & \text { by Frobenius reciprocity of the tre double coset formula and I } \\
=\operatorname{Tr}_{\gamma}^{*}\left(\operatorname{Tr}_{\lambda^{2}}^{*}\left(\rho_{\lambda^{2}}^{*}(u v)\right)\right) & \text { by Frobenius reciprocity } \\
=\operatorname{Tr}_{\gamma}^{*}\left(u v \cdot \operatorname{Tr}_{\lambda^{2}}^{*}(1)\right)=\operatorname{Tr}_{\gamma}^{*}\left(u v \cdot c_{1}^{2^{s}-1}\left(\lambda^{2}\right)\right) & \text { by the formula for } \operatorname{Tr}^{*}(1) \\
=\operatorname{Ta}^{2^{s}-1} & \\
\text { by the definitions of } \alpha, \lambda, \text { and } T .
\end{array}
$$

For the case of $G=G_{37}$ the proof is analogous. We just have to apply Lemma 5.4 and Lemma 5.3 .

The same applies to 11 th and 12 th relations and may be proved for both groups simultaneously. In each case we will arrive at

$$
T\left(a+y_{1}+\sum_{i=1}^{s-1} a^{2^{s}-2^{i}} y_{2}^{2^{i-1}}\right)=a^{2^{s}-1} \operatorname{Tr}^{*}\left(u v^{2}\right)
$$

or

$$
T\left(b+x_{1}+\sum_{i=1}^{s-1} b^{2^{s}-2^{i}} x_{2}^{2^{i-1}}\right)=b^{2^{s}-1} \operatorname{Tr}^{*}\left(u^{2} v\right)
$$

Therefore we will need that for the involution $t \in C_{2}=G / H$ one has by Frobenius reciprocity
i) $\left.\operatorname{Tr}^{*}\left(u^{2} v\right)=\operatorname{Tr}^{*}(u v(u+t u)-v u t u)\right)=\operatorname{Tr}^{*}(u v) x_{1}-\operatorname{Tr}^{*}(v) x_{2}$,
ii) $\left.\operatorname{Tr}^{*}\left(u v^{2}\right)=T r^{*}(u v(v+t v)-u v t v)\right)=\operatorname{Tr}^{*}(u v) y_{1}-\operatorname{Tr}^{*}(u) y_{2}$.

Here is details for $G=G_{37}$. Apply the diagram


Then the above equality i) gives

$$
T\left(b+x_{1}+\sum_{i=1}^{s-1} b^{2^{s}-2^{i}} x_{2}^{2^{i-1}}\right)+b^{2^{s}-1} T x_{1}+b^{2^{s}-1} x_{2}\left(c+y_{1}+\sum_{i=1}^{s-1} c^{2^{s}-2^{i}} y_{2}^{2^{i-1}}\right)=0
$$

Then the second summand is zero by the 6 th relation and (7). The third summand is equal to $b^{2^{s}-1} x_{2}\left(c+y_{1}\right)$ by (7). This gives the 11th relation.

Similarly, applying the diagram

and the above equality ii) we have
$T\left(a+y_{1}+\sum_{i=1}^{s-1} a^{2^{s}-2^{i}} y_{2}^{2^{i-1}}\right)+a^{2^{s}-1} T y_{1}+a^{2^{s}-1} y_{2}\left(c+x_{1}+\sum_{i=1}^{s-1} c^{2^{s}-2^{i}} x_{2}^{2^{i-1}}\right)=0$.
Again the second summand is zero by the 7 th relation and (7) and the third summand is equal to $a^{2^{s}-1} y_{2}\left(c+x_{1}\right)$. The 12 th relation follows.

The relation $c T=0$ is easy:

$$
c T \equiv c \operatorname{Tr}_{\gamma}^{*}(u v)=\operatorname{Tr}_{\gamma}^{*}\left(u v \gamma^{*}(c)\right)=\operatorname{Tr}_{\gamma}^{*}(u v \cdot 0)=0
$$

Now let us prove the 10th relation.
Let $u^{\prime}=t u$ and $v^{\prime}=t v$, where $t$ is the involution as in (6) and $T r^{*}=T r_{\gamma}^{*}$.
Then

$$
\begin{aligned}
\operatorname{Tr}^{*}(u v)+ & \operatorname{Tr}^{*}\left(u v^{\prime}\right)=\operatorname{Tr}^{*}\left(u\left(v+v^{\prime}\right)\right)=\operatorname{Tr}^{*}(u) \operatorname{Tr}^{*}(v) \\
\operatorname{Tr}^{*}(u v) \operatorname{Tr}^{*}\left(u v^{\prime}\right) & =\operatorname{Tr}^{*}\left(u v\left(u v^{\prime}+u^{\prime} v\right)\right) \\
& =\operatorname{Tr}^{*}\left(u^{2} v v^{\prime}\right)+\operatorname{Tr}^{*}\left(v^{2} u u^{\prime}\right)=\operatorname{Tr}^{*}\left(u^{2}\right) y_{2}+\operatorname{Tr}^{*}\left(v^{2}\right) x_{2}
\end{aligned}
$$

Also

$$
\begin{aligned}
& \operatorname{Tr}^{*}\left(u^{2}\right)=\operatorname{Tr}^{*}\left(u\left(u+u^{\prime}\right)-u u^{\prime}\right)=\operatorname{Tr}^{*}(u) x_{1}-\operatorname{Tr}^{*}(1) x_{2} \\
& \operatorname{Tr}^{*}\left(v^{2}\right)=\operatorname{Tr}^{*}\left(v\left(v+v^{\prime}\right)-v v^{\prime}\right)=\operatorname{Tr}^{*}(v) y_{1}-\operatorname{Tr}^{*}(1) y_{2}
\end{aligned}
$$

Now we apply these formulas and take into account $\operatorname{Tr}^{*}(1) x_{1}=\operatorname{Tr}^{*}(1) y_{1}=0$. This gives the quadratic equation in $T=\operatorname{Tr}_{\gamma}^{*}(u v)$

$$
\begin{aligned}
T^{2} & =T\left(c+x_{1}+\sum_{i=1}^{s-1} c^{2^{s}-2^{i}} x_{2}^{2^{i-1}}\right)\left(c+y_{1}+\sum_{i=1}^{s-1} c^{2^{s}-2^{i}} y_{2}^{2^{i-1}}\right) \\
& +x_{2} y_{1}\left(c+y_{1}+\sum_{i=1}^{s-1} c^{2^{s}-2^{i}} y_{2}^{2^{i-1}}\right)+x_{1} y_{2}\left(c+x_{1}+\sum_{i=1}^{s-1} c^{2^{s}-2^{i}} x_{2}^{2^{i-1}}\right)
\end{aligned}
$$

Now to get the 10th relation apply $c T=0$.

The decompositions for $x_{1}$ and $y_{1}$ are the consequences of the formula (4) applied to the determinants of $\rho_{!} \nu$ and $\rho_{!} \lambda$.

We need $\left(x_{1} x_{2}\right)^{2^{s-2}}=0$ and $\left(y_{1} y_{2}\right)^{2^{s-2}}=0$. It follows from the relations ii) of Theorem 3.1 that moreover we have $\left(x_{1} x_{2}\right)^{2^{s}}=\left(y_{1} y_{2}\right)^{2^{s}}=0$ : decompositions $x_{1}^{2^{s}}=(b c)^{2^{s-1}}$ and $y_{1}^{2^{s}}=(a c)^{2^{s-1}}$ imply

$$
x_{1}^{2^{s}} b=x_{1}^{2^{s}} c=x_{1}^{2^{s}} a^{2}=y_{1}^{2^{s}} a=y_{1}^{2^{s}} c=y_{1}^{2^{s}} b^{2}=0
$$

as $a^{2} c=a c^{2}, b^{2} c=b c^{2}$ and $a^{2^{s}}=b^{2^{s}}=c^{2^{s}}=0$.
That is, all the terms of the above decomposition of $x_{2}^{2^{s}}$ annihilate $x_{1}^{2^{s}}$. Similarly for $y_{1}$ and $y_{2}$. It is also clear that for computing the Euler classes of the determinants (see Section 3) in each case we need only the initial fragment of the formal group law $F(x, y)=x+y+(x y)^{2^{s-1}}$.

For $G_{36}$ we have to apply the formula (4) and take into account that $\operatorname{det}(\rho!\lambda)=\gamma$ and $\operatorname{det}(\rho!\nu)=\beta$.

For $G_{37}$ we have $\operatorname{det}\left(\rho_{!} \lambda\right)=1$ and $\operatorname{det}\left(\rho_{!} \nu\right)=\beta \gamma$, hence $e\left(\operatorname{det}\left(\rho_{!} \lambda\right)\right)=1$ and $e\left(\operatorname{det}\left(\rho_{!} \lambda\right)\right)=F(b, F(b, c))=b+c+(b c)^{2^{s-1}}$.

### 5.2. Generators. Here we prove the following

Lemma 5.5. Let $G$ be one of the groups $G_{34}, \ldots, G_{37}$. Then $K(s)^{*}(B G)$ is generated by $c, a, b, x_{2}, y_{2}, T$ as a $K(s)^{*}(p t)$ algebra.

Proof. Recall $G_{34}$ and $G_{35}$ have maximal abelian subgroup $H=\langle\mathbf{a}, \mathbf{b}\rangle \cong C_{4} \times C_{4}$ on which the quotient acts diagonally by inverting $\mathbf{a}$ and $\mathbf{b}$. The dihedral group $D_{8}$ could be written as a semidirect product $C_{4} \rtimes C_{2}$ with that action. Thus as a $\mathcal{C}_{2}$ module $K(s)^{*}(B H)=M \otimes M$, where $M=K(s)^{*}\left(B C_{4}\right)$.

Consider the decomposition of $K(s)^{*}(B H)$ into free and trivial $C_{2}$-modules

$$
\left[K(s)^{*}(B H)\right]^{C_{2}}=(F)^{C_{2}}+T
$$

Denote the restrictions of the generators of Theorem 3.1 to $K(s)^{*}(B H)$ with the same symbols but with bars.

Let $A$ be the subalgebra in $K(s)^{*}(B G)$ generated by $c, a, b, x_{2}, y_{2}, T$. Clearly the composition $\rho^{*} T r^{*}=\operatorname{Norm}=1+$ involution is onto $(F)^{C_{2}}$. But $\operatorname{ImTr}^{*} \subset A$ : As $u^{2}=u \overline{x_{1}}-\overline{x_{2}}$ and $v^{2}=v \overline{y_{1}}-\overline{y_{2}}$ any polynomial in $u, v$ can be uniquely written as $g_{0}+g_{1} u+g_{2} v+g_{3} u v$ where $g_{i}=g_{i}\left(\overline{x_{1}}, \overline{y_{1}}, \overline{x_{2}}, \overline{y_{2}}\right)$. Therefore one can use Frobenius reciprocity of the transfer and write transfer of any element in $T r^{*}(u), T r^{*}(v), T r^{*}(u v)$, and $c, x_{1}, x_{2}, y_{1}, y_{2}$. Then one can use the formulas (2), (3) for $\operatorname{Tr}^{*}(u), \operatorname{Tr}^{*}(v)$ respectively and the decompositions of $x_{1}$ and $y_{1}$ of Theorem 3.1 and see that

$$
\begin{equation*}
\operatorname{ImTr}^{*} \subset A \tag{12}
\end{equation*}
$$

Therefore $\rho^{*}(A)$ is onto $(F)^{C_{2}}$.
Then we have to check whether the invariants in $T$ are also covered by $\rho^{*} A$. Let $\chi_{s}\left(F^{C_{2}}\right)$ and $\chi_{s}(T)$ be the $K(s)^{*}$-Euler characteristics. Then

$$
\chi_{s}(B H)=2 \chi_{s}(F)+\chi_{s}(T), \quad \chi_{s}(B H)^{C_{2}}=\chi_{s}(F)+\chi_{s}(T)
$$

Let $M^{C_{2}}=\left(F^{\prime}\right)^{C_{2}}+T^{\prime}$. Then one can easily read off the bases for $\left(F^{\prime}\right)^{C_{2}}$ and $T^{\prime}$. In notations of [7], p. 3712, the base sets for $T^{\prime}$ and $F^{\prime}$ are as follows

$$
\begin{aligned}
\chi_{s}\left(F^{\prime C_{2}}\right) & =2^{2 s-1}-2^{s-1}, & & \left\{c_{1} c_{2}^{j} \mid 0 \leq j<2^{s-1}, c_{2}^{k}, k \neq j, k \neq j+2^{s-1}\right\} \\
\chi_{s}\left(T^{\prime}\right) & =2^{s}, & & \left\{c_{2}^{i}, x c_{2}^{i} \mid 0 \leq i \leq 2^{s-1}-1\right\} .
\end{aligned}
$$

The latter fact with (12) already completes the proof of Lemma 5.5 if apply Theorem 2.1. Moreover, actually we also computed $\chi_{s}(G)$ : One has

$$
\chi_{s}(T)=\chi_{s}\left(T^{\prime}\right)^{2}=4^{s} \quad \text { and } \quad \chi_{s}(F)=2 \chi_{s}\left(F^{\prime}\right) \chi_{s}\left(T^{\prime}\right)+2 \chi_{s}\left(F^{\prime}\right)^{2}=\frac{1}{2} 16^{s}-\frac{1}{2} 4^{s}
$$

Thus we get already known fact from [19]

$$
\begin{equation*}
\chi_{s}(G)=\chi_{s}(F)+2^{s} \chi_{s}(T)=\frac{1}{2} 16^{s}-\frac{1}{2} 4^{s}+8^{s} \tag{13}
\end{equation*}
$$

$G_{36}$ contains the maximal abelian subgroup $H=\left\langle\mathbf{b}, \mathbf{a}^{2}, \mathbf{c}\right\rangle \cong C_{4} \times C_{2} \times C_{2}$. From the definition one reads off that $K(s)^{*}(B H)=M \otimes N$, where $N=K(s)^{*}\left(B C_{2} \times\right.$ $\left.B C_{2}\right)$, with the switch action. The situation is similar for $G_{37}$ and $H=\langle\mathbf{b}, \mathbf{c}, \mathbf{d}\rangle$.
5.3. End of the proof. After Lemma 5.5] it suffices to verify that for all our groups the defining relations of Theorem 3.1 give us a ring of Euler characteristic (13), already computed in [19]. For each of our groups, we have to choose a basis for $K(s)^{*}(B G)$ as the union of a basis for $K(s)^{*}(B G) /$ ker $^{*}$, and a basis for ker $\rho^{*}$, where $\rho: B H \rightarrow B G$.

Lemma 5.6. $A$ basis for $K(s)^{*}(B G), G=G_{36}, G_{37}$ is

$$
\begin{aligned}
& \left\{x_{1}^{i} y_{1}^{j} x_{2}^{k} y_{2}^{l} \mid i, j<2^{s}, k, l<2^{s-1}\right\} \\
& \left\{a x_{2}^{k+2^{s-1}} y_{2}^{l} \mid k, l<2^{s-1}\right\} \\
& \left\{x_{1}^{i} a x_{2}^{k} y_{2}^{l}, y_{1}^{i} x_{2}^{k+2^{s-1}} y_{2}^{l} \mid i<2^{s}, k, l<2^{s-1}\right\} \\
& \left\{T x_{1}^{i} y_{1}^{j} x_{2}^{k} y_{2}^{l} \mid i, j<2^{s}-1, k, l<2^{s-1}\right\} \\
& \left\{c^{i} x_{2}^{j} y_{2}^{k}, c^{i} a x_{2}^{j} y_{2}^{k} \mid 0<i<2^{s}, j<2^{s}, k<2^{s-1}\right\} .
\end{aligned}
$$

Proof. One can verify that a) the first four sets give a basis for $K(s)^{*}(B G) / k e r \rho^{*}$, and b) the last set forms a basis for $k e r \rho^{*}$.
For a) choose the lexicographic ordering corresponding to ( $T, a, b, y_{2}, x_{2}, y_{1}, x_{1}, c$ ) in that order.
For b) choose the lexicographic ordering corresponding to ( $\left.T, x_{1}, y_{1}, a, b, x_{2}, y_{2}, c\right), l p$. To read off the required basis b) we have to do one more thing. Namely we have replace the monomials $x_{2}^{j} y_{2}^{2^{2 s-1}+k}=x_{2}^{j} y_{2}^{2 s-1} y^{k}$ by ac ${2^{s}-1}^{x_{2}^{j}} y^{k}, j<2^{s}, k<2^{s-1}$. This can be done by relations of Theorem 3.1

$$
y_{2}^{2^{2 s-1}}=y_{1}^{2^{s}}=(a c)^{2^{s-1}}=a c^{2^{s}-1}
$$

Note the last set of Lemma 5.6 is the union of two sets $\left\{c^{i} x_{2}^{j} y_{2}^{k}, c^{i} a x_{2}^{j} y_{2}^{k}, \mid 0<i<2^{s}-1\right\}$, a basis of $k e r \rho^{*} \bigcap K(s)^{*}(B G) / I m T r^{*}$, and $\left\{c^{2^{s}-1} x_{2}^{j} y_{2}^{k}, c^{2^{s}-1} a x_{2}^{j} y_{2}^{k}\right\}$, a basis of $\operatorname{Tr}^{*}(T)$, the image of the trivial module $T$. For the last sentence recall $\operatorname{Tr}^{*}(1)=v_{s} c^{c^{s}-1}$.

## 6. Proof of Theorem 4.1

The proof could be rather short, since it uses the same arguments. Let us give some relations of bundles we will need. Even if no explicit use is made of these relations in the text, it does help to have them on record should the reader be inclined to spell out proofs for $G=G_{34}, G_{35}$.
Bundle relations. Let $\rho: B H \rightarrow B G$ be the double covering $\rho=\rho(H, G)$ and let $\rho_{!} \lambda=\operatorname{Ind}_{H}^{G}(\lambda)$ and $\rho_{!} \nu=\operatorname{Ind}_{H}^{G}(\nu)$ in both cases.

One has for $G=G_{34}$

| Determinants | Product relations |
| :--- | :--- |
| $\operatorname{det}(\rho!\lambda)=\gamma, \operatorname{det}(\rho!\nu)=\gamma ;$ | iii) $\alpha \rho!\lambda=\rho!\lambda, \gamma \rho!\lambda=\rho!\lambda ;$ |
| Restrictions | iv) $\beta \rho!\nu=\rho!\nu, \gamma \rho!\nu=\rho!\nu ;$ |
| i) $\rho^{*} \alpha=\lambda^{2}, \quad \rho^{*} \beta=\nu^{2}, \quad \rho^{*} \gamma=1 ;$ | v) $(\rho!\nu)^{2}=1+\beta+\gamma+\beta \gamma ;$ |
| ii) $\rho^{*}(\rho!\lambda)=\lambda+\lambda^{3}, \quad \rho^{*}(\rho!\nu)=\nu+\nu^{3} ;$ | vi) $(\rho!\lambda)^{2}=1+\alpha+\gamma+\alpha \gamma$. |

$G_{35}$ has the same character table as $G_{34}$. The only difference is in the determinant of $\lambda_{!}$, while restrictions, products, and the two lemmas above are the same.

$$
\operatorname{det}(\rho!\lambda)=1, \operatorname{det}(\rho!\nu)=\gamma
$$

The first relation of iii) suggests that $\rho_{!} \lambda$ should be also transfer of a line bundle over the total space of a 2-covering corresponding to $\alpha$. Namely, we have

Lemma 6.1. Let $G=G_{34}$ and $G=G_{35}$. There exists $H^{\prime} \subset G$ and a line bundle $\lambda^{\prime} \rightarrow B H^{\prime}$ such that $\operatorname{Ind}_{H}^{G}(\lambda)=\operatorname{Ind} H_{H^{\prime}}^{G}\left(\lambda^{\prime}\right)$. The bundle $\alpha$ restricts to trivial bundle over $B H^{\prime}$.

Proof. One can choose the group $H^{\prime}=\left\langle\mathbf{a}^{2}, \mathbf{b}, \mathbf{c}\right\rangle$ and the line bundle $\lambda^{\prime}$ represented by $\lambda^{\prime}\left(\mathbf{a}^{2}\right)=-1, \lambda^{\prime}(\mathbf{b})=1, \lambda^{\prime}(\mathbf{c})=1 ; \mu\left(\mathbf{a}^{2}\right)=1, \mu(\mathbf{b})=1, \mu(\mathbf{c})=-1$.

Similarly we have the following lemma for $\rho_{!} \nu$ by the first equality of iv).
Lemma 6.2. Let $G=G_{34}$ and $G=G_{35}$ There exists $H^{\prime \prime} \subset G$ and a line bundle $\nu^{\prime} \rightarrow B H^{\prime \prime}$ such that $\operatorname{Ind}_{H}^{G}(\nu)=\operatorname{Ind}_{H^{\prime \prime}}^{G}\left(\nu^{\prime}\right)$. The bundle $\beta$ restricts to trivial bundle over $B H^{\prime \prime}$.

Proof. One can choose the group $H^{\prime \prime}=\left\langle\mathbf{a}, \mathbf{b}^{2}, \mathbf{c}\right\rangle$ and the line bundle $\nu^{\prime}$ represented by $\nu^{\prime}(\mathbf{a})=1, \nu^{\prime}\left(\mathbf{b}^{2}\right)=-1, \nu^{\prime}(\mathbf{c})=1$.

End of the proof. Using the above information on complex representations of the groups $G_{34}$ and $G_{35}$, the proof is completely similar to that of Theorem 3.1 and is left to the reader.

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## References

1. J.F. Adams : Infinite loop spaces, Annals of Mathematics Studies, Princeton University Press, Princeton, (1978).
2. M. Bakuradze: Morava K-theory rings of the extensions of $C_{p}$ by the products of good groups under diagonal action, Georgian Math. J., 22 (4)(2015), 451-455.
3. M. Bakuradze : Morava K-theory rings for the modular groups in Chern classes, $K$-theory, 38, 2(2008), 87-94.
4. M. Bakuradze : Induced representations, Transferred Chern classes and Morava rings $K(s)^{*}(B G)$ : some calculations, Proc. Steklov Inst. of Math. 275(2011), 160-168.
5. M. Bakuradze, M. Jibladze : Morava K-theory rings of groups $G_{38}, \ldots, G_{41}$ of order 32, J. K-Theory, 13(2014), 171-198
6. M. Bakuradze, S. Priddy : Transferred Chern classes in Morava K-theory, Proc. Amer. Math. Soc., 132(2004), 1855-1860.
7. M. Bakuradze, V.V. Vershinin : Morava K-theory rings for the dihedral, semi-dihedral and generalized quaternion groups in Chern Classes, Proc. Amer. Math. Soc., 134(2006), 37073714 .
8. V. M. Buchstaber : Modules of differentials of the Atiyah-Hirzebruch spectral sequence, Matem. Sbornik, 78:2(1969), 307-320 .
9. V. M. Buchstaber : Modules of differentials of the Atiyah-Hirzebruch spectral sequence. II, Matem. Sbornik, 83:1(1970), 61-76 .
10. A. Dold : The fixed point transfer of fibre-preserving maps, Math. Zeit., 148(1976), 215-244.
11. T. Gramushnjak and P. Puusemp : Description of a Class of 2-Groups, Journal of Nonlinear Math. Physics 13(2006), 55-65
12. M. Hall and J.K. Senior : The groups of order $2^{n}$, $n \leq 6$, The Macmillan Co., New York; Collier-Macmillan, Ltd., London 1964.
13. M. Hopkins, N. Kuhn, and D. Ravenel: Generalized group characters and complex oriented cohomology theories, J. Amer. Math. Soc., 13, 3(2000), 553-594.
14. J. R. Hunton : Morava K-theories of wreath products, Math. Proc. Camb. Phil. Soc., 107(1990), 309-318.
15. D. C. Jonson, W. S. Wilson : BP operations and Morava's extraordinary K-theories, Math. Z., 144 (1975), 55-75.
16. D. S. Kahn, S. B. Priddy : Applications of the transfer to stable homotopy theory, Bull. Amer. Math. Soc., 78(1972), 981-987.
17. I. Kriz : Morava K-theory of classifying spaces: Some calculations, Topology, 36(1997), 1247-1273.
18. D. C. Ravenel : Morava K-theories and finite groups, Contemp. Math., 12 (1982), 289- 292.
19. B. Schuster : Morava K-theory of groups of order 32, Algebraic and Geometric Topology, 11(2011), 503-521.
20. B. Schuster : $K(n)$ Chern approximations of some finite groups, Algebraic and Geometric Topology, 12, 3 (2012), 1695-1720.
21. B. Schuster : On Morava K-theory of some finite 2-groups, Math. Proc. Camb. Phil. Soc., 121(1997), 7-13.
22. B. Schuster : Morava K-theory of classifying spaces, Habilitationsschrift, 2006, 124 pp .
23. B. Schuster, N. Yagita : On Morava K-theory of extraspecial 2-groups, Proc. Amer. Math. Soc., 132, 4(2004), 1229-1239.
24. M. Tezuka and N. Yagita: Homotopy theory and related topics(Kinosaki, 1988), 57-69, Lecture Notes in Math. 1418, Springer, Berlin, 1990.
25. N. Yagita: Cohomology for groups of $\operatorname{rank}_{p}(G)=2$ and Brown-Peterson cohomology, J. Math. Soc. Japan 45, 4(1993), 627-644.
26. N. Yagita: Note on BP-theory for extensions of cyclic groups by elementary abelian p-groups, Kodai Math. J. 20, 2 (1997), 79-84.
27. N.P. Strickland : Chern Approximations for generalised group cohomology, Topology 40, $\mathbf{6}(2001), 1167-1216$.
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