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All extensions of C_2 by $C_{2^n} \times C_{2^n}$ are good for the Morava K-theory

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Abstract. Let C_m be a cyclic group of order m. We prove that if a group G fits into an extension $1 \to C_{2^{n+1}}^2 \to G \to C_2 \to 1$ for $n \geq 1$ then G is good in the sense of Hopkins-Kuhn-Ravenel, i.e., $K(s)^*(BG)$ is evenly generated by transfers of Euler classes of complex representations of subgroups of G.

1. Introduction and statements

This paper is concerned with analyzing the 2-primary Morava K-theory of the classifying spaces BG of the groups in the title. In particular it answers the question whether transfers of Euler classes suffice to generate $K(s)^*(BG)$. Here $K(s)$ denotes Morava K-theory at prime $p = 2$ and natural number $s > 1$. The coefficient ring $K(s)^*(pt)$ is the Laurent polynomial ring in one variable, $\mathbb{F}_2[v_s, v_s^{-1}]$, where $\mathbb{F}_2$ is the field of 2 elements and $\deg(v_s) = -2(2^s - 1)$ [12]. So the coefficient ring is a graded field in the sense that all its graded modules are free, therefore Morava K-theories enjoy the Künneth isomorphism. In particular, we have for the cyclic group $C_{2^{n+1}}$ that as a $K(s)^*$-algebra

$$K(s)^*(BC_{2^{n+1}}^2) = K(s)^*(BC_{2^{n+1}}) \otimes_{K(s)^*} K(s)^*(BC_{2^{n+1}}),$$

whereas $K(s)^*(BC_{2^m}) = K(s)^*[u]/(u^{2^{ms}})$, so that

$$K(s)^*(BC_{2^{n+1}}^2) = K(s)^*[u, v]/(u^{2^{(n+1)s}}, v^{2^{(n+1)s}}),$$

where u and v are Euler classes of canonical complex linear representations.

The definition of good groups in the sense of [10] is as follows.

(a) For a finite group G, an element $x \in K(s)^*(BG)$ is good if it is a transferred Euler class of a complex subrepresentation of G, i.e., a class of the

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form $Tr^*(e(\rho))$, where ρ is a complex representation of a subgroup $H < G$, $e(\rho) \in K(s)^*(BH)$ is its Euler class (i.e., its top Chern class, this being defined since $K(s)^*$ is a complex oriented theory), and $Tr : BG \to BH$ is the transfer map.

(b) G is called to be good if $K(s)^*(BG)$ is spanned by good elements as a $K(s)^*$-module.

Recall that not all finite groups are good as it was originally conjectured in [10]. For an odd prime p a counterexample to the even degree was constructed in [14]. The problem to construct 2-primary counterexample to the conjecture remains open.

The families of good groups in a weaker sense, i.e., $K(n)^{odd}(BG) = 0$ are listed in [16]. In particular, if G belongs to any of the following families of p-groups, then $K(n)^{odd}(BG) = 0$.

(a) wreath products of the form $H \wr C_p$ with H good [10], [11];

(b) metacyclic p-groups [20];

(c) minimal non-abelian p-groups, i.e., groups all of whose maximal subgroups are abelian [21];

(d) groups of p-rank 2 [22];

(e) elementary abelian by cyclic groups, i.e., the extensions $V \to G \to C$ with V elementary abelian and C cyclic [23], [14];

(f) central product of the form $H \circ C_{p^m}$ with H good [16];

(g) H is a normal subgroup in G of index p, H is good and the integral Morava K-theory $\tilde{K}(s)(BH)$ is a permutation module for the action of G/H [14].

Our main result provides a new series of good groups in the sense of Hopkins-Kuhn-Ravenel.

Theorem 1. *All extensions of* C_2 *by* $C^2_{2^{n+1}}$ *are good for all* $n \geq 0$.

For $n = 0$ and $n = 1$ the statement of the theorem was known. See [2], [4], [16], [18] for detailed discussion and examples. In this particular case, for various examples of groups of order 32, the multiplicative structure of $K^*(BG)$ is also determined in [2], [4] using transfer methods of [5], [6].

The basic tool for the proof is the Serre spectral sequence, which we use throughout the paper. However, if we work in a straightforward way, even for $s = 2$, $n = 1$, this requires a serious computational effort and use of computer, see [17], p. 78. We simplify the task of calculation with invariants by suggesting the special bases for particular C_2-modules $K(s)^*(BH)$, see Lemma 1 and Lemma 2. This simple but comfortable idea is our key tool to prove Theorem 1. We will prove it for the semi-direct products

$$(C_{2^{n+1}} \times C_{2^{n+1}}) \rtimes C_2. \tag{1}$$

Then the general case follows because of the fact that the Serre spectral sequence does not show the difference between the semi-direct products and their non-split versions.

2. Preliminaries

Recall [9] there exist exactly 17 non-isomorphic groups of order 2^{2n+3}, $n \geq 2$, which can be presented as a semidirect product (1). Each such group G is given by three generators **a**, **b**, **c** and the defining relations

$$\mathbf{a}^{2^{n+1}} = \mathbf{b}^{2^{n+1}} = \mathbf{c}^2 = 1, \qquad \mathbf{ab} = \mathbf{ba}, \qquad \mathbf{c}^{-1}\mathbf{ac} = \mathbf{a}^i\mathbf{b}^j, \qquad \mathbf{cbc} = \mathbf{a}^k\mathbf{b}^l$$

for some $i, j, k, l \in \mathbb{Z}/2^{n+1}$ ($\mathbb{Z}/2^m$ denotes the ring of residue classes modulo 2^m). In particular one has the following.

Proposition 1 (See [9]). *Let n be an integer such that $n \geq 2$. Then there exist exactly* 17 *non-isomorphic groups of order* 2^{2n+3} *which can be presented as a semi-direct product* (1). *They are:*

$$G_1 = \langle \mathbf{a}, \mathbf{b}, \mathbf{c} \mid (*), \mathbf{cac} = \mathbf{a}, \mathbf{cbc} = \mathbf{b} \rangle,$$

$$G_2 = \langle \mathbf{a}, \mathbf{b}, \mathbf{c} \mid (*), \mathbf{cac} = \mathbf{a}^{1+2^n}, \mathbf{cbc} = \mathbf{b}^{1+2^n} \rangle,$$

$$G_3 = \langle \mathbf{a}, \mathbf{b}, \mathbf{c} \mid (*), \mathbf{cac} = \mathbf{ab}^{2^n}, \mathbf{cbc} = \mathbf{b} \rangle,$$

$$G_4 = \langle \mathbf{a}, \mathbf{b}, \mathbf{c} \mid (*), \mathbf{cac} = \mathbf{a}^{1+2^n}\mathbf{b}^{2^n}, \mathbf{cbc} = \mathbf{b}^{1+2^n} \rangle,$$

$$G_5 = \langle \mathbf{a}, \mathbf{b}, \mathbf{c} \mid (*), \mathbf{cac} = \mathbf{a}^{-1}, \mathbf{cbc} = \mathbf{b}^{-1} \rangle,$$

$$G_6 = \langle \mathbf{a}, \mathbf{b}, \mathbf{c} \mid (*), \mathbf{cac} = \mathbf{a}^{-1+2^n}, \mathbf{cbc} = \mathbf{b}^{-1+2^n} \rangle,$$

$$G_7 = \langle \mathbf{a}, \mathbf{b}, \mathbf{c} \mid (*), \mathbf{cac} = \mathbf{a}^{-1}\mathbf{b}^{2^n}, \mathbf{cbc} = \mathbf{b}^{-1} \rangle,$$

$$G_8 = \langle \mathbf{a}, \mathbf{b}, \mathbf{c} \mid (*), \mathbf{cac} = \mathbf{a}^{-1+2^n}\mathbf{b}^{2^n}, \mathbf{cbc} = \mathbf{b}^{-1+2^n} \rangle,$$

$$G_9 = \langle \mathbf{a}, \mathbf{b}, \mathbf{c} \mid (*), \mathbf{cac} = \mathbf{ab}^{2^n}, \mathbf{cbc} = \mathbf{a}^{2^n}\mathbf{b}^{1+2^n} \rangle,$$

$$G_{10} = \langle \mathbf{a}, \mathbf{b}, \mathbf{c} \mid (*), \mathbf{cac} = \mathbf{a}, \mathbf{cbc} = \mathbf{b}^{1+2^n} \rangle,$$

$$G_{11} = \langle \mathbf{a}, \mathbf{b}, \mathbf{c} \mid (*), \mathbf{cac} = \mathbf{a}^{-1}\mathbf{b}^{2^n}, \mathbf{cbc} = \mathbf{a}^{2^n}\mathbf{b}^{-1+2^n} \rangle,$$

$$G_{12} = \langle \mathbf{a}, \mathbf{b}, \mathbf{c} \mid (*), \mathbf{cac} = \mathbf{a}^{-1}, \mathbf{cbc} = \mathbf{b}^{-1+2^n} \rangle,$$

$$G_{13} = \langle \mathbf{a}, \mathbf{b}, \mathbf{c} \mid (*), \mathbf{cac} = \mathbf{a}, \mathbf{cbc} = \mathbf{b}^{-1+2^n} \rangle,$$

$$G_{14} = \langle \mathbf{a}, \mathbf{b}, \mathbf{c} \mid (*), \mathbf{cac} = \mathbf{a}^{-1}, \mathbf{cbc} = \mathbf{b}^{1+2^n} \rangle,$$

$$G_{15} = \langle \mathbf{a}, \mathbf{b}, \mathbf{c} \,|\, (*), \mathbf{cac} = \mathbf{b}, \mathbf{cbc} = \mathbf{a} \rangle,$$

$$G_{16} = \langle \mathbf{a}, \mathbf{b}, \mathbf{c} \,|\, (*), \mathbf{cac} = \mathbf{a}, \mathbf{cbc} = \mathbf{b}^{-1} \rangle,$$

$$G_{17} = \langle \mathbf{a}, \mathbf{b}, \mathbf{c} \,|\, (*), \mathbf{cac} = \mathbf{a}^{1+2^n}, \mathbf{cbc} = \mathbf{b}^{-1+2^n} \rangle,$$

where $(*)$ *denotes the collection* $\{\mathbf{a}^{2^{n+1}} = \mathbf{b}^{2^{n+1}} = \mathbf{c}^2 = [\mathbf{a}, \mathbf{b}] = 1\}$ *of defining relations.*

Let H_i and G_i be finite p-groups, $i = 1, \dots, n$, such that H_i is good and G_i fits into an extension $1 \to H_i \to G_i \to C_p \to 1$.

Let G fit into an extension of the form $1 \to H \to G \to C_p \to 1$, with diagonal action of C_p by conjugation on $H = H_1 \times \cdots \times H_n$. Let

$$Tr^* = Tr^*_{\varrho} : K(s)^*(BH) \to K(s)^*(BG)$$

be the transfer homomorphism associated to the p-covering

$$\varrho = \varrho(H, G) : BH \to BG.$$

Let

$$Tr^*_i = Tr^*_{\varrho_i} : K(s)^*(BH_i) \to K(s)^*(BG_i)$$

be the transfer homomorphism associated to the p-covering

$$\varrho_i = \varrho(H_i, G_i) : BH_i \to BG_i, \qquad i = 1, \dots, n.$$

Then

$$(Tr_1 \wedge \cdots \wedge Tr_n)^*$$

is the transfer homomorphism associated to the product $\varrho_1 \times \cdots \times \varrho_n$.

Let

$$\rho_i : BG \to BG_i$$

be the map induced by the projection $p_i : H \to H_i$ on the i-th factor. Consider the map

$$(\rho_1, \dots, \rho_n) : BG \to BG_1 \times \cdots \times BG_n.$$

Then by naturality of the transfer one has

$$(\rho_1, \dots, \rho_n)^* \circ (Tr_1 \wedge \cdots \wedge Tr_n)^* = Tr^* \circ (p_1, \dots, p_n)^*.$$

Therefore $(\rho_1, \dots, \rho_n)^*$ defines the homomorphism

$$\rho^* : K(s)^*(BG_1 \times \cdots \times BG_n)/Im(Tr_1 \wedge \cdots \wedge Tr_n)^* \to K(s)^*(BG)/Im\ Tr^*.$$

In [3] we proved the following.

THEOREM 2. *Let G be a group as above. Then*
i) *If G_i are good, then so is G.*
ii) *As a $K(s)^*(pt)$-module, $K(s)^*(BG)/Im\,Tr^*$ is spanned by elements in $Im\,\rho^*$.*

In particular this implies

COROLLARY 1. *Let $G = G_i$, $i \neq 3, 4, 7, 8, 9, 11$, in Proposition 1. Then G is good in the sense of Hopkins-Kuhn-Ravenel.*

PROOF. G_{15} is good as wreath product [10]. If $i \neq 15$, G_i has maximal abelian subgroup $H_i = \langle \mathbf{a}, \mathbf{b} \rangle$ on which the quotient acts (diagonally) as above. Each of the following groups $C_{2^{n+1}} \times C_2$, the dihedral group $D_{2^{n+2}}$, the quasi-dihedral group $QD_{2^{n+2}}$, the semi-dihedral group $SD_{2^{n+2}}$ could be written as semidirect product $C_{2^{n+1}} \rtimes C_2$ with that kind of action. For all these groups $K(s)^*(BG)$ is generated by transfers of Euler classes, see [19, 20].

We will need the following approximations (see [7], Lemma 2.2) for the formal group law in Morava $K(s)^*$-theory, $s > 1$, where we set $v_s = 1$.

$$F(x, y) \equiv x + y + (xy)^{2^{s-1}} \bmod(y^{2^{2(s-1)}}); \tag{2}$$

$$F(x, y) = x + y + \Phi(x, y)^{2^{s-1}}, \tag{3}$$

where $\Phi(x, y) \equiv xy + (xy)^{2^{s-1}}(x + y) \bmod((xy)^{2^{s-1}}(x + y)^{2^{s-1}})$.

3. Complex representations over BG

Let us define some complex representations over BG we will need.

Let $H = \langle \mathbf{a}, \mathbf{b} \rangle \cong C_{2^{n+1}} \times C_{2^{n+1}}$ be the maximal abelian subgroup in G. Let

$$\pi : BH \to BG \tag{4}$$

be the double covering. Let λ and ν denote the complex line bundles over BH defined by

$$\lambda(\mathbf{a}) = \nu(\mathbf{b}) = e^{2\pi i/2^{n+1}}, \qquad \lambda(\mathbf{b}) = \lambda(\mathbf{c}) = \nu(\mathbf{a}) = \nu(\mathbf{c}) = 1,$$

i.e. the pullbacks of the canonical complex line bundles along the projections onto the first and second factor of H respectively.

Define three line bundles α, β and γ over BG, as follows:

$$\alpha(\mathbf{a}) = \beta(\mathbf{b}) = \gamma(\mathbf{c}) = -1, \qquad \alpha(\mathbf{b}) = \alpha(\mathbf{c}) = \beta(\mathbf{a}) = \beta(\mathbf{c}) = \gamma(\mathbf{a}) = \gamma(\mathbf{b}) = 1.$$

Let us denote Chern classes by

$$x_i = c_i(\pi_!(\lambda)), \qquad y_i = c_i(\pi_!(v)), \qquad i = 1, 2,$$
$$a = c_1(\alpha), \qquad b = c_1(\beta), \qquad c = c_1(\gamma)$$

in $K(s)^*(BG)$, where $\pi_!(-)$ is the induced representation from π.

4. Proof of Theorem 1

Here we prove that all the remaining groups G_i, $i = 3, 4, 7, 8, 9, 11$, not covered by Corollary 1, are also good.

Our tool shall be the Serre spectral sequence

$$E_2 = H^*(BC_2, K(s)^*(BH)) \Rightarrow K(s)^*(BG) \tag{5}$$

associated to a group extension $1 \to H \to G \to C_2 \to 1$.

Here $H^*(BC_2, K(s)^*(BH))$ denotes the ordinary cohomology of BC_2 with coefficients in the $\mathbb{F}_2[C_2]$-module $K(s)^*(BH)$, where the action of C_2 is induced by conjugation in G.

Let $Tr^* : K(s)^*(BH) \to K(s)^*(BG)$ be the transfer homomorphism [1], [13], [8] associated to the double covering $\pi : BH \to BG$.

We use the notations of the previous two sections. In particular let

$$H \cong C_{2^{n+1}} \times C_{2^{n+1}} \cong \langle \mathbf{a}, \mathbf{b} \rangle.$$

The action of the involution $t \in C_2$ on

$$K(s)^*(BH) = K(s)^*[u, v]/(u^{2^{(n+1)s}}, v^{2^{(n+1)s}}) \tag{6}$$

is induced by the conjugation action by $\mathbf{c}$ on H.

As a C_2-module $K(s)^*(BH) = F \oplus T$, where F is C_2-free and T is C_2-trivial.

This gives the decomposition

$$[K(s)^*(BH)]^{C_2} = [F]^{C_2} \oplus T. \tag{7}$$

Clearly the composition $\pi^* Tr^* = 1 + t$, the trace map, is onto $[F]^{C_2}$. Therefore it suffices to check that all elements in T are also represented by good elements.

Note that $\pi^*\pi_!$ is the trace map in complex K-theory, i.e., $\pi^*(\pi_!(\lambda)) = \lambda + t(\lambda)$. Then the Chern classes can be easily computed. In particular for all cases of G let $u = e(\lambda) = c_1(\lambda)$ and $v = e(v) = c_1(v)$ as before. Then

$$\bar{x}_1 = \pi^*(x_1) = c_1(\pi^*(\pi_!(\lambda))) = u + t(u), \qquad \bar{x}_2 = \pi^*(x_2) = c_2(\pi^*(\pi_!(\lambda))) = ut(u),$$
$$\bar{y}_1 = \pi^*(y_1) = c_1(\pi^*(\pi_!(v))) = v + t(v), \qquad \bar{y}_2 = \pi^*(y_2) = c_2(\pi^*(\pi_!(v))) = vt(v).$$

We will need the following.

Lemma 1. *Let G be one of the groups under consideration and $t \in C_2 = G/H$ be the corresponding involution on H. Then there is a set of monomials $\{x^\omega\} = \{\bar{x}_1^i \bar{x}_2^j \bar{y}_1^k \bar{y}_2^l\}$, such that the set $\{x^\omega, x^\omega u, x^\omega v, x^\omega uv\}$ is a $K(s)^*$-basis in $K(s)^*(BH)$. Specifically one can choose $\{x^\omega\}$ as follows:*

$$\{x^\omega\} = \begin{cases} \{\bar{x}_2^j \bar{y}_1^k \bar{y}_2^l \mid j < 2^{ns-1}, k < 2^s, l < 2^{(n+1)s-1}\}, & \text{if } G = G_3, \\ \{\bar{x}_1^i \bar{x}_2^j \bar{y}_1^k \bar{y}_2^l \mid i, k < 2^s, j, l < 2^{ns-1}\}, & \text{if } G = G_4, G_9, \\ \{\bar{x}_1^i \bar{x}_2^j \bar{y}_1^k \bar{y}_2^l \mid i, k < 2^{ns}, j, l < 2^{s-1}\}, & \text{if } G = G_7, G_8, G_{11}. \end{cases}$$

Proof. For any case, the set $\{x^\omega, x^\omega u, x^\omega v, x^\omega uv\}$ generates $K(s)^*(BH)$: using $u^2 = u\bar{x}_1 - \bar{x}_2$ and $v^2 = v\bar{y}_1 - \bar{y}_2$ any polynomial in u, v can be written as $g_0 + g_1 u + g_2 v + g_3 uv$, for some polynomials $g_i = g_i(\bar{x}_1, \bar{y}_1, \bar{x}_2, \bar{y}_2)$. In particular it follows by induction, that

$$v^{2^m} = v\bar{y}_1^{2^m - 1} + \sum_{i=1}^{m} \bar{y}_1^{2^m - 2^i} \bar{y}_2^{2^{i-1}}, \tag{8}$$

and similarly for u^{2^m}.

Now for each case we have to explain the restrictions in $\{x^\omega\}$. Then the restricted set $S = \{x^\omega, x^\omega u, x^\omega v, x^\omega uv\}$ will indeed form a $K^*(s)$-basis in $K^*(s)(BH)$ because of its size $4^{(n+1)s}$.

Consider G_3. For the conditions on l and k we have to take into account (2), (3), (6) and the action of the involution t.

In particular, we have

$$t(\lambda) = \lambda, \qquad t(v) = \lambda^{2^n} v, \qquad \text{and} \qquad t(u) = u.$$

This implies $\bar{x}_1 = u + t(u) = 0$ and $\bar{x}_2 = ut(u) = u^2$.

On the other hand, from (3)

$$t(v) = F(u^{2^{ns}}, v) = v + u^{2^{ns}} + (vu^{2^{ns}})^{2^{s-1}},$$

which implies $\bar{y}_2^{2^{(n+1)s-1}} = 0$ from (6). Similarly

$$\bar{y}_1 = v + t(v) = u^{2^{ns}} + (vu^{2^{ns}})^{2^{s-1}},$$

which implies $\bar{y}_1^{2^s} = 0$.

Thus we have the condition that $k < 2^s$ and $l < 2^{(n+1)s-1}$ in $\{x^\omega\}$.

For the condition on j, that is, the decomposition of $\bar{x}_2^{2^{ns-1}}$ in the suggested basis, note that the formula for $t(v)$ and (8) for $m = s-1$ imply

$$\begin{aligned}\bar{x}_2^{2^{ns-1}} = u^{2^{ns}} &= \bar{y}_1 + (vu^{2^{ns}})^{2^{s-1}} \\ &= \bar{y}_1 + v^{2^{s-1}}(\bar{y}_1 + (vu^{2^{ns}})^{2^{s-1}})^{2^{s-1}} \\ &= \bar{y}_1 + v^{2^{s-1}}\bar{y}_1^{2^{s-1}} \\ &= \bar{y}_1 + \bar{y}_1^{2^{s-1}}\left(v\bar{y}_1^{2^{s-1}-1} + \sum_{i=1}^{s-1}\bar{y}_1^{2^{s-1}-2^i}\bar{y}_2^{2^{i-1}}\right) \\ &= \bar{y}_1 + v\bar{y}_1^{2^s-1} + \bar{y}_1^{2^{s-1}}\sum_{i=1}^{s-1}\bar{y}_1^{2^{s-1}-2^i}\bar{y}_2^{2^{i-1}}.\end{aligned}$$

Here $\bar{x}_2^{2^{ns-1}}$ is represented by $\bar{y}_1^k\bar{y}_2^l$'s, and so we have the condition $j < 2^{ns-1}$.

G_4: The involution acts as follows: $t(\lambda) = \lambda^{2^n+1}$, $t(v) = \lambda^{2^n}v^{2^n+1}$, hence

$$t(u) = F(u, u^{2^{ns}}) = u + u^{2^{ns}} + (uu^{2^{ns}})^{2^{s-1}} \qquad \text{by (2)}, \tag{9}$$

$$t(v) = F(v, F(v^{2^{ns}}, u^{2^{ns}})) = v + F(v^{2^{ns}}, u^{2^{ns}}) + v^{2^{s-1}}(F(v^{2^{ns}}, u^{2^{ns}}))^{2^{s-1}}, \tag{10}$$

so that $\bar{x}_1^{2^s} = \bar{y}_1^{2^s} = 0$.

For the decomposition of $\bar{x}_2^{2^{ns-1}}$, note (9) implies

$$\bar{x}_2^{2^{ns-1}} = (ut(u))^{2^{ns-1}} = u^{2^{ns}}.$$

Then by (9) again

$$\bar{x}_2^{2^{ns-1}} = \bar{x}_1 + (u\bar{x}_2^{2^{ns-1}})^{2^{s-1}} = \bar{x}_1 + (u(\bar{x}_1 + (u\bar{x}_2^{2^{ns-1}})^{2^{s-1}}))^{2^{s-1}} = \bar{x}_1 + u^{2^{s-1}}\bar{x}_1^{2^{s-1}}$$

and apply (8) for $u^{2^{s-1}}$.

Similar arguments work for $\bar{y}_2^{2^{ns-1}}$.

The proof for G_9 is completely analogous as it uses the following similar formulas for the action of the involution:

$$\begin{aligned}&t(\lambda) = \lambda v^{2^n}, \qquad t(v) = \lambda^{2^n}v^{2^n+1}, \\ &t(u) = F(u, v^{2^{ns}}) = u + v^{2^{ns}} + (uv^{2^{ns}})^{2^{s-1}}, \\ &t(v) = F(v, F(u^{2^{ns}}, v^{2^{ns}})).\end{aligned}$$

G_7: Let $\bar{\lambda}$ be the complex conjugate to λ and

$$\bar{u} = [-1]_F(u) = e(\bar{\lambda}), \qquad \bar{v} = [-1]_F(v) = e(\bar{v}).$$

The involution acts as follows:

$$t(\lambda) = \bar{\lambda},$$

$$t(v) = \lambda^{2^n}\bar{v},$$

$$t(u) = \bar{u} \equiv u + (u\bar{u})^{2^{s-1}} \bmod(1+t), \qquad \text{by (3) as } F(u,\bar{u}) = 0$$

$$t(v) = F(\bar{v}, u^{2^{ns}}) = \bar{v} + u^{2^{ns}} + (\bar{v}u^{2^{ns}})^{2^{s-1}}, \qquad \text{by (2).}$$

It follows that

$$0 = u + \bar{u} \bmod(u\bar{u})^{2^{s-1}} \equiv u + \bar{u} \bmod(u^{2^s})$$

therefore

$$\bar{x}_1^{2^{ns}} = (u+\bar{u})^{2^{ns}} = 0, \qquad \text{as } u^{2^{(n+1)s}} = 0.$$

Then as $u\bar{u} = \bar{x}_2$ is nilpotent we can eliminate $\bar{x}_2^{2^i} = (u\bar{u})^{2^i}$ for $i > s-1$ in (3) after finite steps of iteration and write $\bar{x}_2^{2^{s-1}}$ as a polynomial in $u + \bar{u} = \bar{x}_1$. We will not need this polynomial explicitly but only

$$\bar{x}_2^{2^{s-1}} \equiv 0 \bmod(1+t).$$

For $\bar{y}_1^{2^{ns}} = 0$ apply the formula for $t(v)$ and take into account $v + \bar{v} \equiv 0 \bmod v^{2^s}$.

For the decomposition of $\bar{y}_2^{2^{s-1}}$ note we have two formulas for $F(v, t(v)) = e(\lambda^{2^n}) = u^{2^{ns}}$, one is (8) and another is (3). Equating these formulas we have an expression of the form

$$\bar{y}_2^{2^{s-1}} = u\bar{x}_1^{2^{ns}-1} + P(\bar{y}_1, \bar{y}_2), \qquad \text{for some polynomial } P(\bar{y}_1, \bar{y}_2).$$

Again as $\bar{y}_2$ is nilpotent we can eliminate $\bar{y}_2^{2^i}$ for $i > s-1$ in (3) after finite steps of iteration and write $\bar{y}_2^{2^{s-1}}$ in the suggested basis. Again we only will need that

$$\bar{y}_2^{2^{s-1}} \equiv u\bar{x}_1^{2^{ns}-1} \bmod Im(1+t).$$

This completes the proof for G_7. The proofs for G_8 and G_{11} are analogous. Let us sketch the necessary information for the interested reader to produce detailed proofs.

G_8: the action of the involution is as follows:

$$t(\lambda) = \bar{\lambda}\lambda^{2^n}, \qquad t(v) = \bar{v}\lambda^{2^n}v^{2^n},$$

$$t(u) = F(\bar{u}, u^{2^{ns}}),$$

$$t(v) = F(\bar{v}, F(u^{2^{ns}}, v^{2^{ns}})).$$

G_{11}: one has

$$t(\lambda) = \bar{\lambda} v^{2^n}, \qquad t(v) = \bar{v} \lambda^{2^n} v^{2^n},$$

$$t(u) = F(\bar{u}, v^{2^{ns}}),$$

$$t(v) = F(\bar{v}, F(u^{2^{ns}}, v^{2^{ns}})).$$

For both cases to get $\bar{x}_1^{2^{ns}} = 0$ apply formula for $t(u)$ and $u + \bar{u} \equiv 0 \bmod u^{2^s}$. Similarly for $\bar{y}_1^{2^{ns}} = 0$. For the decompositions of $\bar{x}_2^{2^{s-1}}$ and $\bar{y}_2^{2^{s-1}}$ apply (3) and (8). In particular for G_8 we have by (3) $\bar{x}_2^{2^{s-1}} \equiv u^{2^{ns}}$ modulo some $\bar{x}_1 f(\bar{y}_1, \bar{x}_2) \in Im(1+t)$. Therefore $\bar{x}_2^{2^{ns-1}} \equiv 0 \bmod (1+t)$ and by (8) for u, we have

$$\bar{x}_2^{2^{s-1}} \equiv u^{2^{ns}} \equiv \bar{x}_1^{2^{ns}-1} u + \bar{x}_2^{2^{ns-1}} \equiv \bar{x}_1^{2^{ns}-1} u \bmod (1+t).$$

Similarly $\bar{y}_2^{2^{ns-1}} \equiv 0 \bmod (1+t)$ and we get

$$\bar{x}_2^{2^{s-1}} \equiv F(u^{2^{ns}}, v^{2^{ns}}) \equiv \bar{x}_1^{2^{ns}-1} u + \bar{y}_1^{2^{ns-1}} v \bmod (1+t).$$

Thus we obtain

$$\bar{x}_1^{2^{ns}} = \bar{y}_1^{2^{ns}} = 0, \qquad \text{if } G = G_7, G_8, G_9,$$

$$\bar{x}_2^{2^{s-1}} \equiv 0, \qquad \bar{y}_2^{2^{s-1}} \equiv \bar{x}_1^{2^{ns}-1} u \bmod (1+t), \qquad \text{if } G = G_7,$$

$$\bar{x}_2^{2^{s-1}} \equiv \bar{x}_1^{2^{ns}-1} u, \qquad \bar{y}_2^{2^{s-1}} \equiv \bar{x}_1^{2^{ns}-1} u + \bar{y}_1^{2^{ns}-1} v \bmod (1+t), \qquad \text{if } G = G_8,$$

$$\bar{x}_2^{2^{s-1}} \equiv \bar{y}_1^{2^{ns}-1} v, \qquad \bar{y}_2^{2^{s-1}} \equiv \bar{x}_1^{2^{ns}-1} u + \bar{y}_1^{2^{ns}-1} v \bmod (1+t), \qquad \text{if } G = G_{11}.$$

LEMMA 2. *Let* $g = f_0 + f_1 u + f_2 v + f_3 uv \in K(s)^*(BH)$, *where* $f_i = f_i(\bar{x}_1, \bar{y}_1 \bar{x}_2, \bar{y}_2)$ *are some polynomials written uniquely in the monomials* x^ω *of Lemma* 1. *Then* g *is invariant under involution* $t \in G/H$ *iff*

$$f_3 \bar{x}_1 = f_3 \bar{y}_1 = 0; \qquad f_1 \bar{x}_1 = f_2 \bar{y}_1.$$

PROOF. We have g is invariant iff $g \in Ker(1+t)$. Since each f_i is invariant

$$g + t(g) = f_1(u + t(u)) + f_2(v + t(v)) + f_3(uv + t(uv))$$

$$= f_1 \bar{x}_1 + f_2 \bar{y}_1 + f_3(\bar{x}_1 \bar{y}_1 + \bar{x}_1 v + \bar{y}_1 u)$$

and using Lemma 1 the result follows.

To prove Theorem 1 it suffices to see that all invariants are represented by good elements. It is obvious for the elements $a + t(a) = \pi^* Tr^*(a)$ in the free summand $[F]^{C_2}$ in (7). Therefore one can work modulo $Im(1+t)$ and check the elements in the trivial summand T. Let us finish the proof of Theorem 1 using Propositions 2, i). We will turn to Proposition 2 ii) later.

Proposition 2. *Let T' be spanned by the set*

for G_3,

$$\{\bar{x}_2^j\bar{y}_2^l,\bar{x}_2^j\bar{y}_2^l u,\bar{y}_1^{2^s-1}\bar{x}_2^j\bar{y}_2^l v,\bar{y}_1^{2^s-1}\bar{x}_2^j\bar{y}_2^l uv \mid j<2^{ns-1},\ l<2^{(n+1)s-1}\},$$

for G_4, G_9,

$$\{\bar{x}_2^i\bar{y}_2^j,\bar{x}_1^{2^s-1}\bar{x}_2^i\bar{y}_2^j u,\bar{y}_1^{2^s-1}\bar{x}_2^i\bar{y}_2^j v,\bar{x}_1^{2^s-1}\bar{y}_1^{2^s-1}\bar{x}_2^i\bar{y}_2^j uv \mid i,j<2^{ns-1}\},$$

for G_7, G_8, G_{11},

$$\{\bar{x}_2^i\bar{y}_2^j,\bar{x}_1^{2^{ns}-1}\bar{x}_2^i\bar{y}_2^j u,\bar{y}_1^{2^{ns}-1}\bar{x}_2^i\bar{y}_2^j v,\bar{x}_1^{2^{ns}-1}\bar{y}_1^{2^{ns}-1}\bar{x}_2^i\bar{y}^j uv \mid i,j<2^{s-1}\}.$$

Then

i) *All terms in T' are represented by good elements and $T \subset T'$.*
ii) *Moreover, $T = T'$.*

Proof of i). The case of G_3. The basis set of T' above is suggested by Lemma 1 and Lemma 2: it is clear that all its terms are invariants. The terms $\bar{x}_2^j\bar{y}_1^k\bar{y}_2^l \in Im(1+t)$, $k>0$ are omitted as we work modulo $1+t$. Then all the restrictions follow by

$$\bar{y}_1^{2^s}=0, \qquad \bar{x}_1=0, \qquad \bar{y}_2^{2^{(n+1)s-1}}=0, \qquad \bar{x}_2^{2^{ns-1}} \equiv v\bar{y}_1^{2^s-1} \bmod(1+t).$$

Thus $T \subset T'$. Let us check that T' is generated by the images of products of Euler classes under π^*, where π is the double covering (4).

By definitions

$$\pi^*(\alpha)=\lambda^{2^n}, \qquad \pi^*(\det \pi_!(v)\otimes\alpha)=v\lambda^{2^n}v\lambda^{2^n}=v^2,$$
$$\pi^*(v')=v^{2^s}, \qquad \text{where } v'=e(\det \pi_!(v)\otimes\alpha).$$

Taking into account (8), for $m=s$, we get

$$\pi^*(v')=v^{2^s}=v\bar{y}_1^{2^s-1}+\sum_{i=1}^{s}\bar{y}_1^{2^s-2^i}\bar{y}_2^{2^{i-1}}=\bar{y}_2^{2^{s-1}}+v\bar{y}_1^{2^s-1} \bmod(1+t). \quad (11)$$

By definition $\bar{x}_2=\pi^*(x_2)$ and $\bar{y}_2=\pi^*(y_2)$. Combined with (11) this implies that all elements of the first and third parts of the basis set of T' are π^* images of the sums of Euler classes.

For the rest parts of the basis of T' note that the bundle λ can be extended to a bundle over BG, say λ', represented by $\lambda'(\mathbf{a}) = e^{2\pi i/2^{n+1}}$, $\lambda'(\mathbf{b}) = \lambda'(\mathbf{c}) = 1$. So $\pi^*(e(\lambda')) = u$. Then note that the second and last parts are obtained by multiplying by u from the first and third parts respectively. Therefore we can easily read off all elements as π^* images of the sums of Euler classes.

G_4. Again the basis for T' is suggested by Lemma 1: we have $\bar{x}_1^{2^s} = \bar{y}_1^{2^s} = 0$ and $\bar{x}_2^{2^{ns-1}}$ and $\bar{y}_2^{2^{ns-1}}$ are decomposable. Then applying (8) we get

$$\pi^*(\det(\pi_! v) \otimes \alpha) = v^2,$$

$$\pi^*(e(\det(\pi_! v) \otimes \alpha)) = v^{2^s} \equiv v\bar{y}_1^{2^s-1} + \bar{y}_2^{2^{s-1}} \bmod (1+t),$$

$$\pi^*(\det(\pi_! \lambda) \otimes \alpha\beta) = \lambda^2,$$

$$\pi^*(e(\det(\pi_! \lambda) \otimes \alpha\beta)) = u^{2^s} \equiv u\bar{x}_1^{2^s-1} + \bar{x}_2^{2^{s-1}} \bmod (1+t).$$

Thus G_4 is good. The proof for G_9 is completely analogous.

G_7, G_8, G_{11}: It is clear that all of the basis elements for T' are invariants and all restrictions are explained by Lemma 1. It suffices to check that all elements are represented by images of the sums of Euler classes.

G_7. The bundle λ^{2^n} and v^{2^n} can be extended to line bundles over BG, say λ' and v' respectively. Then

$$\pi^*(e(v')) = e(v^{2^n}) = v^{2^{ns}} \quad \text{and} \quad \pi^*(e(\lambda')) = e(\lambda^{2^n}) = u^{2^{ns}}.$$

Applying again (8) we get

$$\begin{aligned} \pi^* e(\lambda') = u^{2^{ns}} &= u\bar{x}_1^{2^{ns}-1} + \sum_{i=1}^{ns} \bar{x}_1^{2^{ns}-2^i} \bar{x}_2^{2^{i-1}} \\ &\equiv u\bar{x}_1^{2^{ns}-1} + \bar{x}_2^{2^{ns-1}} \bmod (1+t) \\ &\equiv u\bar{x}_1^{2^{ns}-1} \bmod (1+t) \end{aligned}$$

by Lemma 1.

Similarly, applying Lemma 1 we have for G_8

$$\pi^*(e(\det(\pi_! \lambda))) = u^{2^{ns}} \equiv \bar{x}_1^{2^{ns}-1} u \bmod (1+t),$$

$$\pi^*(e(\det(\pi_! v))) = F(u^{2^{ns}}, v^{2^{ns}}) \equiv \bar{x}_1^{2^{ns}-1} u + \bar{y}_1^{2^{ns-1}} v \bmod (1+t)$$

and for G_{11}

$$\pi^*(e(\det(\pi_! \lambda))) = v^{2^{ns}} \equiv \bar{y}_1^{2^{ns}-1} v \bmod (1+t),$$

$$\pi^*(e(\det(\pi_! v))) = F(u^{2^{ns}}, v^{2^{ns}}) \equiv \bar{x}_1^{2^{ns}-1} u + \bar{y}_1^{2^{ns}-1} v \bmod (1+t).$$

For the proof of Theorem 1, we only need to see i). This completes the proof of Theorem 1. □

Proposition 2 ii) may have an independent interest. Let us sketch the proof.

Using the Euler characteristic formula of [10], Theorem D, one can compute $K(s)^*$-Euler characteristic

$$\chi_{2,s}(G) = rank_{K(s)^*}\ K(s)^{even}(BG),$$

for the classifying spaces of the groups in the title. The answer is as follows.

group	$\chi_{2,s}$
G_1	$2^{(2n+3)s}$,
G_2, G_4, G_9	$2^{2(n+1)s-1} - 2^{2ns-1} + 2^{(2n+1)s}$,
G_3, G_{10}	$3 \cdot 2^{2(n+1)s-1} - 2^{(2n+1)s-1}$,
$G_5, G_6, G_7, G_8, G_{11}, G_{12}$	$2^{2(n+1)s-1} - 2^{2s-1} + 2^{3s}$,
G_{13}, G_{16}	$2^{2(n+1)s-1} - 2^{(n+2)s-1} + 2^{(n+3)s}$,
G_{14}, G_{15}, G_{17}	$2^{2(n+1)s-1} - 2^{(n+1)s-1} + 2^{(n+2)s}$.

As $T \subset T'$ it suffices to prove $\chi_{2,s}(T) = \chi_{2,s}(T')$. It is easy to check the following relation between the size of the trivial summand $x = \chi_{2,s}(T)$ and $\chi_{2,s}(G)$ for all groups under consideration

$$(\chi_{2,s}(H) - x)/2 + 2^s x = \chi_{2,s}(G), \tag{12}$$

where $\chi_{2,s}(H) = 2^{2s(n+1)}$.

Therefore it suffices to see that the number of basis elements of $T' \subset G$, in Proposition 2 i) is equal to x in (12) for all cases

G	$\chi_{2,s}(T')$
G_3	$2^{(2n+1)s}$,
G_4, G_9	4^{ns},
G_7, G_8, G_{11}	4^s.

□

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References

[1] J. F. Adams, Infinite loop spaces, Annals of Mathematics Studies, Princeton University Press, Princeton, 1978.

[2] M. Bakuradze and N. Gachechiladze, Morava K-theory rings of the extensions of C_2 by the products of cyclic 2-groups, Moscow Math. J., **16 (4)**(2016), 603–619.

[3] M. Bakuradze, Morava $K(s)^*$-rings of the extensions of C_p by the products of good groups under diagonal action, Georgian Math. J., **22 (4)**(2015), 451–455.

[4] M. Bakuradze and M. Jibladze, Morava K-theory rings of groups $G_{38},\dots,G_{41}$ of order 32, J. K-Theory, **13**(2014), 171–198.

[5] M. Bakuradze and S. Priddy, Transferred Chern classes in Morava K-theory, Proc. Amer. Math. Soc., **132**(2004), 1855–1860.

[6] M. Bakuradze and S. Priddy, Transfer and complex oriented cohomology rings, Algebraic and Geometric Topology, **3**(2003), 473–507.

[7] M. Bakuradze and V. V. Vershinin, Morava K-theory rings for the dihedral, semi-dihedral and generalized quaternion groups in Chern classes, Proc. Amer. Math. Soc., **134**(2006), 3707–3714.

[8] A. Dold, The fixed point transfer of fibre-preserving maps, Math. Zeit., **148**(1976), 215–244.

[9] T. Gramushnjak and P. Puusemp, Description of a Class of 2-Groups, Journal of Nonlinear Math. Physics, **13**(2006), 55–65.

[10] M. Hopkins, N. Kuhn, and D. Ravenel, Generalized group characters and complex oriented cohomology theories, J. Amer. Math. Soc., **13, 3**(2000), 553–594.

[11] J. R. Hunton, Morava K-theories of wreath products, Math. Proc. Camb. Phil. Soc., **107**(1990), 309–318.

[12] D. C. Johnson and W. S. Wilson, BP operations and Morava's extraordinary K-theories, Math. Z., **144** (1975), 55–75.

[13] D. S. Kahn and S. B. Priddy, Applications of the transfer to stable homotopy theory, Bull. Amer. Math. Soc., **78**(1972), 981–987.

[14] I. Kriz, Morava K-theory of classifying spaces: Some calculations, Topology, **36**(1997), 1247–1273.

[15] D. C. Ravenel, Morava K-theories and finite groups, Contemp. Math., **12**(1982), 289–292.

[16] B. Schuster, Morava K-theory of groups of order 32, Algebraic and Geometric Topology, **11**(2011), 503–521.

[17] B. Schuster, Morava K-theory of classifying spaces, Habilitationsschrift, 2006.

[18] B. Schuster and N. Yagita, On Morava K-theory of extraspecial 2-groups, Proc. Amer. Math. Soc., **132, 4**(2004), 1229–1239.

[19] M. Tezuka and N. Yagita, Cohomology of finite groups and Brown-Peterson cohomology II, Algebraic Topology (Arcata, Ca, 1986), 396–408. Lecture Notes in Math. **1370**, Springer, Berlin, 1989.

[20] M. Tezuka and N. Yagita, Cohomology of finite groups and Brown-Peterson cohomology II, Homotopy theory and related topics (Kinosaki, 1988), 57–69. Lecture Notes in Math. **1418**, Springer, Berlin, 1990.

[21] N. Yagita, Equivariant BP-cohomology for finite groups, Trans. Amer. Math. Soc., **317, 2**(1990), 485–499.

[22] N. Yagita, Cohomology for groups of $rank_p(G) = 2$ and Brown-Peterson cohomology, J. Math. Soc. Japan, **45, 4**(1993), 627–644.

[23] N. Yagita, Note on BP-theory for extensions of cyclic groups by elementary abelian p-groups, Kodai Math. J. **20, 2**(1997), 79–84.

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