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MOD 2 MORAVA K-THEORY FOR FROBENIUS COMPLEMENTS OF EXPONENT DIVIDING $2^n \cdot 9$

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Abstract

We determine the cohomology rings $K(s)^*(B\mathcal{G})$ at 2 for all finite Frobenius complements $\mathcal{G}$ of exponent dividing $2^n \cdot 9$.

Let V be an abelian group, and let $\mathcal{G}$ be a group of automorphisms of V. If $\mathcal{G}$ has exponent $2^n \cdot 3^k$ for $0 \leqslant n$ and $0 \leqslant k \leqslant 2$ and $\mathcal{G}$ acts freely on V, then $\mathcal{G}$ is finite (see [**6**] Theorem 1.1). Every finite group that acts freely on an abelian group is isomorphic to a Frobenius complement in some finite Frobenius group (see [**6**] Lemma 2.6). By the classification of finite Frobenius complements (see [**7**]) the quotient of $\mathcal{G}$ by its maximal normal 3-subgroup $\mathcal{H}$ is isomorphic to a cyclic 2-group $\mathcal{C}$, a generalized quaternion group Q, the binary tetrahedral group $2\mathcal{T}$ of order 24 (or SL(2,3)), or the binary octahedral group $2\mathcal{O}$ of order 48. Then Atiyah-Hirzebruch-Serre spectral sequence for $\mathcal{H} \lhd \mathcal{G}$ implies that at 2 the ring $K(s)^*(B\mathcal{G})$ is isomorphic to $K(s)^*(B\mathcal{K})$, for $\mathcal{K} = \mathcal{G}/\mathcal{H}$ is either $\mathcal{C}, Q, 2\mathcal{T}, 2\mathcal{O}$. For the cyclic group $\mathcal{C} = \mathbb{Z}/2^k$, $K(s)^*(B\mathbb{Z}/2^k) = \mathbb{F}_2[v_s, v_s^{-1}][u]/(u^{2^{ks}})$. For the generalized quaternion group $Q_{2^{m+2}}$ we have Theorem 1.1 of [**4**]. We deduce Morava K-theory rings at 2 for the groups $2\mathcal{T}$ and $2\mathcal{O}$ as certain subgroups in $K(s)^*(BQ_8)$ and $K(s)^*(BQ_{16})$ respectively (Proposition 5 and Proposition 6.)

In [**3**] we proved the following formula for the first Chern class of the transferred line complex bundle: Let $X \to Y$ be the regular two covering defined by free action of $\mathbb{Z}/2$ on X and let $\theta \to Y$ be the associated line complex bundle; Let $\xi \to X$ be a complex line bundle and let $\zeta \to Y$ be the plane bundle, transferred from ξ by Atiyah transfer [**2**]. Then for $Tr^* : K(s)(X) \to K(s)^*(Y)$, the transfer homomorphism [**1**] for our covering $X \to Y$, one has

$$Tr^*(c_1(\xi)) = c_1(\theta) + c_1(\zeta) + v_s \sum_{i=1}^{s-1} c_1(\theta)^{2^s - 2^i} c_2(\zeta)^{2^{i-1}}. \qquad (1)$$

We show that formula 1 plays major role in the ring structure $K(s)^*(B\mathcal{G})$ at 2 for aforementioned groups and gives another derivations for some related rank one Lie groups.

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Much of our note is written in terms of Theorem 1.1 of [**4**]. Let

$$G = \langle a, b \,|a^{2^{m+1}} = 1,\; b^2 = a^e, bab^{-1} = a^r\rangle, \quad m \geqslant 1$$

and either $e = 0$, $r = -1$ (the dihedral group $D_{2^{m+2}}$ of order 2^{m+2}), $e = 2^m$, $r = -1$ (the generalized quaternion group $Q_{2^{m+2}}$) or $m \geqslant 2$, $e = 0$, $r = 2^m - 1$ (the semidihedral group $SD_{2^{m+2}}$).

Spectral sequence consideration (see [**8**]) imply that $K(s)(BG)$ is generated by following Chern classes $|c| = |x| = 2$, $|c_2| = 4$:

$$c = c_1(\eta_1),\; \eta_1 : G/\langle a\rangle \cong \mathbf{Z}/2 \to \mathbb{C}^*,\; b \mapsto -1;$$
$$x = c_1(\eta_2),\; \eta_2 : G/\langle a^2, b\rangle \cong \mathbf{Z}/2 \to \mathbb{C}^*,\; a \mapsto -1;$$

and $c_2 = c_2(\xi_{\pi_1})$, where $\xi_{\pi_1} \to B\langle a, b\rangle$ is the plane bundle transferred from the canonical line bundle $\xi \to B\langle a\rangle$, for the double covering $\pi_1 : B\langle a\rangle \to B\langle a, b\rangle$ corresponding to η_1.

The ring structure is the result of the formula for transferred first Chern class 1. See [**4**].

Let N be the normalizer of $U(1)$ in S^3. The normalizes of the maximal torus in $SO(3)$ is $O(2) = U(1) \rtimes \mathbb{Z}/2$ and $\mathbb{Z}/2$ acts on $K(s)^*BU(1) = K(s)^*[[u]]$ by $[-1]_F(u)$ as above.

Since $BU(1)\hat{}_p = [colim_n B\mathbb{Z}/(p^n)]\hat{}_p$, we have

$$K(s)^*(BO(2)) = K(s)^*(lim_m(BD_{2^{m+2}})) = K(s)^*(lim_m(BSD_{2^{m+2}}))$$

and

$$K(s)^*(BN) = K(s)^*(lim_m(BQ_{2^{m+2}})).$$

Thus Theorem 1.1 of [**4**] implies

Corollary 1. $K(s)^*(BO(2)) = K(s)^*[[c, c_2]]/(c^{2^s}, v_s c \sum_{i=1}^{s} c^{2^s-2^i} c_2^{2^{i-1}})$, where $c = c_1(det\eta)$ and $c_2 = c_2(\eta)$ are the Chern classes of the bundle $\eta \to BO(2)$, the complexification of canonical $O(2)$ bundle.

Corollary 2. $K(s)^*(BN) = K^*(s)[[c, c_2]]/(c^{2^s}, c^2 + v_s c \sum_{i=1}^{s} c^{2^s-2^i} c_2^{2^{i-1}})$, where $c = c_1(\nu)$ is the Chern class of ν the pullback bundle of the canonical real line bundle by $N \to N/U(1) = \mathbb{Z}/2$ and $c_2 = c_2(p^*(\zeta))$ is the Euler class of the pullback bundle of the canonical quaternionic line bundle by the inclusion $N \subset S^3$.

Then $RP^2 \to BO(2) \to BO(3)$ is the projective bundle of the canonical $SO(3)$ bundle. Hence the pullback of the complexification of this canonical $SO(3)$ bundle splits over $BO(2)$ as $\eta \oplus det\eta$. Note that $c_1(det\eta) = c_1(\eta) + v_s c_2(\eta)^{2^{s-1}}$ modulo transfer for the covering $BU(1) \to BO(2)$. Thus $K(s)^*(BSO(3))$ is subring in $K(s)^*(BO(2))$ generated by $v = c^2 + v_s c c_2^{2^{s-1}} + c_2$ and $w = cc_2$. This implies

Corollary 3. $K(s)^*(BSO(3)) = K(s)^*[[v, w]](f_s(v, w), g_s(v, w))$, where $|v| = 4$, $|w| = 6$, and $f_s = f_s(v, w)$, $g_s = g_s(v, w)$ are determined by $f_2 = vw$, $g_2 = w^2$ and for $s > 2$

$$f_s = \begin{cases} f_{s-1}^2 & s \text{ even}, \\ \frac{f_{s-1}g_{s-1}}{v} + wv^{2^{s-1}-1} & s \text{ odd}, \end{cases}$$

$$g_s = \begin{cases} g_{s-1}^2 & s \text{ odd}, \\ \frac{f_{s-1}g_{s-1}}{v} + wv^{2^{s-1}-1} & s \text{ even}. \end{cases}$$

Our main result is the following.

Let $\mathcal{G}$ be a group acting freely on an abelian group. Let $\mathcal{G}$ be of exponent dividing $2^n \cdot 9$ (hence $\mathcal{G}$ is necessarily finite, as above) and let $\mathcal{H} \lhd \mathcal{G}$ be the maximal normal 3-subgroup.

Theorem 4. *As a ring $K(s)^*(B\mathcal{G})$ has one of the following forms*

(i) If $\mathcal{G}/\mathcal{H}$=Q_8 , then $K(s)^*(B\mathcal{G}) = K(s)^*[c, x, c_2]/R$
and the relations R are determined by

$c^{2^s} = x^{2^s} = 0$, $v_s c c_2^{2^{s-1}} = v_s \sum_{i=1}^{s-1} c^{2^s-2^i+1} c_2^{2^{i-1}} + c^2$, $v_s^2 c_2^{2^s} = c^2 + cx + x^2$, $v_s x c_2^{2^{s-1}} = v_s \sum_{i=1}^{s-1} x^{2^s-2^i+1} c_2^{2^{i-1}} + x^2$.

(ii) If $\mathcal{G}/\mathcal{H}$=$Q_{2^{m+2}}$, $m > 1$, then $K(s)^*(B\mathcal{G}) = K(s)^*[c, x, c_2]/R$,
and the relations R are determined by

$c^{2^s} = x^{2^s} = 0$, $v_s c c_2^{2^{s-1}} = v_s \sum_{i=1}^{s-1} c^{2^s-2^i+1} c_2^{2^{i-1}} + c^2$, $v_s^{2\kappa(m)} c_2^{2^{ms}} = cx + x^2$, $v_s x c_2^{2^{s-1}} = v_s x \sum_{i=1}^{s-1} c^{2^s-2^i} c_2^{2^{i-1}} + \sum_{i=1}^{ms} v_s^{1+\kappa(m)+2^{ms}-2^i} c_2^{(2^{ms}+1)2^{s-1}-(2^s-1)2^{i-1}}$ $+ cx$,

where $\kappa(m) = \frac{2^{ms}-1}{2^s-1}$.

(iii) If $\mathcal{G}/\mathcal{H}$=$2\mathcal{T}$, then $K(s)^*(B\mathcal{G}) = K(s)^*[c_2]/c_2^{(2^s+1)2^{s-1}}$.

(iv) If $\mathcal{G}/\mathcal{H}$=$2\mathcal{O}$, then
$K(s)^*(B\mathcal{G}) = K(s)^*[c, c_2]/(c^{2^s}, c^2 + v_s c \sum_{i=1}^{s} c^{2^s-2^i} c_2^{2^{i-1}}, c_2^{(2^s+1)2^{s-1}})$.

(v) If $\mathcal{G}/\mathcal{H}$=$\mathbb{Z}/2^k$, then $K(s)^*(B\mathcal{G}) = K(s)^*[c]/c^{2^{ks}}$.

Here in all cases $|c| = |x| = 2$, $|c_2| = 4$.

The statement (v) is clear. (i) and (ii) follow from Theorem 1.1 of [**4**] for Q_8 and $Q_{2^{m+2}}$ respectively. What remains is to consider the cases of binary tetrahedral and binary octahedral groups.

Binary Polyhedral groups

As it is known any finite subgroup of $SO(3)$ is either a cyclic group, a dihedral group or one of the groups of a Platonic solid: tetrahedral group $\mathcal{T} \cong A_4$, cube/octahedral group $\mathcal{O} \cong S_4$, or icosahedral group $\mathcal{I} \cong A_5$. We consider the preimages of the latter groups under the covering homomorphism $S^3 \to SO(3)$.

Binary tetrahedral group

Binary tetrahedral group $2\mathcal{T}$ as the group of 24 units in the ring of Hurwitz integers $2\mathcal{T}$ is given by $\{\pm 1, \pm i, \pm j, \pm k, \frac{1}{2}(\pm 1 \pm i \pm j \pm k)\}$.

This group can be written as a semidirect product $2T = Q_8 \rtimes \mathbb{Z}/3$, where Q_8 is the quaternion group consisting of the 8 Lipschitz units $\pm 1, \pm i, \pm j, \pm k$ and $\mathbb{Z}/3$ is the cyclic group generated by $-\frac{1}{2}(1+i+j+k)$. The cyclic group acts on the normal subgroup Q_8 by conjugation. So that the generator of $\mathbb{Z}/3$ cyclically rotates i, j, k.

Consider now Morava K-theory at 2. Then relations of Theorem 1.1 of [**4**] for $K(s)^*(BQ_8)$ imply that its subring of invariants under $\mathbb{Z}/3$ action is generated by c_2: the generator of $\mathbb{Z}/3$ cyclically rotates c, x and $c + x + v_s c^{2^{s-1}} x^{2^{s-1}}$. If ignoring the powers of v_s then the first and second elementary symmetric functions in these three symbols are equal to $c_2^{2^{s-1}}$ and $c_2^{2^s}$ respectively and the third is zero. It follows that $K(s)^*(B2\mathcal{T}) \cong [K(s)^*(BQ_8)]^{\mathbb{Z}/3}$.

Proposition 5. $K(s)^*(B2\mathcal{T}) \cong K(s)^*[c_2]/c_2^{(2^s+1)2^{s-1}}$, *where* $|c_2| = 4$.

Binary octahedral group $2\mathcal{O}$

This group is given as the union of the 24 Hurwitz units $\{\pm 1, \pm i, \pm j, \pm k, \frac{1}{2}(\pm 1 \pm i \pm j \pm k)\}$ with all 24 quaternions obtained from $\frac{1}{\sqrt{2}}(\pm 1 \pm i + 0j + 0k)$ by permutation of coordinates.

The generalized quaternion group Q_{16} forms a subgroup of $2\mathcal{O}$ and its conjugacy classes has 3 members. Therefore by the transfer argument $B2\mathcal{O}$ is a stable wedge summand of BQ_{16} after localized at 2, meaning $K(s)^*(B2\mathcal{O})$ is the subring in $K(s)^*(BQ_{16})$ at 2. We show that this is the subring generated by two symbols c and c_2 of Theorem 1.1 of [**4**]. Namely one has

Proposition 6. $K(s)^*(B2\mathcal{O})$ *is isomorphic to*

$$K(s)^*[c, c_2]/(c^{2^s}, c^2 + v_s c \sum_{i=1}^{s} c^{2^s - 2^i} c_2^{2^{i-1}}, c_2^{(2^s+1)2^{s-1}}),$$

where $|c| = 2$, $|c_2| = 4$.

Binary icosahedral group

$2\mathcal{I}$ is given as the union of the 24 Hutwitz units $\{\pm 1, \pm i, \pm j, \pm k, \frac{1}{2}(\pm 1 \pm i \pm j \pm k)\}$ with all 96 quaternions obtained from $\frac{1}{2}(0 \pm 1 \pm i \pm \varphi^{-1} j \pm \varphi k)$ by even permutation of coordinates. Here $\varphi = \frac{1}{2}(1 + \sqrt{5})$ is the golden ratio. This group is isomorphic to $SL_2(5)$-the group of all 2×2 matrices over $\mathbb{F}_5$ with unit determinant.

Among other subgroups the relevant subgroup is the binary tetrahedral group formed by Hurwitz units. Then coset $2\mathcal{I}/2\mathcal{O}$ has 5 members hence by the transfer argument again $B2\mathcal{I}$ splits off $B2\mathcal{O}$ after localized at 2. Thus we obtain

$$K(s)^*B(2\mathcal{I}) \cong K(s)^*B(2\mathcal{T}).$$

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