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ON THE COEFFICIENT RING OF THE RATIONAL FORMAL GROUP LAW

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Abstract. We calculate the coefficient ring in low dimensions of the formal group law $F_2(x,y)$ given by the member of the Nadiradze family F_n with $n=2$.

რეზიუმე. ნადირაძის ფორმულარულ ჯგუფთა F_n ოჯახის მეორე წევრის $F_2(x,y)$-ისთვის გამოთვლილია კოეფიციენტთა რგოლი დაბალ განზომილებებში.

Introduction

In his habilitation dissertation [1], [2], late Roin Nadiradze formulated several problems concerning the study of formal group laws which can be written in various special forms. One of the most interesting among these was a family of formal groups given by

$$F_n(x,y)=\frac{\sum\limits_{i=0}^{n-2}\left(A_i(y)x^{i+2}-A_i(x)y^{i+2}\right)}{B(y)x-B(x)y}\Psi(xy),$$

where A_i, B and Ψ are formal series of one variable with certain restrictions.

In this paper we will consider the formal power series of two variables $F(x,y)$ given by the member of the family F_n with $n=2$.

$$\frac{A(y)x^2-A(x)y^2}{B(y)x-B(x)y}. \tag{1}$$

Let us thus assume that the series $F(x,y)$, with coefficients in some ring R, is a formal group law. It then follows that there is a (unique) ring homomorphism $f:L\to R$ classifying this formal group law. Here, L is the Lazard ring $\mathbf{Z}[a_1,a_2,\dots]$ of polynomials over integers in variables a_i (of

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degree i) carrying the universal formal group $F_U(x,y)$, and our F is $f(F_U)$. More precisely, if one denotes

$$F_U(x,y) = x + y + \sum_{1\leqslant i,j} \alpha_{U\,ij} x^i y^j$$

and

$$F(x,y) = x + y + \sum_{1\leqslant i,j} \alpha_{ij} x^i y^j$$

then

$$\alpha_{ij} = f(\alpha_{U\,ij})$$

for a unique ring homomorphism f.

There is then a uniquely defined ideal I in L such that for any $f : L \to R$, the resulting formal group $F = f(F_U)$ has form (1) above if and only if I is contained in the kernel of f. The description of low degree defining relations corresponding to this ideal is given in the following theorem.

Theorem 1. *There is a set of polynomial generators* $z_1, z_2, \dots$ *of the Lazard ring for which the low degree defining relations of the ideal* I *corresponding to the formal groups of the form* (1) *above are given by*

$5z_5 = z_2z_3 + 2z_1z_4$,

$2z_6 = 0$,

$7z_7 = z_2z_1z_4 - 8\,z_2{z_1}^2z_3 + 2\,z_2{z_1}^5 + 6\,{z_1}^3{z_2}^2 + z_1{z_2}^3 + 2\,{z_2}^2z_3 + 4\,{z_1}^3z_4 + 3\,z_3z_4 + z_1{z_3}^2$,

$2z_8 = {z_2}^4 + {z_1}^2{z_2}^3 + z_4{z_1}^4 + {z_1}^2{z_3}^2 + z_2{z_3}^2 + {z_4}^2$,

$3z_9 = 2\,{z_2}^2{z_1}^2z_3 + 2\,{z_2}^2z_1z_4 + 2\,{z_1}^4z_2z_3 + 2\,{z_1}^3z_2z_4 - 1856\,z_3z_4{z_1}^2 + 2\,{z_1}^3{z_3}^2 + 12\,{z_2}^2{z_1}^5 + 2\,{z_2}^3{z_1}^3 + 2\,{z_2}^4z_1 + 4\,{z_1}^7z_2 + 2\,{z_1}^5z_4 + 2\,{z_3}^3 - z_1{z_4}^2 + 2\,{z_2}^3z_3$,

$z_{10} = 0$,

$11z_{11} = -19662\,{z_2}^2z_3z_4 + 5\,{z_2}^2z_3{z_1}^4 - 648\,{z_2}^3z_3{z_1}^2 + 7\,{z_1}^3{z_2}^2z_4 + 3\,z_2{z_1}^6z_3 + 10\,z_2{z_1}^3{z_3}^2 - 90238\,z_2{z_1}^5z_4 - 15772\,{z_1}^4z_3z_4 - 596\,{z_2}^4z_3 + 9\,{z_1}^7{z_2}^2 - 10\,{z_1}^5{z_2}^3 + 648822\,{z_1}^3{z_2}^4 + 10\,z_2{z_1}^9 + 5\,{z_1}^5{z_3}^2 + 10\,{z_1}^7z_4 + 5\,z_2{z_1}^2z_3z_4 + 8\,{z_2}^3z_1z_4 + 9\,{z_2}^5z_1 + 8\,{z_1}^2{z_3}^3 + 6\,z_3{z_4}^2 + 7\,{z_1}^3{z_4}^2 + 2\,z_4z_1{z_3}^2 + 9\,z_2z_1{z_4}^2 + 8\,z_2{z_3}^3$,

$z_{12} = 0$,

$13z_{13} = 7\,{z_1}^6z_3z_4 + 12\,{z_3}^3z_4 + 2\,z_1{z_3}^4 + 8\,{z_2}^5z_3 + 5\,{z_2}^2{z_3}^3 + 2\,{z_1}^5{z_2}^4 + {z_1}^7{z_2}^3 + 8\,{z_1}^9{z_2}^2 + 3\,{z_1}^{11}z_2 + 8\,{z_1}^7{z_3}^2 + 3\,{z_1}^4{z_3}^3 + {z_1}^9z_4 + 11\,z_1{z_4}^3 + 2\,{z_2}^4z_1z_4 + 8\,z_3{z_4}^2z_2 + 6\,z_3{z_4}^2{z_1}^2 + 4\,{z_1}^3{z_4}^2z_2 + 12\,{z_2}^2z_1{z_4}^2 + 12\,{z_2}^2{z_3}^2{z_1}^3 + 5\,{z_2}^3{z_3}^2z_1 + 11\,{z_2}^4z_3{z_1}^2 + 3\,{z_2}^3z_3{z_1}^4 + 9\,{z_2}^2z_3{z_1}^6 + 2\,{z_2}^3z_3z_4 + 9\,z_2{z_3}^2{z_1}^5 + 2\,z_2{z_3}^3{z_1}^2 + 9\,z_2z_3{z_1}^8 + 10\,{z_1}^5{z_2}^2z_4 + 8\,{z_1}^7z_2z_4 + 3\,{z_1}^3{z_2}^3z_4 + 9\,{z_1}^3{z_3}^2z_4 + 6\,z_2{z_3}^2z_4z_1 + 2\,z_2z_3z_4{z_1}^4 + 5\,{z_1}^5{z_4}^2$

$\dots$

Proof. First of all note that if the series A and B in (1) have form

$$A(t) = A_0 + A_1t + A_2t^2 + O(t^3)$$

and

$$B(t) = B_0 + B_1 t + B_2 t^2 + O(t^3)$$

then

$$F(x, y) = \frac{A_0}{B_0}(x + y) + O(xy),$$

so if $F(x, y)$ is a formal group law we must have $A_0 = B_0$, and after dividing the numerator and denominator appropriately we may assume that $A_0 = B_0 = 1$.

We then furthermore calculate

$$F(x, y) = x + y + A_1 xy + \sum_{i=2}^{\infty} B_i(x^i y + xy^i) + O(x^2 y^2).$$

We thus have

$$\alpha_{1i} = B_i,\ i \geqslant 2.$$

On the other hand it is well known that, after identifying the Lazard ring with the coefficient ring $\mathbf{MU}^*$ of the complex cobordism (in virtue of the celebrated Quillen theorem), for the universal formal group law one obtains the identity of formal series

$$1 + \sum_{i \geqslant 1} \alpha_{U\,1i} t^i = 1/\mathbf{CP}(t),$$

where

$$\mathbf{CP}(t) := 1 + \sum_{i \geqslant 1} [\mathbf{CP}^i] t^i,$$

with $[\mathbf{CP}^i]$ denoting the cobordism class of the i-dimensional complex projective space.

Thus series F can be written in the form

$$\frac{A(y)x^2 - A(x)y^2}{x/f(\mathbf{CP}(y)) - y/f(\mathbf{CP}(x))}.$$

Let us write the universal formal group law in the similar form:

$$F_U(x, y) = \frac{A(x, y)}{x/\mathbf{CP}(y) - y/\mathbf{CP}(x)}$$

by simply defining

$$A(x, y) := F_U(x, y)\left(x/\mathbf{CP}(y) - y/\mathbf{CP}(x)\right).$$

Denoting

$$A(x, y) = \sum_{i,j} A_{ij} x^i y^j,$$

where $A_{ij} = A_{ij}(a_1, a_2, \dots)$ are certain elements of the Lazard ring, we then obtain that our ring R is the quotient of the Lazard ring by the ideal generated by all A_{ij} with either $i \neq 2$ or $j \neq 2$.

Now direct calculation shows that in fact one has

$$
\begin{aligned}
A(x,y) = x^2 - y^2 + a_1(x^2y - xy^2) + (a_1a_2 - a_3)(x^2y^3 - x^3y^2) + \\
+ (a_1a_3 - a_1^2a_2)(x^2y^4 - x^4y^2) + (3a_5 - 25a_1a_4 - \\
- 66a_2a_3 - 6a_1^2a_3 + 60a_1a_2^2 + 6a_1^3a_2)(x^5y^2 - x^2y^5) + \\
+ (5a_5 - 42a_1a_4 - 111a_2a_3 - 9a_1^2a_3 + 101a_1a_2^2 + \\
+ 9a_1^3a_2)(x^4y^3 - x^3y^4) + \cdots ,
\end{aligned}
$$

so that the first nontrivial elements in the aforementioned ideal are

$$
\begin{aligned}
&A_{43} = -A_{34} = 5a_5 - 42a_1a_4 - 111a_2a_3 - 9a_1^2a_3 + 101a_1a_2^2 + 9a_1^3a_2, \\
&A_{53} = -A_{35} = 2a_6 + 20a_1a_5 + 2a_2a_4 + 2a_3^2 - 172a_1^2a_4 - 448a_1a_2a_3 - \\
&\quad - 40a_1^3a_3 + 40a_1^4a_2 + 406a_1^2a_2^2, \\
&A_{63} = -A_{36} = 14a_7 - 519a_1a_6 - 3544a_2a_5 - 1128a_3a_4 - 1140a_1^2a_5 + \\
&\quad + 31377a_1a_2a_4 - 897a_1a_3^2 + 9866a_1^3a_4 + 78486a_2^2a_3 - 31053a_1^3a_2^2 + \\
&\quad + 34230a_1^2a_2a_3 - 2280a_1^5a_2 - 71440a_1a_2^3 + 2280a_1^4a_3, \\
&A_{54} = -A_{45} = 7a_7 - 259a_1a_6 - 1767a_2a_5 - 564a_3a_4 - 565a_1^2a_5 + \\
&\quad + 15647a_1a_2a_4 - 1130a_1^5a_2 + 4890a_1^3a_4 - 448a_1a_3^2 + 1130a_1^4a_3 - \\
&\quad - 35619a_1a_2^3 + 39132a_2^2a_3 - 15416a_1^3a_2^2 + 16994a_1^2a_2a_3, \\
&\ldots
\end{aligned}
$$

Equating these elements to 0 gives the following relations:

$$5a_5 = (\text{decomposables in } a_1,\ a_2,\ a_3,\ a_4);$$

$$2b_6 = 0,$$

where

$$b_6 = a_6 + a_2a_4 + a_3^2 - 2a_1^3a_3 - 2a_1a_2a_3 - 2a_1^2a_4 + 2a_1^4a_2 + a_1^2a_2^2;$$

$$7a_7 = (\text{decomposables in } a_1,\ a_2,\ a_3,\ a_4,\ b_6);$$

$$a_1b_6 = 0;$$

$$2a_8 = (\text{decomposables in } a_1,\ a_2,\ a_3,\ a_4,\ b_6);$$

$$3a_9 = (\text{decomposables in } a_1,\ a_2,\ a_3,\ a_4,\ b_6);$$

$$a_3b_6 = 0;$$

$$a_{10} = (\text{decomposables in } a_1,\ a_2,\ a_3,\ a_4,\ b_6);$$

$$11a_{11} = (\text{decomposables in } a_1,\ a_2,\ a_3,\ a_4,\ b_6);$$

$$a_5b_6 = 0;$$

$$a_{12} = (\text{decomposables in } a_1,\ a_2,\ a_3,\ a_4,\ b_6).$$

To choose generators z_n, $n = 1, 2, \dots$ of **MU** we then gradually add to the generators a_n decomposable elements in such a way that the resulting relations simplify maximally.

The explicit form of the resulting generators z_n which give the relations of the theorem is given by

$$z_1 = -\mathbf{CP}_1,$$

$$z_2 = \mathbf{CP}_1{}^2 - \mathbf{CP}_2,$$

$$z_3 = -\frac{1}{2}\,\mathbf{CP}_3 + \frac{1}{2}\,\mathbf{CP}_1{}^3,$$

$$z_4 = -\mathbf{CP}_4 + 15\,\mathbf{CP}_1{}^4 - 21\,\mathbf{CP}_2\mathbf{CP}_1{}^2 + 4\,\mathbf{CP}_2{}^2 + 3\,\mathbf{CP}_3\mathbf{CP}_1,$$

$$z_5 = -\frac{1}{6}\,\mathbf{CP}_5 - \frac{13}{2}\,\mathbf{CP}_1{}^5 + \frac{61}{6}\,\mathbf{CP}_2\mathbf{CP}_1{}^3 - \frac{5}{2}\,\mathbf{CP}_1\mathbf{CP}_2{}^2 - \\ - \frac{5}{2}\,\mathbf{CP}_3\mathbf{CP}_1{}^2\mathbf{CP}_4\mathbf{CP}_1 + \frac{1}{2}\,\mathbf{CP}_2\mathbf{CP}_3,$$

$$z_6 = -\mathbf{CP}_6 + 2\,\mathbf{CP}_5\mathbf{CP}_1 - 9\,\mathbf{CP}_1\mathbf{CP}_2\mathbf{CP}_3 - 2\,\mathbf{CP}_2\mathbf{CP}_1{}^4 + \frac{3}{2}\,\mathbf{CP}_3\mathbf{CP}_1{}^3 + \\ + 9\,\mathbf{CP}_1{}^2\mathbf{CP}_2{}^2 - 2\,\mathbf{CP}_4\mathbf{CP}_1{}^2 + 3\,\mathbf{CP}_2\mathbf{CP}_4 + \frac{5}{4}\,\mathbf{CP}_3{}^2 - \frac{3}{4}\,\mathbf{CP}_1{}^6 - \\ - 2\,\mathbf{CP}_2{}^3,$$

$$z_7 = 1/4\,\mathbf{CP}_7 - \frac{2377}{3}\,\mathbf{CP}_2\mathbf{CP}_1{}^5 + \frac{1561}{2}\,\mathbf{CP}_1{}^3\mathbf{CP}_2{}^2 - 225\,\mathbf{CP}_1\mathbf{CP}_2{}^3 + \\ + \frac{1613}{8}\,\mathbf{CP}_1{}^7 + 152\,\mathbf{CP}_1\mathbf{CP}_2\mathbf{CP}_4 + 420\,\mathbf{CP}_3\mathbf{CP}_1{}^4 + \frac{9}{8}\,\mathbf{CP}_1\mathbf{CP}_3{}^2 + \\ + \frac{199}{2}\,\mathbf{CP}_2{}^2\mathbf{CP}_3 - 1/2\,\mathbf{CP}_3\mathbf{CP}_4 - \frac{425}{2}\,\mathbf{CP}_4\mathbf{CP}_1{}^3 - \\ - 442\,\mathbf{CP}_1{}^2\mathbf{CP}_2\mathbf{CP}_3 + \frac{181}{3}\,\mathbf{CP}_5\mathbf{CP}_1{}^2 - 42\,\mathbf{CP}_2\mathbf{CP}_5 - \mathbf{CP}_6\mathbf{CP}_1,$$

$$z_8 = 1/3\,\mathbf{CP}_8 + \frac{135}{4}\,\mathbf{CP}_1{}^8 + 311\,\mathbf{CP}_1{}^2\mathbf{CP}_2{}^3 - \frac{1989}{4}\,\mathbf{CP}_1{}^4\mathbf{CP}_2{}^2 + \\ + \frac{1783}{12}\,\mathbf{CP}_2\mathbf{CP}_1{}^6 - \frac{61}{3}\,\mathbf{CP}_2{}^4 + \frac{1481}{2}\,\mathbf{CP}_1{}^3\mathbf{CP}_2\mathbf{CP}_3 - \\ - \frac{1309}{4}\,\mathbf{CP}_1\mathbf{CP}_2{}^2\mathbf{CP}_3 - 263\,\mathbf{CP}_1{}^2\mathbf{CP}_2\mathbf{CP}_4 + \frac{169}{2}\,\mathbf{CP}_1\mathbf{CP}_3\mathbf{CP}_4 + \\ + 93\,\mathbf{CP}_1\mathbf{CP}_2\mathbf{CP}_5 - \frac{709}{4}\,\mathbf{CP}_1{}^2\mathbf{CP}_3{}^2 - 283\,\mathbf{CP}_3\mathbf{CP}_1{}^5 + \\ + \frac{205}{4}\,\mathbf{CP}_2\mathbf{CP}_3{}^2 + 45\,\mathbf{CP}_6\mathbf{CP}_1{}^2 - 15\,\mathbf{CP}_2\mathbf{CP}_6 + \frac{325}{2}\,\mathbf{CP}_4\mathbf{CP}_1{}^4 + \\ + 40\,\mathbf{CP}_2{}^2\mathbf{CP}_4 - \frac{1195}{12}\,\mathbf{CP}_5\mathbf{CP}_1{}^3 - \frac{57}{4}\,\mathbf{CP}_3\mathbf{CP}_5 - \frac{27}{2}\,\mathbf{CP}_7\mathbf{CP}_1,$$

$$z_9 = 1/10\,\mathbf{CP}_9 - 11/4\,\mathbf{CP}_3{}^3 - \frac{211}{8}\,\mathbf{CP}_1{}^9 - 147\,\mathbf{CP}_8\mathbf{CP}_1 +$$

$$+756\,\mathbf{CP}_2\mathbf{CP}_3\mathbf{CP}_4-\frac{34792}{3}\,\mathbf{CP}_1{}^4\mathbf{CP}_2\mathbf{CP}_3-10530\,\mathbf{CP}_1\mathbf{CP}_2\mathbf{CP}_3{}^2+$$
$$+\frac{146097}{2}\mathbf{CP}_1{}^2\mathbf{CP}_2{}^2\mathbf{CP}_3-4287\mathbf{CP}_1{}^2\mathbf{CP}_3\mathbf{CP}_4-23292\mathbf{CP}_1\mathbf{CP}_2{}^2\mathbf{CP}_4+$$
$$+\frac{44002}{3}\,\mathbf{CP}_1{}^3\mathbf{CP}_2\mathbf{CP}_4+7349\,\mathbf{CP}_1\mathbf{CP}_2\mathbf{CP}_6+\frac{2122}{3}\,\mathbf{CP}_1\mathbf{CP}_3\mathbf{CP}_5-$$
$$-\frac{98981}{6}\,\mathbf{CP}_1{}^2\mathbf{CP}_2\mathbf{CP}_5+\frac{1909}{3}\,\mathbf{CP}_2{}^2\mathbf{CP}_5-\frac{7435}{2}\,\mathbf{CP}_1{}^4\mathbf{CP}_5-$$
$$-\frac{388}{3}\,\mathbf{CP}_4\mathbf{CP}_5-1332\,\mathbf{CP}_2{}^3\mathbf{CP}_3+\frac{15389}{8}\,\mathbf{CP}_1{}^3\mathbf{CP}_3{}^2+$$
$$+\frac{28737}{4}\,\mathbf{CP}_1{}^6\mathbf{CP}_3+2\mathbf{CP}_3\mathbf{CP}_6+\frac{8489}{2}\,\mathbf{CP}_1{}^7\mathbf{CP}_2+\frac{24406}{3}\,\mathbf{CP}_1{}^5\mathbf{CP}_2{}^2-$$
$$-\frac{403457}{6}\,\mathbf{CP}_1{}^3\mathbf{CP}_2{}^3+15666\,\mathbf{CP}_1\mathbf{CP}_2{}^4+1875\,\mathbf{CP}_1{}^3\mathbf{CP}_6+$$
$$+\frac{2149}{4}\,\mathbf{CP}_1{}^2\mathbf{CP}_7-\frac{317}{2}\,\mathbf{CP}_2\mathbf{CP}_7+\frac{7707}{5}\,\mathbf{CP}_1{}^5\mathbf{CP}_4+\frac{1371}{2}\,\mathbf{CP}_1\mathbf{CP}_4{}^2,$$
$$z_{10}=1/12\,\mathbf{CP}_5{}^2+\frac{204111}{8}\,\mathbf{CP}_1{}^{10}-\frac{25}{2}\,\mathbf{CP}_1\mathbf{CP}_3{}^3-\frac{2203}{3}\,\mathbf{CP}_8\mathbf{CP}_1{}^2-$$
$$-\frac{71809}{4}\,\mathbf{CP}_1{}^5\mathbf{CP}_5+\frac{44027}{8}\,\mathbf{CP}_1{}^4\mathbf{CP}_3{}^2+22725\,\mathbf{CP}_1{}^7\mathbf{CP}_3-$$
$$-\frac{40777}{2}\,\mathbf{CP}_1{}^8\mathbf{CP}_2+\frac{657509}{12}\,\mathbf{CP}_1{}^6\mathbf{CP}_2{}^2-\frac{4051295}{12}\,\mathbf{CP}_1{}^4\mathbf{CP}_2{}^3+$$
$$+\frac{935125}{12}\,\mathbf{CP}_1{}^2\mathbf{CP}_2{}^4+9436\,\mathbf{CP}_1{}^4\mathbf{CP}_6+\frac{10651}{4}\,\mathbf{CP}_1{}^3\mathbf{CP}_7+$$
$$+3645\,\mathbf{CP}_1{}^6\mathbf{CP}_4+\frac{6889}{2}\,\mathbf{CP}_1{}^2\mathbf{CP}_4{}^2+5\,\mathbf{CP}_2\mathbf{CP}_4{}^2-1/3\,\mathbf{CP}_2\mathbf{CP}_8-$$
$$-35\,\mathbf{CP}_2{}^3\mathbf{CP}_4+11\,\mathbf{CP}_2{}^2\mathbf{CP}_6+158\,\mathbf{CP}_2{}^2\mathbf{CP}_3{}^2+1/2\,\mathbf{CP}_3\mathbf{CP}_7+$$
$$+7/2\,\mathbf{CP}_3{}^2\mathbf{CP}_4+\mathbf{CP}_4\mathbf{CP}_6+\frac{8043}{2}\,\mathbf{CP}_1\mathbf{CP}_2\mathbf{CP}_3\mathbf{CP}_4-\mathbf{CP}_{10}+$$
$$+\frac{58}{3}\,\mathbf{CP}_2{}^5+1/2\,\mathbf{CP}_9\mathbf{CP}_1-\frac{93415}{3}\,\mathbf{CP}_1{}^5\mathbf{CP}_2\mathbf{CP}_3+$$
$$+\frac{1442859}{4}\,\mathbf{CP}_1{}^3\mathbf{CP}_2{}^2\mathbf{CP}_3-\frac{106713}{2}\,\mathbf{CP}_1{}^2\mathbf{CP}_2\mathbf{CP}_3{}^2-$$
$$-\frac{40387}{2}\,\mathbf{CP}_1{}^3\mathbf{CP}_3\mathbf{CP}_4-116144\,\mathbf{CP}_1{}^2\mathbf{CP}_2{}^2\mathbf{CP}_4+$$
$$+\frac{448063}{6}\,\mathbf{CP}_1{}^4\mathbf{CP}_2\mathbf{CP}_4+36691\,\mathbf{CP}_1{}^2\mathbf{CP}_2\mathbf{CP}_6+$$
$$+\frac{43525}{12}\,\mathbf{CP}_1{}^2\mathbf{CP}_3\mathbf{CP}_5-\frac{331095}{4}\,\mathbf{CP}_1{}^3\mathbf{CP}_2\mathbf{CP}_5+$$
$$+\frac{18517}{6}\,\mathbf{CP}_1\mathbf{CP}_2{}^2\mathbf{CP}_5-\frac{1940}{3}\,\mathbf{CP}_1\mathbf{CP}_4\mathbf{CP}_5-$$
$$-\frac{27355}{4}\,\mathbf{CP}_1\mathbf{CP}_2{}^3\mathbf{CP}_3+5\,\mathbf{CP}_1\mathbf{CP}_3\mathbf{CP}_6-779\,\mathbf{CP}_1\mathbf{CP}_2\mathbf{CP}_7-$$

$$- \frac{277}{4}\,\mathbf{CP}_2\mathbf{CP}_3\mathbf{CP}_5,$$

$$\begin{aligned}
z_{11} = &-\frac{228541039}{6}\,\mathbf{CP}_1{\mathbf{CP}_2}^5 + \frac{10089}{10}\,\mathbf{CP}_9{\mathbf{CP}_1}^2+\\
&+ 97038138\,{\mathbf{CP}_1}^2\mathbf{CP}_2\mathbf{CP}_3\mathbf{CP}_4 - \frac{53883643}{4}\,\mathbf{CP}_1\mathbf{CP}_2\mathbf{CP}_3\mathbf{CP}_5+\\
&+ \frac{20881}{24}\,\mathbf{CP}_1{\mathbf{CP}_5}^2 + \frac{1172227}{2}\,{\mathbf{CP}_1}^2{\mathbf{CP}_3}^3 - \frac{6731}{3}\,\mathbf{CP}_8{\mathbf{CP}_1}^3+\\
&+ 79792066\,{\mathbf{CP}_1}^6\mathbf{CP}_5 + \frac{1106891557}{8}\,{\mathbf{CP}_1}^5{\mathbf{CP}_3}^2+\\
&+ \frac{2514248905}{4}\,{\mathbf{CP}_1}^8\mathbf{CP}_3 - \frac{11422414909}{8}\,{\mathbf{CP}_1}^9\mathbf{CP}_2+\\
&+ \frac{7361141287}{3}\,{\mathbf{CP}_1}^7{\mathbf{CP}_2}^2 - \frac{21321293441}{12}\,{\mathbf{CP}_1}^5{\mathbf{CP}_2}^3+\\
&+ \frac{12427316545}{24}\,{\mathbf{CP}_1}^3{\mathbf{CP}_2}^4 - \frac{2664055}{2}\,{\mathbf{CP}_1}^5\mathbf{CP}_6 + \frac{669287}{2}\,{\mathbf{CP}_1}^4\mathbf{CP}_7-\\
&- \frac{12154749491}{40}\,{\mathbf{CP}_1}^7\mathbf{CP}_4 + 22432669\,{\mathbf{CP}_1}^3{\mathbf{CP}_4}^2 - 276\,\mathbf{CP}_{10}\mathbf{CP}_1+\\
&+ \frac{2230969379}{8}\,{\mathbf{CP}_1}^{11} + \frac{152567}{2}\,\mathbf{CP}_3{\mathbf{CP}_4}^2 - \frac{793}{3}\,\mathbf{CP}_3\mathbf{CP}_8+\\
&+ \frac{1843}{2}\,{\mathbf{CP}_3}^2\mathbf{CP}_5 - \frac{1457}{5}\,\mathbf{CP}_2\mathbf{CP}_9 - \frac{13175}{6}\,\mathbf{CP}_2{\mathbf{CP}_3}^3-\\
&- \frac{14182039}{2}\,{\mathbf{CP}_2}^3\mathbf{CP}_5 + \frac{203643079}{12}\,{\mathbf{CP}_2}^4\mathbf{CP}_3 + 43059\,{\mathbf{CP}_2}^2\mathbf{CP}_7-\\
&- 16022381\,\mathbf{CP}_1\mathbf{CP}_2{\mathbf{CP}_4}^2 + 49436130\,\mathbf{CP}_1{\mathbf{CP}_2}^3\mathbf{CP}_4+\\
&+ \frac{5612}{3}\,\mathbf{CP}_1\mathbf{CP}_2\mathbf{CP}_8 - 168445\,\mathbf{CP}_1{\mathbf{CP}_2}^2\mathbf{CP}_6+\\
&+ 32525669\,\mathbf{CP}_1{\mathbf{CP}_2}^2{\mathbf{CP}_3}^2 + \frac{327931}{4}\,\mathbf{CP}_1\mathbf{CP}_3\mathbf{CP}_7-\\
&- \frac{3404503}{8}\,\mathbf{CP}_1{\mathbf{CP}_3}^2\mathbf{CP}_4 + 107052\,\mathbf{CP}_1\mathbf{CP}_4\mathbf{CP}_6-\\
&- \frac{3146783473}{2}\,{\mathbf{CP}_1}^6\mathbf{CP}_2\mathbf{CP}_3 + \frac{2437547867}{2}\,{\mathbf{CP}_1}^4{\mathbf{CP}_2}^2\mathbf{CP}_3-\\
&- \frac{1170093661}{8}\,{\mathbf{CP}_1}^3\mathbf{CP}_2{\mathbf{CP}_3}^2 - \frac{227791903}{2}\,{\mathbf{CP}_1}^4\mathbf{CP}_3\mathbf{CP}_4-\\
&- 378388896\,{\mathbf{CP}_1}^3{\mathbf{CP}_2}^2\mathbf{CP}_4 + \frac{9745412596}{15}\,{\mathbf{CP}_1}^5\mathbf{CP}_2\mathbf{CP}_4+\\
&+ \frac{5156296}{3}\,{\mathbf{CP}_1}^3\mathbf{CP}_2\mathbf{CP}_6 + 19349356\,{\mathbf{CP}_1}^3\mathbf{CP}_3\mathbf{CP}_5-\\
&- \frac{634004811}{4}\,{\mathbf{CP}_1}^4\mathbf{CP}_2\mathbf{CP}_5 + \frac{982247123}{12}\,{\mathbf{CP}_1}^2{\mathbf{CP}_2}^2\mathbf{CP}_5-\\
&- \frac{38148289}{6}\,{\mathbf{CP}_1}^2\mathbf{CP}_4\mathbf{CP}_5 - \frac{1276858619}{4}\,{\mathbf{CP}_1}^2{\mathbf{CP}_2}^3\mathbf{CP}_3-
\end{aligned}$$

$$
\begin{aligned}
&- \frac{654141}{2}\,\mathbf{CP}_1{}^2\mathbf{CP}_3\mathbf{CP}_6 - \frac{1731595}{4}\,\mathbf{CP}_1{}^2\mathbf{CP}_2\mathbf{CP}_7 -\\
&- \frac{106243}{4}\,\mathbf{CP}_4\mathbf{CP}_7 - \frac{688}{3}\,\mathbf{CP}_5\mathbf{CP}_6 + 1/12\,\mathbf{CP}_{11} -\\
&- \frac{21415747}{2}\,\mathbf{CP}_2{}^2\mathbf{CP}_3\mathbf{CP}_4 + 4423400\,\mathbf{CP}_2\mathbf{CP}_4\mathbf{CP}_5 +\\
&+ 1925\,\mathbf{CP}_2\mathbf{CP}_3\mathbf{CP}_6,
\end{aligned}
$$

$$
\begin{aligned}
z_{12} = {}& \frac{947}{12}\,\mathbf{CP}_3\mathbf{CP}_4\mathbf{CP}_5 + 3\,\mathbf{CP}_4{}^3 - \mathbf{CP}_{12} - \frac{2914}{5}\,\mathbf{CP}_1\mathbf{CP}_2\mathbf{CP}_9 -\\
&- \frac{2725}{6}\,\mathbf{CP}_1\mathbf{CP}_3\mathbf{CP}_8 + \frac{608683}{4}\,\mathbf{CP}_1\mathbf{CP}_3\mathbf{CP}_4{}^2 +\\
&+ \frac{4355}{3}\,\mathbf{CP}_1\mathbf{CP}_3{}^2\mathbf{CP}_5 - \frac{1309889}{2}\,\mathbf{CP}_1{}^3\mathbf{CP}_3\mathbf{CP}_6 -\\
&- \frac{7659947485}{12}\,\mathbf{CP}_1{}^3\mathbf{CP}_2{}^3\mathbf{CP}_3 - \frac{152593691}{12}\,\mathbf{CP}_1{}^3\mathbf{CP}_4\mathbf{CP}_5 +\\
&+ \frac{1473346091}{9}\,\mathbf{CP}_1{}^3\mathbf{CP}_2{}^2\mathbf{CP}_5 - \frac{1268085449}{4}\,\mathbf{CP}_1{}^5\mathbf{CP}_2\mathbf{CP}_5 +\\
&+ \frac{464411797}{12}\,\mathbf{CP}_1{}^4\mathbf{CP}_3\mathbf{CP}_5 + \frac{10341974}{3}\,\mathbf{CP}_1{}^4\mathbf{CP}_2\mathbf{CP}_6 +\\
&+ \frac{19490706367}{15}\,\mathbf{CP}_1{}^6\mathbf{CP}_2\mathbf{CP}_4 - \frac{9081602429}{12}\,\mathbf{CP}_1{}^4\mathbf{CP}_2{}^2\mathbf{CP}_4 -\\
&- \frac{2277923977}{10}\,\mathbf{CP}_1{}^5\mathbf{CP}_3\mathbf{CP}_4 - \frac{14040856049}{48}\,\mathbf{CP}_1{}^4\mathbf{CP}_2\mathbf{CP}_3{}^2 +\\
&+ \frac{29249978491}{12}\,\mathbf{CP}_1{}^5\mathbf{CP}_2{}^2\mathbf{CP}_3 - \frac{151036345303}{48}\,\mathbf{CP}_1{}^7\mathbf{CP}_2\mathbf{CP}_3 +\\
&+ 214063\,\mathbf{CP}_1{}^2\mathbf{CP}_4\mathbf{CP}_6 - \frac{6790473}{8}\,\mathbf{CP}_1{}^2\mathbf{CP}_3{}^2\mathbf{CP}_4 +\\
&+ \frac{1309247}{8}\,\mathbf{CP}_1{}^2\mathbf{CP}_3\mathbf{CP}_7 + \frac{520059285}{8}\,\mathbf{CP}_1{}^2\mathbf{CP}_2{}^2\mathbf{CP}_3{}^2 -\\
&- 336418\,\mathbf{CP}_1{}^2\mathbf{CP}_2{}^2\mathbf{CP}_6 + 3734\,\mathbf{CP}_1{}^2\mathbf{CP}_2\mathbf{CP}_8 +\\
&+ 98868777\,\mathbf{CP}_1{}^2\mathbf{CP}_2{}^3\mathbf{CP}_4 - 32044232\,\mathbf{CP}_1{}^2\mathbf{CP}_2\mathbf{CP}_4{}^2 +\\
&+ \frac{344251}{4}\,\mathbf{CP}_1\mathbf{CP}_2{}^2\mathbf{CP}_7 + \frac{101771114}{3}\,\mathbf{CP}_1\mathbf{CP}_2{}^4\mathbf{CP}_3 -\\
&- \frac{28362945}{2}\,\mathbf{CP}_1\mathbf{CP}_2{}^3\mathbf{CP}_5 + \frac{45409}{48}\,\mathbf{CP}_1\mathbf{CP}_2\mathbf{CP}_3{}^3 -\\
&- \frac{1388}{3}\,\mathbf{CP}_1\mathbf{CP}_5\mathbf{CP}_6 - \frac{212431}{4}\,\mathbf{CP}_1\mathbf{CP}_4\mathbf{CP}_7 -\\
&- \frac{6925857}{8}\,\mathbf{CP}_1{}^3\mathbf{CP}_2\mathbf{CP}_7 + \mathbf{CP}_6{}^2 + 8\,\mathbf{CP}_2\mathbf{CP}_4\mathbf{CP}_6 -\\
&- \frac{769}{2}\,\mathbf{CP}_2\mathbf{CP}_3{}^2\mathbf{CP}_4 + \frac{633}{8}\,\mathbf{CP}_2\mathbf{CP}_3\mathbf{CP}_7 - \frac{1064}{3}\,\mathbf{CP}_2{}^2\mathbf{CP}_3\mathbf{CP}_5 +
\end{aligned}
$$

$$+ 1/6\,\mathbf{CP}_{11}\mathbf{CP}_1 + \frac{4461324565}{8}\,\mathbf{CP}_1{}^{12} + \frac{83}{16}\,\mathbf{CP}_3{}^4 - 1/3\,\mathbf{CP}_4\mathbf{CP}_8 -$$
$$- 1/20\,\mathbf{CP}_3\mathbf{CP}_9 - 7/2\,\mathbf{CP}_3{}^2\mathbf{CP}_6 + \frac{1164372815}{6}\,\mathbf{CP}_1{}^3\mathbf{CP}_2\mathbf{CP}_3\mathbf{CP}_4 -$$
$$- \frac{323195611}{12}\,\mathbf{CP}_1{}^2\mathbf{CP}_2\mathbf{CP}_3\mathbf{CP}_5 - \frac{85606501}{4}\,\mathbf{CP}_1\mathbf{CP}_2{}^2\mathbf{CP}_3\mathbf{CP}_4 +$$
$$+ \frac{26540036}{3}\,\mathbf{CP}_1\mathbf{CP}_2\mathbf{CP}_4\mathbf{CP}_5 + 152\mathbf{CP}_1\mathbf{CP}_2\mathbf{CP}_3\mathbf{CP}_6 + \frac{1}{18}\,\mathbf{CP}_2\mathbf{CP}_5{}^2 -$$
$$- 19\,\mathbf{CP}_2{}^2\mathbf{CP}_4{}^2 + 4/3\,\mathbf{CP}_2{}^2\mathbf{CP}_8 + \frac{469}{3}\,\mathbf{CP}_2{}^4\mathbf{CP}_4 - 52\,\mathbf{CP}_2{}^3\mathbf{CP}_6 +$$
$$+ 798\,\mathbf{CP}_2{}^3\mathbf{CP}_3{}^2 - 76177024\,\mathbf{CP}_1{}^2\mathbf{CP}_2{}^5 + \frac{40359}{20}\,\mathbf{CP}_9\mathbf{CP}_1{}^3 +$$
$$+ \frac{20929}{12}\,\mathbf{CP}_1{}^2\mathbf{CP}_5{}^2 + \frac{18739509}{16}\,\mathbf{CP}_1{}^3\mathbf{CP}_3{}^3 - \frac{28213}{6}\,\mathbf{CP}_8\mathbf{CP}_1{}^4 +$$
$$+ \frac{638303973}{4}\,\mathbf{CP}_1{}^7\mathbf{CP}_5 + \frac{4427335081}{16}\,\mathbf{CP}_1{}^6\mathbf{CP}_3{}^2 +$$
$$+ \frac{20112439837}{16}\,\mathbf{CP}_1{}^9\mathbf{CP}_3 - \frac{45689361171}{16}\,\mathbf{CP}_1{}^{10}\mathbf{CP}_2 +$$
$$+ \frac{117787213321}{24}\,\mathbf{CP}_1{}^8\mathbf{CP}_2{}^2 - \frac{127946032795}{36}\,\mathbf{CP}_1{}^6\mathbf{CP}_2{}^3 +$$
$$+ \frac{12428838833}{12}\,\mathbf{CP}_1{}^4\mathbf{CP}_2{}^4 - 2660476\,\mathbf{CP}_1{}^6\mathbf{CP}_6 + \frac{5358877}{8}\,\mathbf{CP}_1{}^5\mathbf{CP}_7 -$$
$$- \frac{24308647313}{40}\,\mathbf{CP}_1{}^8\mathbf{CP}_4 + \frac{179464091}{4}\,\mathbf{CP}_1{}^4\mathbf{CP}_4{}^2 - 552\mathbf{CP}_{10}\mathbf{CP}_1{}^2 -$$
$$- \frac{292}{3}\,\mathbf{CP}_2{}^6. \qquad \square$$

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