Journal of Mathematical Sciences, Vol. 189, No. 1, February, 2013

TRANSFERRED CHARACTERISTIC CLASSES AND GENERALIZED COHOMOLOGY RINGS

M. Bakuradze UDC 515.16

ABSTRACT. In this paper, we study the interaction between transferred Chern classes and Chern classes of transferred bundles. We calculate the algebra $BP^*(X^p_{h\Sigma_p})$ and show that its multiplicative structure is completely determined by the Frobenius reciprocity. We also give some tables of the initial segments of the formal group law in the Morava K-theory which are often useful in calculations.

CONTENTS

1. Introduction

The stable transfer map and its applications is already a subject of a whole branch of algebraic topology. For the constructions and formal properties of the transfer map, the best reference is the book [1] where J. F. Adams suggested that all topologists become acquainted with the transfer map as a major tool in applications and proofs. For axiomatic approach, see [10].

Our aim here is to give some applications of the transfer map in the presentation of the cohomology rings in the explicit form.

A considerable part of the material is collected from joint publications with Stewart Priddy, Vladimir Vershinin, and Mamuka Jibladze [3–7].

For various examples of finite groups, the complex oriented cohomology ring coincides with its subring generated by Chern classes [36, 39, 40]. Even more groups are good in the sense that their Morava K-theory is generated by transferred Chern classes of complex representations of subgroups [23]. Special effort was needed to find an example of a group not good in this sense [28]. Thus, the relations in the complex oriented cohomology ring of a finite group derived from formal properties of the transfer should play a major role. The purpose of the first part is to elucidate for finite coverings the interaction between transferred Chern classes and Chern classes of transferred bundles.

Translated from Sovremennaya Matematika i Ee Prilozheniya (Contemporary Mathematics and Its Applications), Vol. 77, Complex Analysis and Topology, 2012.

 1072–3374/13/1891–0010

Let p be a prime and let $G \leq \Sigma_p$ be a subgroup of the symmetric group. In the first part we consider the complex oriented cohomology of homotopy orbit spaces $X^p_{hG} = EG{\times}_G X^p$. Several authors have calculated these cohomology groups [23–25]; however, we are particularly interested in the ring structure and thereby explicit formulas for the transfer. Thus we are led to consider Frobenius reciprocity, the relation between cup products and transfer:

$$\mathrm{Tr}^*(x)y = \mathrm{Tr}^*(x\rho^*(y))$$

where $\rho : EG{\times}X^p \to X^p_{hG}$ is the covering projection and

$$\mathrm{Tr}^* : E^*(X^p) \to E^*(X^p_{hG})$$

is the associated transfer homomorphism.

Let $\pi \leq \Sigma_p$ be the subgroup of cyclic permutations of order p. Our results for $MU^*(X^p_{h\pi})$, $X = \mathbf{C}P^\infty$, and the canonical complex line bundle $\xi \to \mathbf{C}P^\infty$, provide a universal example which enables us to write explicitly the Chern classes $c_1, \ldots, c_{p-1}$ of the transferred bundle ξ_π as a certain formal power series in the Euler class $c_p(\xi_\pi)$ with coefficients in $E^*(B\pi)$ plus certain transferred classes of the bundle ξ. We give an algorithm for calculating these coefficients. In particular for $E = BP$, Brown–Peterson cohomology, the coefficients of this formal power series are invariant under the action of the normalizer of π in Σ_p. This enables us to give similar results for Σ_p coverings. Moreover, we calculate the algebra $BP^*(X^p_{h\Sigma_p})$ and show that its multiplicative structure is completely determined by Frobenius reciprocity. In addition to the Morava K-theory $E = K(s)$, calculations become easier: we show that the formal power series in the algorithm above descends to polynomials. We derive an alternative way for calculation and give some examples. Finally we give some tables of the initial segments of the formal group law in the Morava K-theory which are often useful in calculations.

The author would like to thank Mamuka Jibladze for Maple programs used in the examples.

2. Transferred Chern Classes

2.1. Preliminaries. We recall that a multiplicative cohomology theory E^* is called *complex oriented* if there exists a Thom class, that is, a class $u \in E^2(\mathbf{C}P^\infty)$ that restricts to a generator of the free one-dimensional E^* module $E^2(\mathbf{C}P^1)$. The universal example is the complex cobordism MU^*. Then

$$E^*(\mathbf{C}P^\infty) = E^*[[x]],$$

where x is the Euler class of the canonical complex line bundle ξ over $\mathbf{C}P^\infty = BU(1)$. Further,

$$E^*(BU(1)^p) = E^*[[x_1, \ldots, x_p]],$$

where $x_i = c_1(\xi_i)$ and ξ_i is the pullback bundle over $BU(1)^p$ by the projection $BU(1)^p \to BU(1)$ on the ith factor.

Much of our paper is written in terms of transfer maps [1, 27] and formal group laws. Let us give a brief review of formal properties of the transfer. For a finite covering

$$\rho : X \to X/G,$$

there is a stable transfer map

$$\mathrm{Tr} = \mathrm{Tr}(\rho) : X/G^+ \to X^+.$$

For any multiplicative cohomology theory E^*, the *Frobenius reciprocity* holds, i.e., the induced map Tr^* is a map of $E^*(X/G)$ modules possessing the following properties:

(i) $\mathrm{Tr}^*(x\rho^*(y)) = \mathrm{Tr}^*(x)y$, $x \in E^*(X)$, $y \in E^*(X/G)$.

For example,

(ii) $\mathrm{Tr}^*(\rho^*(y)) = \mathrm{Tr}^*(1)y$.

The element $\mathrm{Tr}^*(1) \in E^0(X/G)$ is called the *index or stable Euler class* of the covering ρ. The following additional properties of the transfer will be used:

(iii) the transfer is natural with respect to pullbacks;

(iv) $\mathrm{Tr}(\rho_1 \times \rho_2) = \mathrm{Tr}(\rho_1) \wedge \mathrm{Tr}(\rho_2)$;

(v) if $\rho = \rho_2\rho_1$, then $\mathrm{Tr}(\rho) = \mathrm{Tr}(\rho_2)\,\mathrm{Tr}(\rho_1)$.

More generally, for a covering projection

$$\rho_{H,G} : X/H \to X/G$$

with $H \leq G$, there is a stable transfer map

$$\mathrm{Tr}_{H,G} : X/G^+ \to X/H^+.$$

To simplify notation, we write projection and transfer in equivalent ways $\rho = \rho_G$ and $\mathrm{Tr} = \mathrm{Tr}(\rho) = \mathrm{Tr}_G$ if $H = e$.

The reverse composition to (ii) is as follows:

(vi) (double coset formula) if $K, H \leq G$, then

$$\rho^*_{K,G}\,\mathrm{Tr}^*_{H,G} = \sum_x \mathrm{Tr}^*_{K\cap H^x,K} \circ x^{-1*} \circ \rho^*_{K^{x^{-1}}\cap H,H},$$

where the sum is taken over a set of double coset representatives $x \in K\backslash G/H$ and $H^x = xHx^{-1}$.

For a regular covering $\rho_{H,G}$, i.e., $H \trianglelefteq G$,

$$\rho^*_{H,G}\,\mathrm{Tr}^*_{H,G}(x) = N(x) = \sum_{g\in G/H} g^*(x),$$

where $N(x)$ is called the *norm* or *trace* of x.

In subsequent sections, the reduced transfer

$$\mathrm{Tr}_{H,G} : X/G \to X/H$$

is used.

We recall Quillen's formula [16, 30]. First,

$$E^*(B\mathbf{Z}/p) = E^*[[z]]/([p](z)),$$

where x is the Euler class of a faithful one-dimensional complex representation of $\mathbf{Z}/p$ and $[p](z)$ is the p-series or p-fold iterated formal sum. Then

$$\mathrm{Tr}^*_{\mathbf{Z}/p}(1) = [p](z)/z, \tag{2.1}$$

where $\mathrm{Tr}^*_{\mathbf{Z}/p}$ is the transfer homomorphism for the universal $\mathbf{Z}/p$-covering $E\mathbf{Z}/p \to B\mathbf{Z}/p$. The relation $[p](z) = 0$ is equivalent to the transfer relation

$$z\,\mathrm{Tr}^*_{\mathbf{Z}/p}(1) = \mathrm{Tr}^*_{\mathbf{Z}/p}(c_1(\mathbf{C})) = \mathrm{Tr}^*_{\mathbf{Z}/p}(0) = 0$$

obtained by applying (ii). Of course, since the transfer is natural, Quillen's formula enables us to calculate the stable Euler class for any regular $\mathbf{Z}/p$ covering.

In this spirit, let

$$\pi = \langle t\rangle \leq \Sigma_p$$

be the subgroup of cyclic permutations of order p. For a given free action of π on a space Y with a given complex line bundle $\eta \to Y$ we have an equivariant map

$$\eta_\pi = (g_1, \ldots, g_p) : Y \to BU(1)^p,$$

where g_i classifies the line bundle $t^{i-1}\eta$.

So by the naturality of the transfer, the calculation of transferred Chern classes $\mathrm{Tr}^*(c_1^i(\eta))$, $i \geq 1$, for cyclic coverings can be reduced to the covering

$$\rho_\pi : E\pi \times (BU(1))^p \to E\pi \times_\pi (BU(1))^p,$$

as the universal example.

For the symmetric group, the arguments are similar.

Let ξ be the canonical complex line bundle over $\mathbf{C}P^\infty = BU(1)$ and ξ_i be the pullback bundle over $BU(1)^p$ by the projection on the ith factor as before. Then

$$MU^*(BU(1)^p) = MU^*[[x_1, \ldots, x_p]],$$

$x_i = c_1(\xi_i)$ and $x_1 \cdots x_p$ is the Euler class of the bundle $\xi^{\times p} = \bigoplus \xi_i$.

Note that by transfer property (v), $\mathrm{Tr}(\rho_\pi)^*$ has the same value on the Chern classes $x_1, \ldots, x_p$: the group π permutes the x_i and $\rho_\pi t = \rho_\pi$, $t \in \pi$. Thus in calculations of the transfer we sometimes write these Chern classes in an equivalent way $x, tx, \ldots, t^{p-1}x$.

For the sphere bundle $S(\xi^{\times p})$, we have

$$MU^*(S(\xi^{\times p})) = MU^*[[x_1, \ldots, x_p]]/(x_1 \cdots x_p). \tag{2.2}$$

Then for the trace map

$$N = 1 + t + \cdots + t^{p-1}$$

we have $\ker N = \mathrm{Im}(1-t)$, $t \in \pi$ in $MU^*BU(1)^p$, and after restricting N to $MU^*(S(\xi^{\times p}))$ we have the exact sequence

$$\cdots \leftarrow MU^*(S(\xi^{\times p})) \xleftarrow{N} MU^*(S(\xi^{\times p})) \xleftarrow{1-t} MU^*(S(\xi^{\times p})) \xleftarrow{N} MU^*(S(\xi^{\times p})) \leftarrow \ldots. \tag{2.3}$$

Then let $\xi_\pi = E\pi \times_\pi \xi^{\times p}$ be the Atiyah transfer bundle [2],

$$S(\xi_\pi) = E\pi \times_\pi S(\xi^{\times p}) \tag{2.4}$$

be its sphere bundle, and

$$D(\xi_\pi) = E\pi \times_\pi D(\xi^{\times p}) \tag{2.5}$$

be its disk bundle. Let $X = \mathbf{C}P^\infty$; then $D(\xi_\pi)$ is homotopy equivalent to $X^p_{h\pi} = B(\pi \wr U(1))$.

The cofibration $D(\xi_\pi)/S(\xi_\pi) = (X^p_{h\pi})^{\xi_\pi}$ gives a long exact sequence

$$\cdots \leftarrow MU^*(S(\xi_\pi)) \leftarrow MU^*(X^p_{h\pi}) \xleftarrow{\times c_p} MU^*((X^p_{h\pi})^{\xi_\pi}) \leftarrow \ldots \tag{2.6}$$

where $(X^p_{h\pi})^{\xi_\pi}$ is the Thom space of the bundle ξ_π and the right homomorphism is multiplication by the Euler class $c_p = c_p(\xi_\pi)$.

Since the diagonal of $BU(1)^p$ is fixed under the permutation action of π, the inclusion

$$E\pi \to E\pi \times BU(1)^p, \quad x \to (x, \text{fixed point})$$

defines the inclusions

$$i : B\pi \to X^p_{h\pi}, \quad i_0 : B\pi \to S(\xi_\pi).$$

The projection $\varphi : X^p_{h\pi} \to B\pi$ induced by $\pi \wr U(1) \to \pi$ defines the projection

$$\varphi_0 : S(\xi_\pi) \to B\pi \tag{2.7}$$

and the compositions $\varphi_0 i_0$ and φi are the identities. We can consider $S(\xi_\pi)$ as a bundle over $B\pi$ with fiber $S(\xi^{\times p})$.

Let η be the canonical line bundle over $B\pi$ and

$$\theta = \varphi^*(\eta) \to X^p_{h\pi}$$

be the pullback bundle. Thus

$$i^*(\theta) = \eta, \quad i^*(\xi_\pi) = \mathbf{C} + \eta + \cdots + \eta^{p-1}.$$

Consider the pullback diagram

$$\begin{tikzcd} E\pi \times S(\xi^{\times p}) \arrow[r] \arrow[d, "\rho_0"'] & E\pi \times BU(1)^p \arrow[d, "\rho"] \\ S(\xi_\pi) \arrow[r] & X^p_{h\pi} \end{tikzcd}$$

Let $\mathrm{Tr} = \mathrm{Tr}(\rho)$ be the transfer of the covering ρ, and $\mathrm{Tr}_0 : S(\xi_\pi) \to S(\xi^{\times p})$ the transfer map of ρ_0.

We will often refer to the following lemma which follows from (2.3) and the Frobenius reciprocity.

Lemma 2.1. *In $MU^*(X^p_{h\pi})$, we have*

$$\mathrm{Im}\,\mathrm{Tr}^* \cap \mathrm{Ker}(\rho^*) = 0.$$

Proof. We have

$$\rho^*(\mathrm{Tr}^*(a)) = N(a) = 0 \;\Rightarrow\; a \in \mathrm{Im}(1-t) \;\Rightarrow\; \mathrm{Tr}^*(a) = 0.$$

The lemma is proved. □

Remark 2.2. Lemma 2.1 is valid only in a complex oriented cohomology E^* with a torsion-free coefficient ring. This lemma is used in the proof of Theorem 2.3 in a complex cobordism and in the second statement of Theorem 3.6 in the Brown–Peterson cohomology. By the naturality, these results hold for all E^* in the first case and all p-local E^* in the second.

2.2. Transferred Chern classes for cyclic coverings. In this section, we prove our main result for cyclic coverings, Theorem 2.4.

In the notation of the previous section, the kth Chern class of the bundle $\xi^{\times p}$ is an elementary symmetric function $\sigma_k(x_1, \ldots, x_p)$ in the Chern classes x_i and is the sum of $\binom{p}{k}$ elementary monomials. The action of π on the set of these monomials yields $p^{-1}\binom{p}{k}$ orbits, and the transfer homomorphism is constant on orbits by transfer property (v) or (iii).

Let E^* be a complex oriented cohomology theory. For $k = 1, \ldots, p-1$, let

$$\omega_k = \omega_k(x_1, \ldots, x_p) \in E^*(BU(1)^p)$$

be the sum of representative monomials one from each of these orbits. The value of $\mathrm{Tr}^*(\omega_k)$ does not depend on the choice of ω_k since ω_k is defined modulo $\mathrm{Im}(1-t)$ and on the elements of $\mathrm{Im}(1-t)$ the transfer homomorphism is zero again by (v). In other words, we can take any ω_k for which $N\omega_k = \sigma_k(x_1, \ldots, x_p)$ holds. As we shall explain in Corollary 2.8 of Theorem 2.3, the following result enables us to calculate the transfer on all elements whose norm is symmetric.

To simplify the notation, we set $X = \mathbf{C}P^\infty$ and $c_j = c_j(\xi_\pi)$, $j = 1, \ldots, p$.

Theorem 2.3. *We can construct explicit elements*

$$\delta_i^{(k)} \in \tilde{E}^*(B\pi), \quad k = 1, \ldots, p-1,$$

such that

$$\mathrm{Tr}^*(\omega_k) = c_k + \sum_{i \geq 0} \varphi^*(\delta_i^{(k)}) c_p^i$$

for the transfer of the covering $\rho : X^p \to X^p_{h\pi}$.

Before constructing the elements $\delta_i^{(k)}$ by proving Theorem 2.3, we prove their existence.

2.3. Complex cobordism of $(\mathbf{C}P^\infty)^p_{h\pi}$.

Theorem 2.4. *In $MU^*(X^p_{h\pi})$, we have:*

(a) *the annihilator of the Chern class $c = c_1(\theta)$ coincides with* $\operatorname{Im}\operatorname{Tr}^*$;
(b) *multiplication by $c_p = c_p(\xi_\pi)$ is a monomorphism;*
(c) *any element of* $\operatorname{Ker}(\rho^*)$ *has the form*

$$\sum_{k\geq 0} \varphi^*(\delta_k)c_p^k$$

for some elements $\delta_k \in \tilde{MU}^*(B\pi)$;
(d) *for $\pi = \mathbf{Z}/2$,*

$$MU^*B(\pi \wr U(1)) = MU^*[[c, c_1, c_2]]/(c_1 - c_1^*, c_2 - c_2^*) = MU^*(B\pi)[[\mathrm{Tr}^*(x), c_2]]/(cTr^*(x)),$$

where $c_i = c_i(\xi_\pi)$, $c_i^* = c_i(\xi_\pi \otimes_{\mathbf{C}} \theta)$, and x are Chern characteristic classes with

$$x \in MU^*(BU(1)^2) = MU^*[[x, tx]].$$

Lemma 2.5. *The left homomorphism in the long exact sequence* (2.6) *is an epimorphism and thus gives a short exact sequence*

$$0 \leftarrow MU^*(S(\xi_\pi)) \leftarrow MU^*(X^p_{h\pi}) \xleftarrow{\times c_p} MU^*(X^p_{h\pi})^{\xi_\pi} \leftarrow 0.$$

Moreover, there is a space X_π and a stable equivalence

$$\varphi_0 \vee f_\pi : S(\xi_\pi) \to B\pi \vee X_\pi,$$

with f_π factoring through the composite map

$$S(\xi_\pi) \to X^p_{h\pi} \xrightarrow{\mathrm{Tr}} E\pi \times BU(1)^p$$

and φ_0 as in (2.7).

Proof. Consider the Serre spectral sequence for the fibration (2.7):

$$S(\xi^{\times p}) \to S(\xi_\pi) \xrightarrow{\varphi_0} B\pi,$$

$E_2^{i,j} = H^i(\pi, H^j(S(\xi^{\times p}); \mathbf{F}_q))$ with the action of π on $H^*(S(\xi^{\times p}); \mathbf{F}_q)$ by permutations of the cohomological Chern classes.

When $q = p$,

$$E_2^{0,j} = H^j(S(\xi^{\times p}); \mathbf{F}_p)^\pi, \quad E_2^{i,0} = H^i(B\pi; \mathbf{F}_p).$$

Then in positive dimensions

$$H^*(S(\xi^{\times p}); \mathbf{F}_q) = \mathbf{F}_q[x_1, \ldots, x_p]/(x_1 \cdots x_p)$$

is a permutation representation of π acting on monomials which have degree zero in at least one indeterminate. This is a free $\mathbf{F}_q[\pi]$-module since all the monomials that are fixed under this action have been factored out after quotienting by the ideal $(x_1 \cdots x_p)$. Hence the cohomology of π with coefficients in this module is trivial in positive dimensions, i.e., $E_2^{i,j} = 0$ when $i, j > 0$. Thus the spectral sequence collapses and we have

$$H^*(S(\xi_\pi); \mathbf{F}_p) \approx H^*(B\pi; \mathbf{F}_p) \oplus \tilde{H}^*(S(\xi^{\times p}); \mathbf{F}_p)^\pi.$$

Also if $q \neq p$, we have

$$H^*(S(\xi_\pi); \mathbf{F}_q) \approx H^*(S(\xi^{\times p}); \mathbf{F}_q)^\pi.$$

Let X_π be a stable summand of $BU(1)^p$ defined as follows. The action of π on $BU(1)^p$ induces an action of π on the stable decomposition of $BU(1)^p$ as a wedge of all smash products of length

$1, \dots, p-1$, say Y_π, and a smash product of length p. Then choose X_π such that $NX_\pi = Y_\pi$, where $N = 1 + t + \cdots + t^{p-1}$. By the stable equivalence

$$S(\xi^{\times p}) \to BU(1)^p \to Y_\pi, \tag{2.8}$$

we can consider X_π as a stable summand of $S(\xi^{\times p})$. For any choice of X_π, consider the composition of stable maps

$$f_\pi : S(\xi_\pi) \to X^p_{h\pi} \xrightarrow{\mathrm{Tr}} E\pi \times BU(1)^p \to BU(1)^p \to X_\pi. \tag{2.9}$$

We must show that the stable map $\varphi_0 \vee f_\pi$ induces an isomorphism in cohomology for any group of coefficients $\mathbf{F}_q$, q is a prime, and hence gives a stable equivalence by the stable Whitehead lemma. It follows from the above arguments that

$$\tilde{H}^*(S(\xi_\pi); \mathbf{F}_p) = \varphi_0^* \tilde{H}^*(B\pi; \mathbf{F}_p) \oplus \mathrm{Tr}_0^* \tilde{H}^*(S(\xi^{\times p}); \mathbf{F}_p),$$

and

$$\tilde{H}^*(S(\xi_\pi); \mathbf{F}_q) = \mathrm{Tr}_0^* \tilde{H}^*(S(\xi^{\times p}); \mathbf{F}_q),$$

when $q \neq p$. The restriction of Tr_0 on X_π induces a monomorphism on $\mathrm{Im}\,\mathrm{Tr}_0^*$ since by the transfer property (iv),

$$\rho_0^* \mathrm{Tr}_0^* = N,$$

and the restriction of N on $\tilde{H}^*(X_\pi; \mathbf{F}_q)$ is a monomorphism. Hence $(\varphi_0 \vee \mathrm{Tr}_0 | X_\pi)^*$ is an isomorphism and so is $(\varphi_0 \vee f_\pi)^*$ by the commutative diagram

$$\begin{tikzcd} S(\xi_\pi) \arrow[r] \arrow[d, "\mathrm{Tr}_0"'] & X^p_{h\pi} \arrow[d, "\mathrm{Tr}"] \\ S(\xi^{\times p}) \arrow[r] & E\pi \times BU(1)^p \end{tikzcd} \tag{2.10}$$

This proves Lemma 2.5. □

Proof of Theorem 2.4. (a) We consider the restriction of any element $y \in MU^*(X^p_{h\pi})$ to $MU^*(S(\xi_\pi))$. By Lemma 2.5, we see that this restriction has the form $\varphi_0^*(u) + f_\pi^*(w)$ for some $u \in \tilde{MU}^*(B\pi)$, $w \in MU^*(X_\pi)$. Since the composition

$$S(\xi_\pi) \to X^p_{h\pi} \xrightarrow{\varphi} B\pi$$

coincides with φ_0, $\varphi^*(u)$ also restricts to $\varphi_0^*(u)$. By diagram (2.10), there is an element $v \in MU^*(BU(1)^p)$ such that $\mathrm{Tr}^*(v)$ restricts to $f_\pi^*(w)$. By the exactness

$$y = \varphi^*(u) + \mathrm{Tr}^*(v) + y_1 c_p,$$

for some $y_1 \in MU^*(X^p_{h\pi})$. For use in the proof of (c) we observe that (2.2) and (2.8) imply that v can be chosen in the direct summand $MU^*[[x_1, \dots, x_p]]/(x_1 \cdots x_p)$. Thus we can assume that this expression for y is unique and if $v \neq 0$ then $\mathrm{Tr}^*(v)$ restricts nontrivially in $MU^*(S(\xi_\pi))$.

Then we suppose that $cy = 0$. We know that $\rho^*(\theta) = \mathbf{C}$, hence $\rho^*(c) = 0$ and

$$c\,\mathrm{Tr}^*(v) = \mathrm{Tr}^*(\rho^*(c)v) = 0$$

by the Frobenius reciprocity. So we have

$$c\varphi^*(u) + cy_1 c_p = 0.$$

We prove that

$$\varphi^*(u) \in \mathrm{Im}\,\mathrm{Tr}^*.$$

Applying i^* we have

$$0 = i^*(c\varphi^*(u)) + i^*(cy_1 c_p) = zu$$

since

$$i^*\varphi^* = \mathrm{id}, \quad c = \varphi^*(z), \quad i^*(c_p) = 0.$$

Hence

$$u \in \mathrm{Ann}(z) = \mathrm{Im}\,\mathrm{Tr}^*_{\mathbf{Z}/p}$$

by the naturality of the transfer $\varphi^*(u) \in \mathrm{Im}\,\mathrm{Tr}^*$. Thus, $c\varphi^*(u) = 0$ and therefore $cy_1c_p = 0$. Multiplication by c_p is injective by Lemma 2.5, hence $cy_1 = 0$. Since $\dim(y_1) = \dim(y) - 2p$, iterating this argument gives us statement (a).

Item (b) follows from the fact that the right homomorphism in the short exact sequence from Lemma 2.5 is multiplication by the Euler class $c_p(\xi_\pi)$.

(c) Let $y \in \mathrm{Ker}\,\rho^*$. Since $\varphi\rho = *$ we have

$$0 = \rho^*(\mathrm{Tr}^*(v)) + \rho^*(y_1c_p).$$

If the first summand is not zero, it restricts nontrivially in $MU^*(S(\xi^{\times p}))$ by the definition of v. However, the second summand restricts to zero since

$$\rho^*(c_p) = x_1\cdots x_p, \quad \rho^*(y_1c_p) = \rho^*(y_1)x_1\cdots x_p,$$

$x_1\cdots x_p$ restricts to zero as the Euler class. Hence both summands are zero. Furthermore, multiplication by $x_1\cdots x_p$ is a monomorphism hence $\rho^*(y_1) = 0$. Thus

$$y = \varphi^*(u) + y_1c_p = \varphi^*(u) + (\varphi^*(u_1) + y_2c_p)c_p = \varphi^*(u) + \varphi^*(u_1)c_p + y_2c_p^2.$$

Repetition of this process proves c).

(d) The fact that c, c_1, and c_2 multiplicatively generate $MU^*B(\pi \wr U(1))$ follows from Lemma 2.5. The relations $c_1 = c_1^*$, $c_2 = c_2^*$ follow from the bundle relation

$$\xi_\pi \otimes_{\mathbf{C}} \theta = (\xi \otimes_{\mathbf{C}} \rho^*(\theta))_\pi = \xi_\pi,$$

which in turn follows from transfer property (i).

So we must prove that the Chern classes c, c_1, and c_2 with these relations are a complete system of generators and relations. Let us use the splitting principle to write formally

$$\xi_\pi = \eta_1 + \eta_2, \quad u_1 = c_1(\eta_1), \quad u_2 = c_1(\eta_2).$$

Let

$$F(x,y) = \sum \alpha_{ij}x^iy^j$$

be a formal group law. Using the bundle relation above and applying the Whitney formula for the first and second Chern classes, we obtain two relations of the form

$$F(u_1,c) + F(u_2,c) = c_1, \quad F(u_1,c)F(u_2,c) = c_2; \tag{2.11}$$

or in terms of c, $c_1 = u_1 + u_2$, and $c_2 = u_1u_2$,

$$F(u_1,c) + F(u_2,c) - c_1 = c\left(2 + \sum \beta_{ijk}c^ic_1^jc_2^k\right) = 0, \tag{2.12}$$

$$F(u_1,c)F(u_2,c) - c_2 = c\left(c_1 + \sum \gamma_{ijk}c^ic_1^jc_2^k\right) = 0 \tag{2.13}$$

for some coefficients β_{ijk}, $\gamma_{ijk} \in MU^*(pt)$.

We claim that relations (2.12) and (2.13) are equivalent to the following two obvious transfer relations for $\mathrm{Tr}^* : MU^*[[x, tx]] \to MU^*(B(\pi \wr U(1)))$:

$$c\,\mathrm{Tr}^*(1) = 0, \quad c\,\mathrm{Tr}^*(x) = 0.$$

Rewrite relations (2.12) and (2.13) as follows:

$$ca = 0, \quad \text{where} \quad a = 2 + \alpha_{11}c_1 + \sum_{k\ge 2} \alpha_{k1}(u_1^k + u_2^k) + o(c),$$

$$cb = 0, \quad \text{where} \quad b = c_1 + 2\alpha_{11}c_2 + \sum_{k\ge 2} \alpha_{k1}(u_1^{k-1} + u_2^{k-1})c_2 + o(c),$$

and the α_{ij} are the coefficients of the formal group law.

By the first part of Theorem 2.4, $a \in \operatorname{Im}\operatorname{Tr}^*$. Also by transfer property (vi)

$$\rho^*(a) = \rho^*(\operatorname{Tr}^*(1) + \alpha_{11}\operatorname{Tr}^*(x) + \sum_{k\ge 2} \alpha_{k1}\operatorname{Tr}^*(x^k)).$$

Thus by Lemma 2.5

$$\operatorname{Tr}^*(1) + \alpha_{11}\operatorname{Tr}^*(x) + \sum_{k\ge 2} \alpha_{k1}\operatorname{Tr}^*(x^k) = \frac{F(u_1, c) + F(u_2, c) - c_1}{c};$$

similarly, $b \in \operatorname{Im}\operatorname{Tr}^*$ and

$$\operatorname{Tr}^*(x) + \alpha_{11}\operatorname{Tr}^*(1)c_2 + \sum_{k\ge 2} \alpha_{k1}\operatorname{Tr}^*(x^{k-1})c_2 = \frac{F(u_1, c)F(u_2, c) - c_2}{c}.$$

Now since

$$x^k = x^{k-1}(x + tx) - x^{k-2}(xtx),$$

transfer property (i) and the calculation of $\operatorname{Tr}^*(x)$ is sufficient for the calculation of $\operatorname{Tr}^*(x^k)$, $k \ge 2$ (see also Corollary 2.8 and Remark 2.2). So we have

$$\operatorname{Tr}^*(1)(1 + g_0) + \operatorname{Tr}^*(x)h_0 = \frac{F(u_1, c) + F(u_2, c) - c_1}{c}, \tag{2.14}$$

$$\operatorname{Tr}^*(1)g_1 + \operatorname{Tr}^*(x)(1 + h_1) = \frac{F(u_1, c)F(u_2, c) - c_2}{c}, \tag{2.15}$$

where $g_0, h_0, g_1, h_1 \in MU^*(B(\pi \wr U(1)))$. This proves (d).

This completes the proof of Theorem 2.4. □

Formula (2.15) for calculating $\operatorname{Tr}^*(x)$ is complicated; let us give a simpler form. Consider again (2.13). Note that the coefficient $\gamma_{00k} \in MU^*(pt)$ contains a factor 2: the element

$$c_1 + \sum \gamma_{ijk} c^i c_1^j c_2^k$$

annihilates c and hence belongs to $\operatorname{Im}\operatorname{Tr}^*$. On the other hand,

$$\rho^*\operatorname{Tr}^* = 1 + t, \quad \rho^*(c) = 0, \quad \rho^*(c_1) = x + tx, \quad \rho^*(c_2) = xtx,$$

hence applying ρ^* we obtain that

$$x + tx + \sum \gamma_{0jk}(x + tx)^j (xtx)^k$$

belongs to $\operatorname{Im}(1 + t)$. So

$$\gamma_{00k}(xtx)^k = 2\gamma_k(xtx)^k,$$

that is, $\gamma_{00k} = 2\gamma_k$ for some coefficient γ_k.

Recall that, on the other hand, $F(c, c) = 0$, that is, $2c = o(c^2)$. So $\gamma_{00k}c = o(c^2)$, hence taking into account the relation $F(c, c) = 0$ we can rewrite (2.13) after division by

$$1 + \sum_{i,k\ge 0} \gamma_{i1k} c^i c_2^k$$

(the coefficient at cc_1) as follows:

$$cc_1 = d_0c + d_2cc_1^2 + \cdots + d_ncc_1^n + \ldots, \tag{2.16}$$

where $d_k = d_k(c, c_2) \in MU^*[[c, c_2]]$ and $d_0(0, c_2) = 0$; the subscript n indicates the coefficient at cc_1^n. Since

$$\rho^*(\mathrm{Tr}^*(x) - c_1) = 0,$$

it follows from Theorem 2.4c) that there exist elements

$$\delta_j \in \tilde{MU}^*(B\pi)$$

such that

$$\mathrm{Tr}^*(x) = c_1 + \sum_{j\geq 0} \varphi^*(\delta_j)c_2^j.$$

Using the inclusion $i : B\pi \to B(\pi \wr U(1))$ we have

$$i^*(c_1) = i_0^*(c), \quad i^* \mathrm{Tr}^*(x) = 0, \quad i^*(c_2) = 0;$$

therefore,

$$\varphi^*(\delta_0) = -c.$$

For the calculation of the other elements δ_j recall that $cTr^*(x) = 0$, hence

$$cc_1^n = -c\varphi^*(\delta^n), \quad n \geq 1, \tag{2.17}$$

where

$$\delta = -c + \sum_{j\geq 1} \delta_j c_2^j.$$

Combining (2.16) and (2.17), we obtain the following assertion.

Proposition 2.6. *The elements δ_j, $j > 0$, can be determined from the recurrence relations which arise from the following formula in $MU^*(B\pi)[[c_2]]$:*

$$\delta = d_0 + \sum_{i\geq 2} d_i\delta^i.$$

Proof. By definition, the element

$$\delta - d_0 - \sum_{i\geq 2} d_i\delta^i$$

belongs to $\mathrm{Ker}\,\rho^*$. On the other hand, this element is annihilated by c, hence

$$\delta - d_0 - \sum_{i\geq 2} d_i\delta^i \in \mathrm{Im}\,\mathrm{Tr}^* \cap \mathrm{Ker}(\rho^*) = 0$$

by Lemma 2.5. □

Corollary 2.7. *For the elements $\delta_j \in \tilde{MU}^*(B\mathbf{Z}/2)$ constructed in Proposition 2.6, the following formula holds in $MU^*B(\mathbf{Z}/2 \wr U(1))$:*

$$\mathrm{Tr}^*(x) = c_1 - c + \sum_{j\geq 1} \varphi^*(\delta_j)c_2^j.$$

In fact, we have proved Theorem 2.3 for $p = 2$. The general case, similar but more cumbersome, will be examined below.

Proof of Theorem 2.3. Note that by the definition of ω_k, the difference $\mathrm{Tr}^*(\omega_k) - c_k$ is an element of $\mathrm{Ker}(\rho^*)$. Thus, Theorem 2.4(c) implies the existence of the elements $\delta_i^{(k)}$ in Theorem 2.3.

First, let us elucidate the meaning of the relations

$$\xi_\pi \otimes_{\mathbf{C}} \theta = \xi_\pi$$

in the general case of $B(\pi \wr U(1))$.

Again, we can use the splitting principle and write formally

$$\xi_\pi = \eta_1 + \eta_2 + \cdots + \eta_p, \quad u_m = c_1(\eta_m), \quad m = 1, \ldots, p.$$

Applying the Whitney formula for the relation

$$\eta_1 \otimes_{\mathbf{C}} \theta + \cdots + \eta_p \otimes_{\mathbf{C}} \theta = \eta_1 + \cdots + \eta_p$$

and taking into account the fact that $c_m = c_m(\xi_\pi)$ is an elementary symmetric function $\sigma_m(u_1, \ldots, u_p)$, we have

$$\sigma_m(F(u_1, c), \ldots, F(u_p, c)) = c_m, \quad m = 1, \ldots, p, \tag{2.18}$$

or, in terms of $c, c_1, \ldots, c_p$,

$$c\left(p + \sum \beta^0_{i_0, i_1, \ldots, i_p} c^{i_0} c_1^{i_1} \ldots c_p^{i_p}\right) = 0$$

and

$$c\left((p-k)c_k + \sum \beta^k_{i_0, i_1, \ldots, i_p} c^{i_0} c_1^{i_1} \ldots c_p^{i_p}\right) = 0 \tag{2.19}$$

for $k = 1, \ldots, p-1$ and some $\beta^0_{i_0, i_1, \ldots, i_p}$, $\beta^k_{i_0, i_1, \ldots, i_p} \in MU^*(pt)$.

We claim that these relations are equivalent to the obvious relations

$$c\,\mathrm{Tr}^*(1) = 0, \quad c\,\mathrm{Tr}^*(\omega_k) = 0$$

for the elements $\omega_k \in MU^*(BU(1))^p$, $k = 1, \ldots, p-1$, defined above.

For the proof of our claim, multiply the kth relation from (2.19) by $p_k = (p-k)^{-1}$ in $\mathbf{F}_p$. Then by Theorem 2.4, $\mathrm{Ann}(c)$ coincides with $\mathrm{Im}\,\mathrm{Tr}^*$ and hence (2.18) implies that

$$p_k\big(\sigma_{k+1}(F(u_1, c), \ldots, F(u_p, c)) - c_{k+1}\big)/c = \mathrm{Tr}^*(a_k)$$

for some a_k which we have to find. Let us write

$$\begin{aligned}
\rho^*\Big(p_k\big(\sigma_{k+1}(F(u_1, c), \ldots, F(u_p, c))c_{k+1}\big)/c\Big) &= g^{(k)}(\sigma_1, \ldots, \sigma_p) \\
&= \sigma_k\big(1 + g_k^{(k)}(\sigma_1, \ldots, \sigma_p)\big) + \sum_{\substack{j \neq k \\ 1 \le j \le p-1}} \sigma_j g_j^{(k)}\big(\sigma_j, \sigma_{j+1}, \ldots, \check{\sigma_k}, \ldots, \sigma_p\big) \\
&= N(\omega_k)\big(1 + g_k^{(k)}(\sigma_1, \ldots, \sigma_p)\big) + \sum_{\substack{j \neq k \\ 1 \le j \le p-1}} N(\omega_j) g_j^{(k)}\big(\sigma_j, \sigma_{j+1}, \ldots, \check{\sigma_k}, \ldots, \sigma_p\big).
\end{aligned}$$

Here the symbol $\check{\sigma_k}$ indicates absence of the corresponding term. So we have

$$\begin{aligned}
&p_k\big(\sigma_{k+1}(F(u_1, c), \ldots, F(u_p, c)) - c_{k+1}\big)/c \\
&\quad = \mathrm{Tr}^*(\omega_k)\big(1 + g_k^{(k)}(c_1, \ldots, c_p)\big) + \sum_{\substack{j \neq k \\ 1 \le j \le p-1}} \mathrm{Tr}^*(\omega_j) g_j^{(k)}\big(c_j, c_{j+1}, \ldots, \check{c_k}, \ldots, c_p\big)
\end{aligned}$$

and

$$\big[\sigma_1(F(u_1, c), \ldots, F(u_p, c)) - c_1\big]/c$$

$$= \mathrm{Tr}^*(1)\big(1 + g_0^{(0)}(c_1, \dots, c_p)\big) + \sum_{1\le j\le p-1} \mathrm{Tr}^*(\omega_j) g_j^{(0)}(c_j, c_{j+1}, \dots, c_p).$$

This proves our claim.

For calculating $\delta_i^{(k)}$, we start with Eqs. (2.19) and rewrite them as follows:

$$cf_k(c, c_1, \dots, c_p) = 0, \quad k = 1, \dots, p-1. \tag{2.20}$$

These are equations in a power series algebra $MU^*(B\pi)[[c_p]]$, since we know $cc_k \in cMU^*(B\pi)[[c_p]]$.

Now we find explicitly formal series

$$\delta^{(k)}(c_p) = \sum_{i\ge 0} \delta_i^{(k)}(c) {c_p}^i \tag{2.21}$$

such that

$$\mathrm{Tr}^*(\omega_k) = c_k + \delta^{(k)}(c_p)$$

and hence

$$cc_k^j = -c(\delta^{(k)}(c_p))^j, \quad j \ge 1. \tag{2.22}$$

For this, we replace Eqs. (2.20) by the equations

$$c\widetilde{f}_k(c, \delta^{(1)}(c_p), \dots, \delta^{(p-1)}(c_p), c_p) = 0, \tag{2.23}$$

where $\widetilde{f}_k \in \mathrm{Ker}\,\rho_\pi^*$ is a series whose coefficient at $\delta^{(k)}$ is invertible.

In fact, $\widetilde{f}_k = 0$ since we know that

$$\mathrm{Ann}(c) = \mathrm{Im}\,\mathrm{Tr}^*, \quad \mathrm{Ker}(\rho^*) \cap \mathrm{Im}\,\mathrm{Tr}^* = 0$$

by Lemma 2.5.

Then equating each coefficient of the resulting series

$$g_k(c_p) = \widetilde{f}_k(c, \delta^{(1)}(c_p), \dots, \delta^{(p-1)}(c_p), c_p) = 0 \tag{2.24}$$

in the ring $MU^*(B\pi)[[c_p]]$ to zero, we obtain $p-1$ infinite strings of equations in $MU^*(B\pi)$. Assuming that $\delta_i^{(l)}$ are already found for $i < n$, we get

$$\delta_n^{(k)} = \psi_{n,k}\Big((\delta_i^{(1)})_{i\le n}, \ \dots, \ (\delta_i^{(k)})_{i<n}, \ \dots, \ (\delta_i^{(p-1)})_{i\le n}\Big), \tag{2.25}$$

which is a system of linear equations in $\delta_n^{(l)}$, $l = 1, \dots, p-1$, with invertible determinant and coefficients in $MU^*[[c]]$. Since $\delta_0^{(l)}$ are already known as lth Chern classes of the bundle $1 + \theta + \dots + \theta^{p-1}$, by induction on n we can solve formally (2.25) to get

$$\delta_n^{(k)}(c) = \tilde{\psi}_{n,k}((\delta_i^{(l)})_{i<n}). \tag{2.26}$$

This yields

$$\delta_n^{(k)} = \delta_n^{(k)}(z) \in MU^*(B\pi),$$

obviously satisfying our equations.

Now for the remaining equation (2.23) we proceed as follows. Let us look at the term $cf_k(0, 0, \dots, 0, c_p)$ in Eqs. (2.20). Note that $f_k(0, 0, \dots, 0, c_p)$ is divisible by p:

$$f_k \in \mathrm{Ann}(c) = \mathrm{Im}\,\mathrm{Tr}^* \ \Rightarrow \ \rho_\pi^* f_k \in \mathrm{Im}\, N \ \Rightarrow \ f_k(0, \dots, 0, \sigma_p) \in \mathrm{Im}\, N \ \Rightarrow \ f_k(0, \dots, 0, \sigma_p).$$

Next using the relation $[p]_F(c) = 0$ we know that pc is divisible by c^2; hence each occurrence of pc in these equations can be replaced by terms with higher powers of c. So $cf_k(0, 0, \dots, 0, c_p)$ can be replaced by a term divisible by c^2.

Also the kth relation from (2.20) contains the term $c(p-k)c_k$, and for the condition (2.24) we multiply the kth equation from (2.20) by $(p-k)^{-1}$, the inverse of $p-k$ in $\mathbf{F}_p$, and as above we can replace $c(p-k)c_k$ by cc_k+(terms divisible by c^2). Then we use (2.22) and substitute the series $\delta^{(k)}$ in the resulting equations, thus obtaining (2.23).

This completes the proof for $E = MU$, which is the universal example of complex oriented cohomology theories. From this result we can descend to all E. □

We now turn to calculation of Tr^* in general.

Corollary 2.8. *For all primes p, Theorem 2.3 enables us to explicitly calculate the transfer homomorphism for those polynomials $a \in \tilde{MU}^*[[x_1, \dots, x_p]]$ for which $Na = a + ta + \cdots + t^{p-1}a$ is symmetric in $x_1, \dots, x_p$.*

Proof. If

$$Na = \sigma_1 a_1(\sigma_1, \dots, \sigma_p) + \cdots + \sigma_{p-1} a_{p-1}(\sigma_1, \dots, \sigma_p),$$

then

$$\mathrm{Tr}^*(a) = \mathrm{Tr}^*(\omega_1) a_1(c_1, \dots, c_p) + \cdots + \mathrm{Tr}^*(\omega_{p-1}) a_{p-1}(c_1, \dots, c_p)).$$

To see this, we set

$$\hat{a} = \omega_1 a_1(\sigma_1, \dots, \sigma_p) + \cdots + \omega_{p-1} a_{p-1}(\sigma_1, \dots, \sigma_p).$$

Then $N(a - \hat{a}) = 0$, that is, $a - \hat{a} \in \mathrm{Im}(1-t)$, hence $\mathrm{Tr}^*(a) = \mathrm{Tr}^*(\hat{a})$. □

Remark 2.9. For $p = 2$, we have the recurrence formulas for $\mathrm{Tr}^*(x^k)$, $k \geq 1$:

$$\mathrm{Tr}^*(x) = \mathrm{Tr}^*(\omega_1), \quad \mathrm{Tr}^*(x^k) = \mathrm{Tr}^*(x^{k-1})c_1 - \mathrm{Tr}^*(x^{k-2})c_2.$$

This follows from the formula

$$x^k = x^{k-1}(x + tx) - x^{k-2}(xtx).$$

3. Transferred Chern Classes for Σ_p-Coverings

If we consider a p-local, complex, oriented cohomology E^*, then by standard transfer arguments (see Lemma 3.2 below), $E^*(B\Sigma_p)$ is isomorphic to the subring of $E^*(B\pi)$ invariant under the action of the normalizer of π in Σ_p. The results of this section imply that the elements $\delta_i^{(k)} \in \tilde{E}^*(B\pi)$ from Theorem 2.3 are invariant under this action. This defines elements $\tilde{\delta}_i^{(k)} \in \tilde{E}^*(B\Sigma_p)$ which we use for calculating the transfer.

In this section, we consider $BP^*(X^p_{h\Sigma_p})$ for $X = \mathbf{C}P^\infty$, and for the covering projection

$$\rho_{\Sigma_p} : E\Sigma_p \times X^p \to X^p_{h\Sigma_p},$$

we give the formula for the transfer homomorphism

$$\mathrm{Tr}_{\Sigma_p}{}^* : BP^*(X^p) \to BP^*(X^p_{h\Sigma_p}) \tag{3.1}$$

using the elements $\tilde{\delta}_i^{(k)}$.

We need definitions similar to those of Sec. 2, with the cyclic group replaced by the symmetric group. The p-fold product $\xi^{\times p}$ of the canonical line bundle over X^p extends to an p-dimensional bundle

$$\xi_{\Sigma_p} = E\Sigma_p \times_{\Sigma_p} \xi^{\times p} \tag{3.2}$$

over $X^p_{h\Sigma_p}$ classified by the inclusion

$$X^p_{h\Sigma_p} = B(\Sigma_p \wr U(1)) \hookrightarrow BU(p).$$

Let $c_i = c_i(\xi_{\Sigma_p})$. Then

$$\rho_{\Sigma_p}{}^*(c_i) = c_i(\xi^{\times p}) = \sigma_i,$$

the ith symmetric polynomial in the x_j, where $BP^*(X^p) = BP^*[[x_1, \dots, x_p]]$.

Then we have the projection

$$\varphi : X^p_{h\Sigma_p} \to B\Sigma_p \tag{3.3}$$

induced by the factorization $\Sigma_p \wr U(1)/U(1)^p = \Sigma_p$ and the inclusion

$$i : B\Sigma_p \to X^p_{h\Sigma_p}, \tag{3.4}$$

induced by the inclusion of Σ_p in $\Sigma_p \wr U(1)$.

Definition 3.1. Let $\tilde{c}_i = \mathrm{Tr}_{\Sigma_p}{}^*(x_1 x_2 \cdots x_i)$ for $i = 1, \dots, p-1$.

Lemma 3.2. *The following relation holds*:

$$\rho_{\Sigma_p}{}^*(\tilde{c}_i) = i!(p-i)!\sigma_i.$$

Proof. We have

$$\rho_{\Sigma_p}{}^*(\tilde{c}_i) = \rho_{\Sigma_p}{}^* \mathrm{Tr}^*_{\Sigma_p}(x_1 x_2 \cdots x_i) = N_{\Sigma_p}(x_1 x_2 \cdots x_i).$$

For each subset of i integers $\{j_1, j_2, \dots, j_i\}$ such that $1 \le j_k \le p$, there are $i!$ bijections

$$\{1, 2, \dots, i\} \to \{j_1, j_2, \dots, j_i\}$$

and $(p-i)!$ bijections

$$\{i+1, i+2, \dots, p\} \to \{1, 2, \dots, p\} \backslash \{j_1, j_2, \dots, j_i\}.$$

Thus, there are $i!(p-i)!$ summands of $x_{j_1} x_{j_2} \dots x_{j_i}$ in $N_{\Sigma_p}(x_1 x_2 \dots x_i)$. □

We recall that

$$BP^*(B\pi) = BP^*[[z]]/([p]z)$$

with $|z| = 2$. The corresponding calculation for $BP^*(B\Sigma_p)$ is also known (see [35]). For the reader's convenience, we derive this result in a form useful for our purposes.

Lemma 3.3. *As a BP^*-algebra,*

$$\text{(i)} \qquad BP^*(B\Sigma_p) = BP^*[[y]]/(yTr^*_{\Sigma_p}(1)),$$

where y and $\mathrm{Tr}^*_{\Sigma_p}(1)$ are uniquely determined by $\rho^*_{\pi,\Sigma_p}(y) = z^{p-1}$, and

$$\text{(ii)} \qquad \rho^*_{\pi,\Sigma_p}(\mathrm{Tr}^*_{\Sigma_p}(1)) = (p-1)!\,\mathrm{Tr}^*_\pi(1) = (p-1)![p](z)/z.$$

In particular, $|y| = 2(p-1)$.

Proof. (ii) If we apply the double coset formula (transfer property (vi)) to

$$BP^*(Be) \xrightarrow{\mathrm{Tr}^*_{e,\Sigma_p}} BP^*(B\Sigma_p) \xrightarrow{\rho^*_{\pi,\Sigma_p}} BP^*(B\pi),$$

the statement follows from Quillen's formula.

(i) The relation $y\,\mathrm{Tr}^*_{\Sigma_p}(1) = 0$ is a consequence of the Frobenius reciprocity. To see that it is the defining relation, we recall that the cohomology of $B\Sigma_p$ with simple coefficients in $\mathbf{Z}_{(p)}$ is

$$H^*(B\Sigma_p; \mathbf{Z}_{(p)}) = \mathbf{Z}_{(p)}[y]/(py),$$

where $|y| = 2(p-1)$. This easily follows from the mod-p cohomology and the Bockstein spectral sequence.

Also

$$H^*(B\pi; \mathbf{Z}_{(p)}) = \mathbf{Z}_{(p)}[z]/(pz),$$

where $|z| = 2$. The map

$$\rho_{\pi,\Sigma_p} : B\pi \to B\Sigma_p$$

yields

$$\rho^*_{\pi,\Sigma_p}(y) = x^{p-1}.$$

Now the Atiyah–Hirzebruch–Serre spectral sequence for $BP^*(B\Sigma_p)$ is

$$E_2 = H^*(B\Sigma_p; BP^*) = BP^*[y]/(py) \Longrightarrow BP^*(B\Sigma_p).$$

Since y is even-dimensional, the sequence collapses at $E_2 = E_\infty$. Thus, $BP^*(B\Sigma_p)$ is generated by y as a BP^*-algebra.

For the group $W = N_{\Sigma_p}(\pi)/\pi \approx \mathbf{Z}/(p-1)$, $|W|$ is prime to p, hence by the standard transfer argument

$$\rho^*_{\pi,\Sigma_p} : BP^*(B\Sigma_p) \to BP^*(B\pi)$$

is an injective map of BP^* algebras. Since

$$\rho^*_{\pi,\Sigma_p}(y \operatorname{Tr}^*_{\Sigma_p}(1)) = p! z^{p-1} + \text{terms of higher filtration},$$

we see that

$$y \operatorname{Tr}^*_{\Sigma_p}(1) = 0$$

is the unique relation. $\square$

Relating π and Σ_p, we have a lift of ρ_{π,Σ_p}

$$\begin{tikzcd}
X^p_{h\pi} \arrow[r, "\tilde{\rho}_{\pi,\Sigma_p}"] \arrow[d, "\varphi"'] & X^p_{\Sigma_p} \arrow[d, "\varphi"] \\
B\pi \arrow[r, "\rho_{\pi,\Sigma_p}"] & B\Sigma_p
\end{tikzcd}$$

Lemma 3.4. *We have*

$$\tilde{\rho}^*_{\pi,\Sigma_p}(\widetilde{c}_k) = k!(p-k)! \operatorname{Tr}^*_\pi(\omega_k).$$

Proof. Note that modulo $\operatorname{Im}(1-t)$, we have

$$\Sigma g^*(x_1 x_2 \cdots x_k) = k!(p-k)! \omega_k$$

summed over Σ_p/π. Applying the double coset formula we have

$$\rho^*_{\pi,\Sigma_p}(\widetilde{c}_k) = \rho^*_{\pi,\Sigma_p} \operatorname{Tr}^*_{\Sigma_p}(x_1 x_2 \cdots x_k) = \operatorname{Tr}^*_\pi \sum_{g \in \Sigma_p/\pi} g^*(x_1 x_2 \cdots x_k) = k!(p-k)! \operatorname{Tr}^*_\pi(\omega_k).$$

The lemma is proved. $\square$

Let

$$c = \varphi^*(y) \in BP^{2(p-1)}(X^p_{h\Sigma_p}).$$

Lemma 3.5. $\operatorname{Im} \operatorname{Tr}^*_{\Sigma_p}$ *is contained in the* BP^**-algebra generated by* $c, \tilde{c}_1, \ldots, \tilde{c}_{p-1}, c_p$.

Proof. By the Künneth isomorphism,

$$BP^*(X^p) = BP^*(X)^{\otimes p} = F \oplus T \tag{3.5}$$

as a π-module, where F is free and T is trivial. Explicitly, a BP^*-basis for T is $\{x_1^i \cdots x_p^i,\ i \geq 0\}$, while a BP^*-basis for F is $\{x_1^{i_1} \cdots x_p^{i_p},\ i_j \geq 0\}$, where not all the exponents are equal.

By Lemma 3.3, $\operatorname{Tr}^*_{\Sigma_p}(1)$ is a power series in c. Now recall (see [22, p. 44]) that we can consider $BP^*(X^p)$, $X = \mathbf{C}P^\infty$, as a free $BP^*[[\sigma_1, \ldots, \sigma_p]]$-module generated by 1 and the elements $x_1^{i_1} \cdots x_p^{i_p} \in F$, with $0 \leq i_j \leq p-j$. So by the Frobenius reciprocity, it suffices to calculate the transfer on these monomials. Summed over the symmetric group $\sum g^*(x_1^{i_1}\cdots x_p^{i_p})$ is a symmetric function and hence has the form

$$\sum_{\pi}\sum_{\Sigma_p/\pi} g^*(x_1^{i_1}\cdots x_p^{i_p}) = \sigma_1 s_1 + \cdots + \sigma_{p-1}s_{p-1} = \sum_{\pi}(\omega_1 s_1 + \cdots + \omega_{p-1}s_{p-1})$$

for the elements ω_k from Theorem 2.4 and some symmetric functions $s_1,\ldots,s_{p-1}$. Hence modulo $\ker N_\pi = \mathrm{Im}(1-t)$, $t\in\pi$, we have the following equation in F:

$$\sum_{\Sigma_p/\pi} g^*(x_1^{i_1}\cdots x_p^{i_p}) = \omega_1 s_1 + \cdots + \omega_{p-1}s_{p-1}.$$

The left sum consists of $(p-1)!$ elements each having the same transfer value. Also ω_k is the sum of $p^{-1}\binom{p}{k}$ elements $x_{i_1}\cdots x_{i_k}$; on each of these elements the transfer evaluates to $\mathrm{Tr}^*_{\Sigma_p}(x_1\cdots x_k) = \tilde{c}_k$. Thus, the Frobenius reciprocity and Lemma 3.4 are all that is needed for calculating $\mathrm{Tr}^*_{\Sigma_p}$. $\square$

Recall the elements $\delta_i^{(k)} \in \tilde{BP}^*(B\pi)$ derived from Theorem 2.3 by the naturality. By the standard transfer argument again, the map induced by

$$\tilde{\rho}_{\pi,\Sigma_p} : X^p_{h\pi} \to X^p_{h\Sigma_p},$$

the lift of $\rho_{\pi,\Sigma_p} : B\pi \to B\Sigma_p$, is also injective. Moreover, for $BP^*(X^p_{h\Sigma_p})$ the ring structure is completely determined by the following theorem.

Theorem 3.6. *As a BP^*-algebra,*

$$BP^*(X^p_{h\Sigma_p}) = BP^*[[c,\tilde{c}_1,\ldots,\tilde{c}_{p-1},c_p]]/(c\,\mathrm{Tr}^*_{\Sigma_p}(1), c\tilde{c}_i)$$

and we have the formula

$$\widetilde{c}_k - k!(p-k)!c_k = \Sigma_{i\geq 0}\varphi^*(\tilde{\delta}_i^{(k)})c_p^i, \quad k=1,\ldots,p-1,$$

where the elements $\tilde{\delta}_i^{(k)} \in \tilde{BP}^*(B\Sigma_p)$ are determined by

$$\rho^*_{\pi,\Sigma_p}(\tilde{\delta}_j^{(k)}) = k!(p-k)!\delta_j^{(k)}, \quad j\geq 0.$$

For the proof, we follow that of Theorem 2.4. Let

$$S(\xi_{\Sigma_p}) = E\Sigma_p \times_{\Sigma_p} S(\xi^{\times p})$$

be the sphere bundle of the bundle ξ_{Σ_p} of (3.2)). $X^p_{h\Sigma_p}$ is homotopy equivalent to the disk bundle

$$D(\xi_{\Sigma_p}) = E\Sigma_p \times_{\Sigma_p} D(\xi^{\times p}).$$

Then we have the obvious inclusion

$$i_0 : B\Sigma_p \to S(\xi_{\Sigma_p})$$

and the projection

$$\varphi_0 : S(\xi_{\Sigma_p}) \to B\Sigma_p$$

with fiber $S(\xi^{\times p})$; $\varphi_0 i_0$ is the identity. Thus, stably $B\Sigma_p$ is a wedge summand of $S(\xi_{\Sigma_p})$. As for the other summand, let

$$X_{\Sigma_p} = \vee_{i=1}^{p-1} E\Sigma_i \times_{\Sigma_i} BU(1)^{\wedge i}.$$

By the standard transfer argument localized at p, X_{Σ_p} is a stable summand of

$$\bigvee_{i=1}^{p-1} E\Sigma_i \times BU(1)^{\wedge i}$$

and hence of $E\Sigma_p \times BU(1)^{\times p}$. From this we derive the following result.

Lemma 3.7. *One has a stable equivalence localized at p*

$$\varphi_0 \vee f_{\Sigma_p} : S(\xi_{\Sigma_p}) \to B\Sigma_p \vee X_{\Sigma_p},$$

with f_{Σ_p}, the composition of stable maps

$$f_{\Sigma_p} : S(\xi_{\Sigma_p}) \to X^p_{h\Sigma_p} \xrightarrow{\mathrm{Tr}_{\Sigma_p}} E\Sigma_p \times BU(1)^p \to X_{\Sigma_p}.$$

Proof. The inclusion i_0 splits off $\varphi_0^* H^*(B\Sigma_p)$ in $H^*(S(\xi_{\Sigma_p}))$. Furthermore, in mod-p cohomology

$$H^*(S(\xi^{\times p})) = \mathbf{F}_p[x_1, \ldots, x_p]/(\sigma_p),$$

hence

$$H^*(S(\xi^{\times p}))^{\Sigma_p} \approx \mathbf{F}_p[\tilde{c}_1, \ldots, \tilde{c}_{p-1}],$$

by Lemma 3.2.

Then $H := \tilde{H}^*(S(\xi^{\times p}))$ is a free π module and $H^*(\Sigma_p; H) \subseteq H^*(\pi; H)$. Thus

$$H^*(\Sigma_p; H) = \begin{cases} H^{\Sigma_p} & \text{if } * = 0, \\ 0 & \text{if } * > 0. \end{cases}$$

Therefore, there is an isomorphism

$$H^*(S(\xi_{\Sigma_p})) \xrightarrow{\rho^* \oplus i_0^*} \tilde{H}^*(S(\xi^{\times p}))^{\Sigma_p} \oplus H^*(B\Sigma_p),$$

where $\rho : S(\xi^{\times p}) \to S(\xi_{\Sigma_p})$ is the projection. We have to prove that the first summand is $f^*_{\Sigma_p} \tilde{H}^*(X_{\Sigma_p})$.

By the naturality of the transfer we have the commutative diagram

$$\begin{tikzcd} S(\xi^{\times p}) \arrow[r] & BU(1)^p \arrow[r] & X_{\Sigma_p} \\ S(\xi_{\Sigma_p}) \arrow[u, "\mathrm{Tr}_0"] \arrow[r] & X^p_{h\Sigma_p} \arrow[u, "\mathrm{Tr}_{\Sigma_p}"'] & \end{tikzcd}$$

Thus, f_{Σ_p} coincides with $\tilde{f}_{\Sigma_p}$, the map Tr_0 followed by the horizontal maps in the above diagram. We show that the restriction of Tr_0^* to the image of $H^*(X_{\Sigma_p})$ is an isomorphism onto $H^*(S(\xi^{\times p}))^{\Sigma_p}$.

Now considering the transfer for the Σ_i coverings

$$E\Sigma_i \times BU(1)^{\wedge i} \to E\Sigma_i \times_{\Sigma_i} BU(1)^{\wedge i},$$

it follows from transfer properties (ii) and (vi) that $H^*(E\Sigma_i \times_{\Sigma_i} BU(1)^{\wedge i})$ is a submodule of $H^*(E\Sigma_i \times BU(1)^{\wedge i})$ generated by Σ_i norms of monomials in $x_1, x_2, \ldots, x_i$, with nonincreasing degrees. From this it is straightforward that $H^*(X_{\Sigma_p})$ and $H^*(S(\xi^{\times p}))^{\Sigma_p}$ have the same ranks in each dimension. Thus, we are reduced to showing the desired map is injective.

However, for any monomial x in $x_1, x_2, \ldots, x_i$, we have

$$\mathrm{Tr}_0^*(N_{\Sigma_i}(x)) = i!\,\mathrm{Tr}_0^*(x)$$

by the naturality of the transfer. Thus, the restriction of Tr_0^* to the image of $H^*(X_{\Sigma_p})$ will be a monomorphism if Tr_0^* is nonzero on polynomials consisting of monomials with nonincreasing degrees. This in turn will follow if the norm N_{Σ_p} is non-zero on such polynomials. In fact we claim:

(1) N_{Σ_p} is nonzero on any monomial $x^I = x_1^{i_1} \cdots x_{p-1}^{i_{p-1}}$, and
(2) different monomials with nonincreasing degrees in $x_1, \ldots, x_i$, $i < p$ are in different Σ_p orbits.

Claim (2) is obvious. To prove (1), we set $J = (j_1, \ldots, j_p)$ and $x^J = x_1^{j_1} \cdots x_p^{j_p}$, all of whose exponents are not equal. Then we will show that the coefficient of x^J in $N_{\Sigma_p}(x^J)$ is prime to p. The isotropy subgroup of x^J is the finite product $\Sigma_{n_1} \times \Sigma_{n_2} \times \cdots < \Sigma_p$, where n_j is the number of terms of J equaling j. This group has order $n_1!n_2!\ldots$, which is prime to p. Hence

$$N_{\Sigma_p}(x) = (n_1!n_2!\ldots)x^J + \text{other monomials}.$$

This proves the claim. Thus, $\varphi_0 \vee f_{\Sigma_p}$ induces an isomorphism and hence is a p-local stable equivalence. $\square$

This implies

Lemma 3.8. *The long exact sequence for the pair $(D(\xi_{\Sigma_p}), S(\xi_{\Sigma_p}))$ gives the following short exact sequence:*

$$0 \leftarrow BP^*(S(\xi_{\Sigma_p})) \leftarrow BP^*(X^p_{h\Sigma_p}) \leftarrow BP^*((X^p_{h\Sigma_p})^{\xi_{\Sigma_p}}) \leftarrow 0.$$

Indeed, the left arrow is an epimorphism by Lemma 3.7 and hence the right arrow is a monomorphism.

Now the proof of Theorem 3.6 is completely analogous to that of Theorem 2.4 taking into account additionally that any element $y \in BP^*(X^p_{h\pi})$ has the form

$$y = \varphi^*(u) + g(\tilde{c}_1, \ldots, \tilde{c}_{p-1}) + y_1 c_p$$

for some $u \in BP^*(X^p_{h\pi})$, where g is a formal power series and $y_1 \in BP^*(X^p_{h\pi})$. This follows by Lemmas 3.7 and 8.4.

4. Calculation of the Elements $\delta_i^{(k)}$ and $\tilde{\delta}_i^{(k)}$ in the Morava K-Theory

In this section, we work in the Morava K-theory $K(s)^*$ and give an alternative, better algorithm for explicit calculations.

Fix a prime p and an integer $s \geq 0$; then

$$K(s)^* = \mathbf{F}_p[v_s, {v_s}^{-1}],$$

where $|v_s| = -2(p^s - 1)$. By a result of Würgler [38], there is no restrictions on p: although $K(s)$ is not a commutative ring spectrum for $p = 2$, we shall consider only spaces whose Morava K-theory is even-dimensional. This implies the deviation from commutativity is zero.

We recall that

$$K(s)^*(B\pi) = K(s)^*[z]/(z^{p^s}),$$

where $|z| = 2$.

Similarly to Lemma 3.3, we obtain the following assertion.

Lemma 4.1. (i) *The map $\rho_{\pi,\Sigma_p} : B\pi \to B\Sigma_p$ induces an isomorphism of $K(s)^*$-algebras*

$$\rho^*_{\pi,\Sigma_p} : K(s)^*(B\Sigma_p) \xrightarrow{\approx} \{K(s)^*(B\pi)\}^W,$$

where

$$W = N_{\Sigma_p}(\pi)/\pi \approx \mathbf{Z}/(p-1).$$

Computing invariants yields

$$K(s)^*(B\Sigma_p) = K(s)^*[y]/(y^{m_s}),$$

where

$$\rho^*_{\pi,\Sigma_p}(y) = z^{p-1}, \quad m_s = [(p^s - 1)/(p-1)] + 1.$$

(ii) $\mathrm{Tr}^*_{\Sigma_p}(1) = -v_s y^{m_s - 1}$.

Then combining Theorem 3.6 and Remark 2.2 we obtain

$$K(s)^*(X^p_{h\Sigma_p}) = K(s)^*[[c, \tilde{c}_1, \dots, \tilde{c}_{p-1}, c_p]]/(c^{m_s}, c\tilde{c}_i).$$

Our main result in this section is the following proposition.

Proposition 4.2. *We can construct explicit elements $\delta_i^{(k)} \in K(s)^*(B\pi)$ such that the following assertions hold.*

(1) *in $K(s)^*(X^p_{h\pi})$, we have*

$$c_k(\xi_\pi) = \mathrm{Tr}^*_\pi(\omega_k) - \sum_{0\le i\le p^s} \varphi^*_\pi(\delta_i^{(k)}) c_p^i(\xi_\pi);$$

(2) *in $K(s)^*(X^p_{h\Sigma_p})$, we have*

$$c_k(\xi_{\Sigma_p}) = \mathrm{Tr}^*_{\Sigma_p}(x_1 \dots x_k) - \sum_{0\le i\le p^s} \varphi^*_{\Sigma_p}(\tilde{\delta}_i^{(k)}) c_p^i(\xi_{\Sigma_p})$$

with

$$\tilde{\rho}^*_{\pi,\Sigma_p}(\tilde{\delta}_i^{(k)}) = k!(p-k)!\delta_i^{(k)} k;$$

(3) *the value of $\mathrm{Tr}^*_\pi(c_1(\xi_i))$ is determined by*

$$c_1(\xi_\pi) = \mathrm{Tr}^*_\pi(c_1(\xi_i)) + v_s \sum_{1\le j\le s-1} c^{p^s-p^j} c_p^{p^{j-1}}(\xi_\pi),$$

where ξ_i is the pullback of the canonical line bundle ξ by projection $BU(1)^p \to BU(1)$ on the ith factor.

We are grateful to D. Ravenel for supplying us with the proof of the following result.

Lemma 4.3. *For the formal group law in the Morava K-theory $K(s)$, $s > 1$, we have*

$$F(x, y) \equiv x + y - v_s \sum_{0<j<p} p^{-1}\binom{p}{j}(x^{p^{s-1}})^j (y^{p^{s-1}})^{p-j}$$

modulo $x^{p^{2(s-1)}}$ (or modulo $y^{p^{2(s-1)}}$).

Proof. This result can be derived from the recursive formula for the FGL given in [31, 4.3.9]. For the FGL in the Morava K-theory it reads

$$F(x, y) = \sum_{i\ge 0}^{F} v_s^{e_i} w_i(x, y)^{p^{i(s-1)}},$$

where w_i is a certain homogeneous polynomial of degree p^i defined in [31, 4.3.5] and

$$e_i = \frac{p^{is} - 1}{p^s - 1}.$$

In particular, $\omega_0 = x + y$,

$$w_1 = -\sum_{0<j<p} p^{-1}\binom{p}{j} x^j y^{p-j},$$

and $w_i \notin (x^p, y^p)$.

We find it more convenient to express $F(x, y)$ as

$$F(x, y) = F\Big(x + y,\ v_s w_1(x, y)^{p^{s-1}},\ v_s^{e_2} w_2(x, y)^{p^{2(s-1)}}, \dots\Big).$$

Then for $s > 1$, we can reduce modulo the ideal $v_s^{e_2}(x^{p^{2(s-1)}}, y^{p^{2(s-1)}})$ and get

$$
\begin{aligned}
F(x,y) &\equiv F\Big(x+y,\ v_s w_1(x,y)^{p^{s-1}}\Big)\\
&= F\Big(x+y+v_s w_1(x,y)^{p^{s-1}},\ v_s w_1\big(x+y,\ v_s w_1(x,y)^{p^{s-1}}\big)^{p^{s-1}},\ \dots\Big)\\
&\equiv F\Big(x+y+v_s w_1(x,y)^{p^{s-1}},\ v_s w_1\big(x^{p^{s-1}}+y^{p^{s-1}},\ v_s^{p^{s-1}} w_1(x,y)^{p^{2(s-1)}}\big)\Big),
\end{aligned}
$$

and modulo $v_s^{1+p^{s-1}}(x^{p^{2(s-1)}}, y^{p^{2(s-1)}})$ we have

$$F(x,y) \equiv x+y+v_s w_1(x,y)^{p^{s-1}}.$$

The lemma is proved. □

For brevity, we write $\sigma_k = \sigma_k(x, F(x,z), \dots, F(x,(p-1)z))$.

Corollary 4.4. *In $K(s)^*(BU(1)\times B\pi)$, the following formula holds*:

$$\sigma_k = -\sum_{0\le i\le p^s} \lambda_i^{(k)}\sigma_p^i + p^{-1}\binom{p}{k} x^k v_s z^{p^s-1},$$

where $\lambda_i^{(k)} = \lambda_i^{(k)}(z^{p-1})$ are polynomials in z^{p-1} and $\lambda_0^{(j)} = 0$, $j=1,\dots,p-2$, $\lambda_0^{(p-1)} = -z^{p-1}$.

Proof. For $1\le k\le p-1$, equating the coefficients of x^{ip}, $1\le i\le p^s$, we obtain a system of linear equations with invertible matrix of the form Id +nilpotent. Thus, the elements $\lambda_1^{(k)},\dots,\lambda_{p^s}^{(k)}$ can be defined as the solution of this system. Of course, equating the coefficients at x^i for $i\ne p, 2p, \dots, p^{s+1}$, we obtain other equations for $\lambda_j^{(k)}$, $j=1,\dots,p^s$. But these equations are derived from the old equations above. These additional equations make the matrix upper triangular. □

Now we prove Proposition 4.2 and show that one necessarily has $\delta_i^{(k)} = \lambda_i^{(k)}$, $i=0,\dots,p^s$ for $\lambda_i^{(k)}$ encountered in Corollary 4.4. Thus by Lemma 4.1, $\delta_i^{(k)}$ is invariant under the action of W and we can define $\tilde\delta_i^{(k)}$ by

$$\tilde\rho^*_{\pi,\Sigma_p}(\tilde\delta_i^{(k)}) = k!(p-k)!\delta_i^{(k)}.$$

The diagonal map $\Delta : BU(1)\to BU(1)^p$ induces an inclusion $B\pi\times BU(1)\to X^p_{h\pi}$ and the commutative diagram

$$
\begin{tikzcd}
E\pi\times BU(1) \arrow[r] \arrow[d, "\pi\times 1"'] & E\pi\times BU(1)^p \arrow[d, "\rho_\pi"] \\
B\pi\times BU(1) \arrow[r] & X^p_{h\pi}
\end{tikzcd}
$$

Then

$$(1\times\Delta)^*(\omega_k) = p^{-1}\binom{p}{k}x^k, \quad x = c_1(\xi).$$

Hence by transfer properties (i) and (iv), we have for the transfer $\mathrm{Tr} = \mathrm{Tr}(\pi\times 1)$:

$$\mathrm{Tr}^*((1\times\Delta)^*(\omega_k)) = p^{-1}\binom{p}{k}x^k\,\mathrm{Tr}^*(1) = p^{-1}\binom{p}{k}x^k v_s z^{p^s-1}.$$

On the other hand, by the existence of the elements $\delta_i^{(k)}$ we have

$$
\begin{aligned}
&\mathrm{Tr}^*((1\times\Delta)^*(\omega_k))\\
&\quad= \sigma_k\big(x, F(x,z),\dots,F(x,(p-1)z)\big) + \sum_{i\ge 0}\delta_i^{(k)}\sigma_p^i\big(x,F(x,z),\dots,F(x,(p-1)z)\big);
\end{aligned}
$$

ξ_π restricts to $\sum_i \xi \otimes \theta^i$ on $BU(1) \times B\pi$, and thus $c_k(\xi_\pi)$ to

$$\sigma_k\big(x, F(x,z), \ldots, F(x,(p-1)z)\big);$$

by Lemma 4.3 and the fact that $z^{p^s} = 0$, $[i]z$ can be replaced by iz. By Corollary 4.4,

$$\begin{aligned} &\sigma_k\big(x, F(x,z), \ldots, F(x,(p-1)z)\big) \\ &\qquad = -\sum_{0 \le i \le p^s} \lambda_i^{(k)} \sigma_p^i\big(x, F(x,z), \ldots, F(x,(p-1)z)\big) + p^{-1}\binom{p}{k} x^k v_s z^{p^s-1}. \end{aligned}$$

Then the restriction of $(1 \times \Delta)^*$ to $\operatorname{Ker}\rho^*$ is a monomorphism (see [24]). This proves Proposition 4.2 and shows that $\delta_i^{(k)} = \lambda_i^{(k)}$ for $0 \le i \le p^s$ and zero otherwise. Statement (2) follows from Lemma 3.4. Then statement (3) follows from the following explicit formula for σ_1.

Lemma 4.5. *In $K(s)^*(B\pi \times BU(1))$, we have*

$$\sigma_1 = v_s\left(z^{p^s-1}x + \sum_{i=1}^{s-1} z^{p^s-p^i}\sigma_p^{p^{i-1}}\right).$$

Proof. One has

$$\begin{aligned} \sigma_1 &= x + F(x,z) + \cdots + F(x,(p-1)z) \\ &= x + x + z + v_s w_1\big(x^{p^{s-1}}, z^{p^{s-1}}\big) + \cdots + x + (p-1)z + v_s w_1\big(x^{p^{s-1}}, ((p-1)z)^{p^{s-1}}\big) \\ &= px + \frac{p(p-1)}{2}z + v_s\left(\sum_{i=1}^{p-1} w_1(x, iz)\right)^{p^{s-1}} = v_s\left(\sum_{i=1}^{p-1}\sum_{j=1}^{p-1} -p^{-1}\binom{p}{j} i^j x^{p-j} z^j\right)^{p^{s-1}} \\ &\qquad = v_s\left(\sum_{j=1}^{p-1} -\left(\sum_{i=1}^{p-1} i^j\right) p^{-1}\binom{p}{j} z^j x^{p-j}\right)^{p^{s-1}}. \end{aligned}$$

Now $\sum_{i=1}^{p-1} i^j$ is an integral linear combination of $\sigma_k(1,2,\ldots,p-1)$ with $k \le i$, hence by it is zero for $i < p-1$ and for $i = p-1$ it is $p-1$. Thus,

$$\sigma_1 = -v_s\left((p-1)p^{-1}\binom{p}{p-1} z^{p-1} x^{p-(p-1)}\right)^{p^{s-1}} = v_s z^{p^s-p^{s-1}} x^{p^{s-1}}. \tag{4.1}$$

Now since $F(x,z)^p = x^p + z^p$, we have

$$\sigma_p^p = (x(x+z)\ldots(x+(p-1)z))^p.$$

But again we have

$$x(x+z)\ldots(x+(p-1)z) = x^p - xz^{p-1}.$$

Substituting this we obtain

$$\begin{aligned} &v_s\left(z^{p^s-1}x + \sum_{i=1}^{s-1} z^{p^s-p^i}\sigma_p^{p^{i-1}}\right) \\ &\qquad = v_s\left(z^{p^s-1}x + z^{p^s-p}\sigma_p + \sum_{i=2}^{s-1} z^{p^s-p^i}(x^p - z^{p-1}x)^{p^{i-1}}\right) \end{aligned}$$

$$= v_s \left(z^{p^s-1}x + z^{p^s-p}\sigma_p + \sum_{i=2}^{s-1} z^{p^s-p^i}\left(x^{p^i} - z^{(p-1)p^{i-1}}x^{p^{i-1}}\right)\right).$$

But it is straightforward to see that

$$\sum_{i=2}^{s-1} z^{p^s-p^i}(x^{p^i} - z^{(p-1)p^{i-1}}x^{p^{i-1}}) = z^{p^s-p^{s-1}}x^{p^{s-1}} - z^{p^s-p}x^p.$$

Hence

$$v_s \left(z^{p^s-1}x + \sum_{i=1}^{s-1} z^{p^s-p^i}\sigma_p^{p^{i-1}} \right) = v_s \left(z^{p^s-1}x + z^{p^s-p}\sigma_p + z^{p^s-p^{s-1}}x^{p^{s-1}} - z^{p^s-p}x^p \right). \tag{4.2}$$

Now we have

$$z^{p^s-p}F(x,kz) = z^{p^s-p}\big(x + kz + v_s w_1(x^{p^{s-1}}, (kz)^{p^{s-1}})\big) = z^{p^s-p}(x+kz),$$

hence

$$z^{p^s-p}\sigma_p = z^{p^s-p}x(x+z)\dots(x+(p-1)z) = z^{p^s-p}(x^p - z^{p-1}x).$$

Substituting this into (4.2) we obtain

$$v_s \left(z^{p^s-1}x + \sum_{i=1}^{s-1} z^{p^s-p^i}\sigma_p^{p^{i-1}} \right) = v_s z^{p^s-p^{s-1}}x^{p^{s-1}},$$

which is σ_1 by (4.1). $\square$

Now we calculate some of the elements $\delta_i^{(k)}$ and $\tilde{\delta}_i^{(k)}$.

First, recall (see [21, 31]) that generators for

$$\begin{array}{ccc} \pi_* BP & \subset & H_* BP \\ \downarrow & & \downarrow \\ \mathbf{Z}_{(p)}[v_1, v_2, \dots] & \subset & \mathbf{Z}_{(p)}[m_1, m_2, \dots] \end{array}$$

where

$$|v_n| = 2(p^n - 1) = |m_n|,$$

are given by

$$v_n = pm_n - \sum_{i=1}^{n-1} m_i v_{n-i}^{p^i}.$$

Given a formal group law over a graded ring R_*,

$$F(x,y) = \sum_{i,j} \alpha_{ij}^R x^i y^j \in R_*[[x,y]], \quad \alpha_{ij}^R \in R_{2(i+j-1)},$$

there is a ring map

$$g : MU_* \to R_*,$$

which induces the formal group law, that is,

$$g^*(\alpha_{ij}^{MU}) = \alpha_{ij}^R.$$

We use also the following well-known formulas:

$$F(x,y) = \exp(\log x + \log y), \quad \log x = \sum_{n \geq 0} m_n x^{n+1}$$

for calculating the elements δ_i in BP theory by the algorithm of Sec. 2.

Example 4.6. For $\delta_1 \in BP^*(B\mathbf{Z}/2) = BP^*[[z]]/([2](z))$, we have modulo z^8:

$$\delta_1 = v_1^2 z^2 + (v_1^3 + v_2)z^3 + v_1 z^4 + (v_1^6 + v_1^3 v_2)z^6 + (v_1^4 v_2 + v_2^2 + v_3)z^7.$$

5. Transfer and $K(s)^*(X^p_{h\Sigma_p})$

Let X be a CW-complex whose Morava K-theory $K(s)^*(X)$ is even-dimensional and finitely generated as a module over $K(s)^*$.

In this section, we study the transfer homomorphism in this more general context. We extend some results of Hopkins–Kuhn–Ravenel [23] to spaces. We consider the Atiyah–Hirzebruch–Serre (AHS) spectral sequence:

$$E_2{}^{*,*}(\pi, X) = H^*(\pi; K(s)^* X^p) \Rightarrow K(s)^*(X^p_{h\pi}). \tag{5.1}$$

By the Künneth isomorphism,

$$K(s)^* X^p \xrightarrow[\approx]{(} K(s)^* X)^{\otimes p}. \tag{5.2}$$

Then $K(s)^* X^p$ is a π-module, where π acts by permuting factors (see [23, Theorem 7.3]).

An element $x \in K(s)^*(X)$ is said to be *good* if there is a finite cover $Y \to X$ together with an Euler class $y \in K(s)^*(Y)$ such that $x = \mathrm{Tr}^*(y)$, where $\mathrm{Tr}^* : K(s)^*(Y) \to K(s)^*(X)$ is the transfer. The space X is said to be *good* if $K(s)^*(X)$ is spanned over $K(s)^*$ by good elements.

Let $\gamma = \varphi^*(z)$, where $\varphi : X^p_{h\pi} \to B\pi$ is the projection, and let $\{x_j, j \in \mathcal{J}\}$ be a $K(s)^*$-basis for $K(s)^*(X)$. Hunton showed [25] that if $K(s)^*(X)$ is concentrated in even dimensions, then so is $K(s)^*(X^p_{h\pi})$. We adopt a stronger hypothesis that X is good and derive a stronger result, following the argument of [23, Theorem 7.3] for classifying spaces.

Proposition 5.1. *Let X be a good space.*

(i) *As a $K(s)^*$-module, $K(s)^*(X^p_{h\pi})$ is free with basis*

$$\left\{\gamma^i \otimes (x_j)^{\otimes p} \;\middle|\; 0 \le i < p^s,\ j \in \mathcal{J}\right\}$$

and

$$\left\{\sum_{(i_1, i_2, \ldots, i_p) = I} 1 \otimes x_{i_1} \otimes x_{i_2} \otimes \cdots \otimes x_{i_p} \;\middle|\; I \in \mathcal{P}_p\right\},$$

where $I = \{(i_1, i_2, \ldots, i_p)\}$ runs over the set $\mathcal{P}_p$ of π-equivalence classes of p-tuples of indices $i_j \in \mathcal{J}$ at least two of which are not equal.

(ii) *$X^p_{h\pi}$ is good.*

Proof. (i) By the Künneth isomorphism,

$$K(s)^*(X)^{\otimes p} = F \oplus T \tag{5.3}$$

as a π-module, where F is free and T is trivial. Explicitly, a $K(s)^*$-basis for T is $\{(x_i)^{\otimes p},\ i \in \mathcal{J}\}$, while a $K(s)^*$-basis for F is $\{x_{i_1} \otimes x_{i_2} \otimes \cdots \otimes x_{i_p},\ i_j \in \mathcal{J}\}$, where not all the factors are equal. Then

$$H^*(\pi; F) = \begin{cases} F^\pi & \text{if } * = 0, \\ 0 & \text{if } * > 0 \end{cases}$$

and

$$H^*(\pi; T) = H^*(B\pi) \otimes T.$$

Thus,

$$E_2{}^{0,*}(\pi, X) = K(s)^*(X^p_{h\pi})^\pi = F^\pi \oplus T.$$

To continue the proof, we recall the covering projection

$$\rho_\pi : E\pi \times X^p \to X^p_{h\pi},$$

its associated transfer homomorphism

$$\mathrm{Tr}^* = \mathrm{Tr}^*_\pi : K(s)^*(X^p) \to K(s)^*(X^p_{h\pi}), \tag{5.4}$$

and the induced homomorphism

$$\rho_\pi{}^* : K(s)^*(X^p_{h\pi}) \to K(s)^*(X^p).$$

Similar maps are defined for the group Σ_p. Then $\rho_\pi{}^* \, \mathrm{Tr}^* = N$, where $N = N_\pi$ is the trace map.

Thus, we have established the following lemma.

Lemma 5.2. *If $y \in K(s)^*(X^p)$ is good, then there exists a good element $z \in K(s)^*(X^p_{h\pi})$ such that $\rho^*_\pi(z) = N(y)$.*

Lemma 5.3. *If $x \in K(s)^*(X)$ is good, then there is a good element $z \in K(s)^*(X^p_{h\pi})$ such that $\rho^*_\pi(z) = x^{\otimes p}$.*

Proof. By assumption, there is a finite covering $f : Y \to X$ and an Euler class $e \in K(s)^*(Y)$ such that $x = \mathrm{Tr}^*(e)$. Now consider the covering

$$\phi = f \times \cdots \times f : Y^p \to X^p,$$

which extends to a covering

$$1 \times \phi : Y^p_{h\pi} \to X^p_{h\pi}$$

and yields a map of coverings

$$\begin{tikzcd} E\pi \times Y^p \arrow[r] \arrow[d, "1\times\phi"'] & Y^p_{h\pi} \arrow[d, "1\times\phi"] \\ E\pi \times X^p \arrow[r] & X^p_{h\pi} \end{tikzcd}$$

The class $e^{\otimes p}$ is an Euler class for Y^p. Since the transfer is natural and commutes with tensor products, we have

$$\rho^*_\pi \, \mathrm{Tr}^*(1 \otimes e^{\otimes p}) = \mathrm{Tr}^* \, \rho_\pi{}^*(1 \otimes e^{\otimes p}) = \mathrm{Tr}^*(e^{\otimes p}) = \mathrm{Tr}^*(e) \otimes \cdots \otimes \mathrm{Tr}^*(e) = x^{\otimes p}.$$

The lemma is proved. $\square$

Corollary 5.4. *$E_2{}^{0,*}(\pi, X)$ consists of permanent cycles that are good.*

Thus, as differential graded $K(s)^*$ modules, there is an isomorphism of spectral sequences

$$\big(E_r{}^{*,*}(\pi, pt) \otimes_{K(s)^*} T\big) \oplus F^\pi \xrightarrow{\approx} E_r{}^{*,*}(\pi, X).$$

It follows that as a $K(s)^*$-algebra $K(s)^*(X^p_{h\pi})$ is generated by $K(s)^*(B\pi)$, T, and F^π.

(ii) The proof of [23, Theorem 7.3] carries over. This completes the proof of Proposition 5.1. $\square$

Remark 5.5. From the periodicity of the cohomology of a cyclic group [15, Proposition XII, 11.1], we have the isomorphisms

$$H^t(\pi; K(s)^*(X^p)) \xrightarrow{\cdot z} H^{t+2}(\pi; K(s)^*(X^p))$$

for $t > 0$ and

$$H^0(\pi; K(s)^*(X^p))/\operatorname{Im}(N) \xrightarrow{\cdot z} H^2(\pi; K(s)^*(X^p)).$$

Thus, multiplication by z is also injective on T at the E_2 term.

Remark 5.6. $\rho_\pi{}^* \operatorname{Tr}^* = N$; thus, modulo $\ker(\rho_\pi{}^*)$ we have

$$\operatorname{Tr}^*(x_{i_1} \otimes x_{i_2} \otimes \cdots \otimes x_{i_p}) = \sum_{\sigma \in \pi} 1 \otimes x_{\sigma(i_1)} \otimes x_{\sigma(i_2)} \otimes \cdots \otimes x_{\sigma(i_p)}. \tag{5.5}$$

Note that if the i_j in (5.5) are equal, the right-hand side is zero. However,

$$\operatorname{Tr}^*(x_j{}^{\otimes p}) = 1 \otimes x_j{}^{\otimes p} \cdot \operatorname{Tr}^*(1).$$

Now we turn to $K(s)^*(X^p_{h\Sigma_p})$.

Let $c = \varphi^*(y)$, where $\varphi : E\Sigma_p \times_{\Sigma_p} X^p \to B\Sigma_p$ is the projection.

Proposition 5.7. *Let X be a good space. As a $K(s)^*$-module, $K(s)^*(X^p_{h\Sigma_p})$ is free with basis*

$$\left\{ c^i \otimes (x_j)^{\otimes p} \;\middle|\; 0 \le i < m_s,\ j \in \mathcal{J} \right\}$$

and

$$\left\{ \sum_{(i_1, i_2, \ldots, i_p) = I} 1 \otimes x_{i_1} \otimes x_{i_2} \otimes \cdots \otimes x_{i_p} \;\middle|\; I \in \mathcal{E}_p \right\},$$

where $I = \{(i_1, i_2, \ldots, i_p)\}$ runs over the set $\mathcal{E}_p$ of Σ_p-equivalence classes of p-tuples of indices $i_j \in \mathcal{J}$ at least two of which are not equal.

Since $|W|$ is prime to p, the result follows from the AHS spectral sequence, as in the proof of Proposition 5.1.

6. Transferred Chern Classes in the Morava K-Theory

6.1. Statements. Let p be a prime and π be the cyclic group of order p. For a given action of π on a space X, consider the regular covering

$$\rho : E\pi \times X \to E\pi \times_\pi X.$$

For brevity we write $X_{h\pi} := E\pi \times_\pi X$. For the permutation action of π on $X = Y^p$, we have

$$Y^p_{h\pi} = E\pi \times_\pi Y^p.$$

Let t be a generator of π and $N_\pi = 1 + t + \cdots + t^{p-1}$ be the trace map. For a n-plane bundle η^n, the corresponding classifying map

$$f : X \to BU(n)$$

induces the classifying map $(f, tf, \ldots, t^{p-1}f)$ for the bundle $N_\pi \eta^n$ and thereby a map of orbit spaces

$$f_{\eta^n} : X_{h\pi} \to BU(n)^p_{h\pi}.$$

For the covering

$$\rho_\pi : E\pi \times BU(n)^p \to BU(n)^p_{h\pi}$$

and the universal n-plane bundle $\xi^n \to BU(n)$, we consider the Atiyah transfer bundle (see [2])

$$\xi^n_\pi \to BU(n)^p_{h\pi},$$

i.e., the np-plane bundle

$$\xi^n_\pi = E\pi \times_\pi (\xi^n)^{\times p}.$$

Then the map f_{η^n} classifies the Atiyah transfer bundle for η^n and ρ. So by the naturality of the transfer [1] we can consider ρ_π as the universal example.

Let $K(s)^*$, $s \ge 1$, be the sth Morava K-theory at p. We recall that, by the Künneth isomorphism, $K(s)^*(BU(n)^p) = F \oplus T$ as a π-module, where F is free and T is trivial.

Definition 6.1. Let $\omega_k^{(n)} \in F$ be defined modulo $\ker N_\pi = \mathrm{Im}(1-t)$ by

$$N_\pi(\omega_k^{(n)}) = c_k(N_\pi(\xi^n)),$$

where c_k are Chern classes, $k \in \{1, \dots, np\} \setminus \{p, 2p, \dots, np\}$.

By the naturality of the transfer $\mathrm{Tr}_\pi^* = \mathrm{Tr}_\pi^* t$, where

$$\mathrm{Tr}_\pi^* : K(s)^*(E\pi \times BU(n)^p) \to K(s)^*(BU(n)^p_{h\pi}),$$

hence $\mathrm{Tr}_\pi^*(\omega_k^{(n)})$ is well defined.

We write $\omega_k(\eta^n)$ for the pullback by map f_{η^n} of orbit spaces defined above.

Recall that

$$K(s)^*(B\pi) = F_p[v_s, v_s^{-1}][z]/(z^{p^s}),$$

where $z = c_1(\theta)$ is the Chern class of the canonical complex line bundle over $B\pi$.

Lemma 6.2. *We can define a polynomial in $n+1$ variables*

$$A_k^{(n)}(z^{p-1}, Z_1, \dots, Z_n) \in K(s)^*[z, Z_1, \dots, Z_n]$$

uniquely by the equation in $K(s)^* B(\pi \times U(n))$:

$$C_k - v_s z^{p^s-1} p^{-1} \sum_{\substack{i_1+2i_2\cdots+ni_n=k \\ i_0+i_1+\cdots+i_n=p}} \binom{p}{i_0, i_1, \dots, i_n} c_1^{i_1} \dots c_n^{i_n} = A_k^{(n)}(z^{p-1}, C_p, \dots, C_{pn}),$$

where

$$C_i = c_i(\xi^n \oplus \theta \otimes \xi^n \oplus \cdots \oplus \theta^{p-1} \otimes \xi^n),$$

$c_j = c_j(\xi^n)$ *are Chern classes, and* $k \in \{1, \dots, np\} \setminus \{p, \dots, np\}$.

For example, in $K(s)^*(B\pi \times BU(1))$, $p > 2$, we have

$$C_1 = v_s \left(z^{p^s-1} c_1 \sum_{i=1}^{s-1} z^{p^s - p^i} C_p^{p^{i-1}} \right);$$

thus

$$A_1^{(1)}(z^{p-1}, Z_1) = v_s \sum_{i=1}^{s-1} z^{p^s-p^i} Z_1^{p^{i-1}}.$$

Then using the polynomials $A_k^{(n)}$, we evaluate the transferred classes $\omega_k(\eta^n)$ for regular coverings.

Theorem 6.3. *Let $\rho : X \to X/\pi$ be the regular cyclic covering of prime index p defined by a free action of π on X and $\mathrm{Tr}^* = \mathrm{Tr}_\rho^*$ be the associated transfer homomorphism. Let $\eta^n \to X$ be a complex n-plane bundle, $\eta_\pi^n \to X/\pi$ be the pn-plane bundle defined by Atyiah transfer, and $\psi \to X/\pi$ be the complex line bundle associated with ρ. Then*

$$c_k(\eta_\pi^n)) - \mathrm{Tr}^*(\omega_k(\eta^n)) = A_k^{(n)}\Big(c_1^{p-1}(\psi),\ c_p(\eta_\pi^n),\ \dots,\ c_{pn}(\eta_\pi^n)\Big),$$

where $k \in \{1, \dots, np\} \setminus \{p, \dots, np\}$.

Example 6.4. $\omega_1 = c_1(\eta^n)$; hence if $k = 1$, we have

$$c_1(\eta_\pi^n)) - \mathrm{Tr}^*(c_1(\eta^n)) = A_1^{(n)}\Big(c_1^{p-1}(\psi),\ c_p(\eta_\pi^n),\ \dots,\ c_{pn}(\eta_\pi^n)\Big).$$

If, in addition, $n = 1$, then by above example we have for the line bundle $\eta \to X$ and the transferred Chern class

$$c_1(\eta_\pi) - \mathrm{Tr}^*(c_1(\eta)) = v_s \sum_{i=1}^{s-1} c_1(\psi)^{p^s - p^i} c_p(\eta_\pi)^{p^{i-1}}.$$

For $p = 2$, we have

$$A_1^{(1)}(c_1^2(\psi), c_2) = c_1(\psi) + sum_{i=1}^{s-1} c_1(\psi)^{2^s - 2^i} c_2(\eta_\pi)^{2^{i-1}}.$$

6.2. Proofs.

Remark 6.5. Neil Strickland suggested an elegant proof of Lemma 6.2 completely within the context of subgroups of formal groups. This material is available from `N.P.Strickland@sheffield.ac.uk`. It uses rather different techniques than our proof for which we now offer some motivation. The lemma says that modulo image of the transfer

$$\mathrm{Tr}^* : K(s)^*B(U(n)) \to K(s)^*B(\pi \times U(n))$$

for the covering $\pi \times 1 : E\pi \times BU(n) \to B\pi \times BU(n)$, the class C_k can be written as a polynomial in z^{p-1} and $C_p, \dots, C_{np}$, that this expression vanishes when restricted to $K(s)^*BU(n)$, and that the indeterminacy is obtained by applying the transfer to $p^{-1}C_k$ after restricting to $K(s)^*(BU(n))$. This uses Frobenius reciprocity and the fact that

$$\mathrm{Tr}^*(1) = [p](z)/z = v_s z^{p^s - 1}$$

in the Morava K-theory.

Proof of Lemma 6.2. We begin by considering the case $n = 1$. Let σ_i be the ith symmetric functions in p variables. Then

$$C_i = \sigma_i\big(x, F(x, z), \dots, F(x, (p-1)z)\big),$$

where $x = c_1(\xi^1)$ and $z = c_1(\theta)$ are Chern classes in $K(s)^*B(\pi \times U(1)) = K(s)^*[[z, x]]/(z^{p^s})$. Consider the equation

$$C_k = -\sum_{0 \le i \le p^s} \lambda_{ki} C_p^i + p^{-1}\binom{p}{k} x^k z^{p^s - 1}, \quad 1 \le k \le p - 1.$$

We prove that such λ_{ki} uniquely exist as elements in $K(s)^*[[z]]/(z^{p^s})$ and calculate these elements as polynomials in z^{p-1}. Then

$$A_k^{(1)}(z^{p-1}, C_p) = \sum_{0 \le i \le p^s} \lambda_{ki} C_p^i.$$

Since C_k is the Chern class of the bundle $\xi \otimes (1 + \theta + \theta^2 + \dots + \theta^{p-1})$, it can be written as a series in the Chern classes of ξ, i.e., x, and the Chern classes of $1 + \theta + \theta^2 + \dots + \theta^{p-1}$. But the Chern classes of the latter bundle are elementary symmetric functions in $z, 2z, \dots, (p-1)z$, the Chern classes of $\theta, \theta^2, \dots, \theta^{p-1}$ all of which vanish except the $(p-1)$th class which is $-z^{p-1}$. Hence we can write the classes C_k as series in x and z^{p-1}. Lemma 4.1 allows us to write C_k as explicit polynomials in x and z^{p-1}. Now noting that $C_p = x^p \mod z^{p-1}$, we obtain from the above equation a system of linear equations in variables λ_{kj} by equating the coefficients at x^i, $i \ge 0$. Vanishing also implies $\lambda_{k0} = 0$, $k = 1, \dots, p-2$, and

$$\lambda_{p-10} = c_{p-1}(1 + \theta + \dots + \theta^{p-1}) = -z^{p-1}.$$

Then equating the coefficients at $x^p, \dots, x^{p^{s+1}}$ in the above equation after rewriting it in terms of x and z^{p-1} as above, we have a system of p^s linear equations in p^s variables λ_{ki}, $i = 1, \dots, p^s$. The determinant of this system is invertible since the diagonal coefficients are invertible and all other coefficients lie in the (nilpotent) augmentation ideal. Thus the elements λ_{ki} are uniquely defined.

Of course, equating the coefficients at x^i for $i \neq p, 2p, \dots, p^{s+1}$ will produce other equations in λ_{kj}, $j = 1, \dots, p^s$. But these equations are derived from the old equations above. These additional equations make the matrix upper triangular. This defines $A_k^{(1)}$.

In the general n case we proceed analogously, noting that $C_{ip} = c_i^p \mod z^{p-1}$, $i = 1, \dots, n$.

Our additional claim again is that the $A_k^{(n)}$ are polynomials, that is, only a finite number of elements $\lambda_{k,i_1,\dots,i_n}$ are nontrivial. Here we need the splitting principle and Lemma 4.3 to express explicitly the elements C_i, $i = 1, \dots, np$, in terms of polynomials in z^{p-1} and $c_1, \dots, c_n$. For $k \in \{1, \dots, np\} \setminus \{p, \dots, np\}$, let

$$A_k^{(n)}(z^{p-1}, C_p, \dots, C_{pn}) = \sum_{0 \leq i_1, \dots, i_n \leq p^s} \lambda_{k,i_1,\dots,i_n} C_p^{i_1} \dots C_{np}^{i_n}.$$

We define

$$\lambda_{k,0\dots 0} = c_k(n + n\theta + \dots + n\theta^{p-1})$$

again by looking at reductions to $K(s)^*(B\pi)$. The other $n(p^s + 1) - 1$ elements $\lambda_{k,i_1,\dots,i_n}$ can be defined as the solution of a system of $n(p^s + 1) - 1$ linear equations with an invertible determinant. This system is obtained from the equation after using Lemma 4.3 to rewrite it in terms of z^{p-1} and $c_1, \dots, c_n$ and equating coefficients of $c_1^{pi_1} \dots c_n^{pi_n}$. The solution defines $\lambda_{k,i_1,\dots,i_n}$, $0 \leq i_j \leq p^s$. Again, the additional equations in these elements arise from the coefficients of other monomials and are not new. The desired polynomials are thus uniquely defined. $\square$

Proof of Theorem 6.3. Consider the homotopy orbit space

$$BU(n)^p_{h\pi} = E\pi \times_\pi BU(n)^p$$

as the universal example. The diagonal map $BU(n) \to BU(n)^p$ induces the inclusion

$$i : B\pi \times BU(n) \to E\pi \times_\pi BU(n)^p.$$

We use [24, Proposition 4.2], which implies that since $BU(n)$ is a unitary-like space (i.e., $K(s)^*BU(n)$ has no nilpotent elements), the map $(i \vee \rho_\pi)^*$ is a monomorphism. Since $\rho_\pi^* \operatorname{Tr}^* = N_\pi$, the difference

$$c_k(\eta_\pi^n) - \operatorname{Tr}^*(\omega_k^{(n)})$$

belongs to $\ker \rho_\pi^*$ and hence is detected by i^*. The result now follows from Lemma 6.2. $\square$

Note we can replace the cyclic group by the symmetric group Σ_p and use the polynomials $A_k^{(n)}$ to evaluate the disparity or "gap" between the Chern class $c_k(\xi^n_{\Sigma_p})$ and $\operatorname{Im} \operatorname{Tr}^*_{\Sigma_p}$, for

$$\rho_{\Sigma_p} : E\Sigma_p \times U(n)^p \to BU(n)^p_{h\Sigma_p}.$$

Namely, the Euler characteristic of the coset space

$$\Sigma_p \wr U(n)/\pi \wr U(n) = (p-1)!$$

is prime to p. Hence the inclusion

$$\rho_{\pi,\Sigma_p} : \pi \to \Sigma_p$$

induces a monomorphism

$$K(s)^*(B(\Sigma_p \wr U(n))) \to K(s)^*(B(\pi \wr U(n))).$$

Hence Hunton's result above holds for $BU(n)^p_{h\Sigma_p}$.

Now let $\varsigma_k^{(n)} \in F$ be defined modulo $\ker N_{\Sigma_p}$ by

$$N_{\Sigma_p}(\varsigma_k^{(n)}) = c_k(N_\pi(\xi^n)).$$

Again $\operatorname{Tr}^*_{\Sigma_p}(\varsigma_k^{(n)})$ is well defined: as above, $\operatorname{Ker} N_\pi = \operatorname{Im}(1 - t^*)$; therefore,

$$a \in F \cap \ker N_{\Sigma_p} \;\Rightarrow\; \sum_{g \in \Sigma_p/\pi} g^* a \in \operatorname{Im}(1 - t^*)$$

$$\Rightarrow\; \operatorname{Tr}^*_{\Sigma_p}\left(\sum_{g \in \Sigma_p/\pi} g^* a\right) = 0 \;\Rightarrow\; (p-1)!\operatorname{Tr}^*_{\Sigma_p}(a) = 0 \;\Rightarrow\; \operatorname{Tr}^*_{\Sigma_p}(a) = 0.$$

Then for the np-plane bundle over $BU(n)^p_{h\Sigma_p}$

$$\xi^n_{\Sigma_p} = E\Sigma_p \times_{\Sigma_p} (\xi^n)^{\times p},$$

the difference

$$c_k(\xi^n_{\Sigma_p}) - \operatorname{Tr}^*_{\Sigma_p}(\varsigma^{(n)}_k)$$

belongs to $\ker \rho^*_{\Sigma_p}$ and hence is detected by the polynomials

$$A^{(n)}_k(y, c_p(\xi^n_{\Sigma_p}), \ldots, c_{np}(\xi^n_{\Sigma_p})),$$

where

$$y \in K(s)^*(B(\Sigma_p)) = K(s)^*[[y]]/(y^{m_s})$$

and $|y| = 2(p-1)$ and $m_s = [(p^s - 1/(p-1))] + 1$.

Theorem 6.6.

$$c_k(\xi^n_{\Sigma_p}) - \operatorname{Tr}^*_{\Sigma_p}(\varsigma^{(n)}_k) = A^{(n)}_k\Big(y,\; c_p(\xi^n_{\Sigma_p}),\; \ldots,\; c_n p(\xi^n_{\Sigma_p})\Big)$$

for $k \in \{1, \ldots, np\} \setminus \{p, \ldots, np\}$.

7. Stable Euler Classes

Now we turn to $G_n = \pi \wr (\mathbf{Z}/p^n)$, where $\pi = \mathbf{Z}/p$. Then $BG_n = X^p_{h\pi}$ for $X = B\mathbf{Z}/p^n$. Consider the AHS spectral sequence for

$$B(\mathbf{Z}/p^n)^p \to BG_n \xrightarrow{\varphi} B\pi.$$

Then

$$E_2^{p,q} = H^*\big(\pi; K^*(s)(B(\mathbf{Z}/p^n)^p)\big),$$

where

$$K^*(s)\big(B(\mathbf{Z}/p^n)^p\big) = \big(K(s)^*[z]/(z^{p^{ns}})\big)^{\otimes p} = F \oplus T$$

and F and T, as in (5.3) above, are free (respectively, trivial) π-modules.

Let $\gamma = \varphi^*(z)$, where $K(s)^*(B\pi) = K(s)^*[z]/(z^{p^s})$ as above.

Proposition 7.1. *As a $K(s)^*$-module, $K(s)^*(BG_n)$ is free with basis*

$$\Big\{\gamma^i \otimes (z^j)^{\otimes p},\; 0 \le i < p^s,\; 0 \le j < p^{ns}\Big\}$$

and

$$\left\{ \sum_{(i_1, i_2, \ldots, i_p) = I} 1 \otimes z^{i_1} \otimes z^{i_2} \otimes \cdots \otimes z^{i_p} \;\middle|\; I \in \mathcal{P}_p(n) \right\},$$

where $I = \{(i_1, i_2, \ldots, i_p)\}$ runs over the set $\mathcal{P}_p(n)$ of π-equivalence classes of p-tuples of integers $\{0 \le i_j < p^{ns}\}$ at least two of which are not equal.

Proof. This spectral sequence calculation is exactly analogous to that of Proposition 5.1. ☐

Remark 7.2. (i) For $X = \mathbf{C}P^\infty$, Proposition 7.1 gives another derivation of $K(s)^*(X^p_{h\pi})$. Since

$$\mathbf{C}P^\infty{}^\wedge_p = \left[\operatorname*{colim}_n B(\mathbf{Z}/p^n)\right]^\wedge_p,$$

we have

$$K(s)^*(X^p_{h\pi}) = \lim_n \; K(s)^*(BG_n).$$

(ii) G_n is good for $K(s)^*$ by [23, Theorem 7.3].

By analogy with Sec. 4 we have the following lemma.

Lemma 7.3. (i) $\mathrm{Im}(\mathrm{Tr}^*) \cdot \gamma = 0$.
(ii) $\mathrm{Tr}^*(1) = v_s \gamma^{p^s-1}$.
(iii) *If* $y \in T$, *then* $\mathrm{Tr}^*(y) = y \cdot \mathrm{Tr}^*(1)$.

7.1. p-Groups with cyclic subgroup of index p. In this section, we consider the class of p-groups with a (necessarily normal) cyclic subgroup of index p. It is known (see [13, Theorem 4.1, Chap. IV]) that every p-group of this form is isomorphic to one of the following groups:

(a) $\mathbf{Z}/q$ ($q = p^n$, $n \geq 1$);
(b) $\mathbf{Z}/q \times \mathbf{Z}/p$ ($q = p^n$, $n \geq 1$);
(c) $\mathbf{Z}/q \rtimes \mathbf{Z}/p$ ($q = p^n$, $n \geq 2$), where the canonical generator of $\mathbf{Z}/p$ acts on $\mathbf{Z}/q$ as multiplication by $1 + p^{n-1}$. This group is called the modular group if $p \geq 3$ and the quasi-dihedral group if $p = 2$ and $n \geq 4$.

For $p = 2$, there are three additional families.

(d) *Dihedral* 2*-groups* $D_{2m} = \mathbf{Z}/m \rtimes \mathbf{Z}/2$ ($m \geq 2$), where the generator of $\mathbf{Z}/2$ acts on $\mathbf{Z}/m$ as multiplication by -1. If $m = 2^n$, D_{2m} is a 2-group. Note that D_4 belongs to (b) and D_8 belongs to (c).
(e) *Generalized quaternion* 2*-groups.* Let $\mathbf{H}$ be the algebra of quaternions $\mathbf{R} \oplus \mathbf{R}i \oplus \mathbf{R}j \oplus \mathbf{R}k$. For $m \geq 2$, the generalized quaternion group Q_{4m} is defined as the subgroup of the multiplicative group $\mathbf{H}^*$ generated by $x = e^{\pi i/m}$ and $y = j$. The subgroup $\mathbf{Z}/2m$ generated by x is normal and has index 2. If m is a power of 2, Q_{4m} is a 2-group. In the extension

$$0 \to \mathbf{Z}/2m \to Q_{4m} \to \mathbf{Z}/2 \to 0,$$

the generator of $\mathbf{Z}/2$ acts on $\mathbf{Z}/2m$ as -1. In particular, Q_8 is the group of quaternions $\{\pm 1, \pm i, \pm j, \pm k\}$.
(f) *Semi-dihedral groups* $\mathbf{Z}/q \rtimes \mathbf{Z}/2$ ($q = p^n$, $n \geq 3$), where the generator of $\mathbf{Z}/2$ acts on $\mathbf{Z}/q$ as multiplication by $-1 + 2^{n-1}$.

Now we consider the problem of calculation of the stable Euler class, $\mathrm{Tr}^*_G(1)$, for the universal G-covering $EG \to BG$.

For the case (a), there is the well-known formula of Quillen (2.1)

$$\mathrm{Tr}^*_{\mathbf{Z}/q}(1) = [q]_F(z)/z$$

in $MU^*(B\mathbf{Z}/q) = MU^*[[z]]/([q]_F(z))$.

For the case (b), the answer follows from transfer property (ii):

$$\mathrm{Tr}_G = \mathrm{Tr}_{\mathbf{Z}/q} \wedge \mathrm{Tr}_{\mathbf{Z}/p}.$$

In the cases (d), (e), and (f), Tr^*_G is the composition of two transfers $\mathrm{Tr}^*_{\mathbf{Z}/q}$ and $\mathrm{Tr}^*_{C,G} : MU^*(BC) \to MU^*(BG)$, where C is the corresponding cyclic subgroup. So we must calculate $\mathrm{Tr}^*_{C,G}(z^i)$, $i \geq 1$, and we can apply our results for $B\mathbf{Z}/2 \wr U(1)$, namely Remark 2.9.

Similarly for the case (c), Tr_G is the composition $\mathrm{Tr}_{\mathbf{Z}/q,G} \mathrm{Tr}_{\mathbf{Z}/q}$ and we can apply Corollary 2.8.

This task is trivial for wreath products $\mathbf{Z}/p \wr \mathbf{Z}/p^n$ since $\mathrm{Tr}^*_{\mathbf{Z}/p^n}(1)$ is symmetric in $z_1, \ldots, z_p$ in the ring

$$MU^*((B\mathbf{Z}/p^n)^p) = MU^*[[z_1, \ldots, z_p]]/([p^n](z_1), \ldots, [p^n](z_p))$$

and hence invariant under the $\mathbf{Z}/p$ action. So in this case we need only Quillen's formula.

Finally, we note that if G is the modular group of case (c), Brunetti [12] has completely calculated the ring $K(s)^*(BG)$. The relations are quite simple, but the generators are technically complicated. In a future paper, we plan to use transferred Chern classes to give a more natural presentation.

7.2. Other examples. Consider the semi-direct products

$$G = (\mathbf{Z}/p)^n \rtimes \mathbf{Z}/p,$$

where the generator α of $\mathbf{Z}/p$ acts on $H_n = \mathbf{Z}/p[T]/(T^n)$ by $1 - \alpha = T$, $1 \leq n \leq p$. Then every $\mathbf{Z}/p[\mathbf{Z}/p]$-module is a direct sum of the modules H_n. As shown by Yagita [40] and Kriz [28], these semi-direct products are good in the sense of Hopkins–Kuhn–Ravenel.

We recall that

$$K(s)^*(B(\mathbf{Z}/p)^n) = K(s)^*[[z_1, \ldots, z_n]]/(z_i^{p^s}),$$

where z_i is the Euler class of a faithful complex line bundle θ_i on the ith factor. Then $\mathbf{Z}/p$ acts on $K(s)^*[z_1, \ldots, z_n]/(z_i^{p^s})$ by

$$\alpha : z_i \to F_{K(s)}(z_i, z_{i+1}), \quad z_{n+1} := 0,$$

where $F_{K(s)}$ denotes the formal group law for the Morava K-theory.

Our aim is to show how to calculate the stable Euler classes in terms of characteristic classes and the formal group law.

The transfer $\mathrm{Tr}^* : K(s)^*EG \to K(s)^*BG$ is the composition of two transfers

$$\mathrm{Tr}_1^* : K(s)^*E((\mathbf{Z}/p)^n) \to K(s)^*B((\mathbf{Z}/p)^n)$$

and

$$\mathrm{Tr}_2^* : K(s)^*B((\mathbf{Z}/p)^n) \to K(s)^*BG.$$

Recall also that

$$\mathrm{Tr}_1^*(1) = z_1^{p^s-1} \cdots z_n^{p^s-1}.$$

It is easy to see that in $K(s)^*((B\mathbf{Z}/p)^n)$, we have

$$e^{p^s-1}(\alpha^{i_1}(\theta_1)) \cdots e^{p^s-1}(\alpha^{i_n}(\theta_1)) = z_1^{p^s-1} \cdots z_n^{p^s-1},$$

where e is the Euler class and $1 \leq i_1 < \cdots < i_n \leq p$. Then recall the elements ω_n from Theorem 2.3 and let $\omega_n(l)$ be the sum of the same monomials after raising to the power l. Since $\omega_n(l)$ consist of $p^{-1}\binom{p}{n}$ summands and $p^{-1}\binom{p}{n} = \dfrac{(-1)^n}{n}$ mod p, we have that in $K(s)^*((BZ/p)^n)$

$$\eta_\pi^*(\omega_n(p^s - 1)) = (-1)^n z_1^{p^s-1} \cdots z_n^{p^s-1}/n,$$

where the map η_π defined in Sec. 2 sends $\xi_i = t^{i-1}\xi_1$ to $\alpha^{i-1}\theta_1$. Hence

$$\mathrm{Tr}_G^*(1) = \mathrm{Tr}_2^*(\mathrm{Tr}_1^*(1)) = \mathrm{Tr}_2^*((-1)^n n \eta_\pi^*(\omega_n(p^s - 1))) = (-1)^n n \,\mathrm{Tr}_2^*(\eta_\pi^*(\omega_n(p^s - 1))),$$

and we apply Corollary 2.7.

8. Morava K-Theory Rings for p-Groups with Maximal Cyclic Subgroup in Chern Classes

The rank of the Morava K-theory ring of the classifying space of a finite group as a free $K(s)^*$ module is given by the Hopkins–Kuhn–Ravenel generalized character theory [23]. For many finite p-groups, the Morava K-theory ring is generated by transferred Chern classes (in general, this is invalid; see [28]).

In [4, 5] we studied the Chern classes of a transferred bundle in terms of transferred classes of the bundle. As an application, we derived formulas for the stable Euler classes $\mathrm{Tr}_G(1)$ for these groups and sought to simplify presentations of $K(s)^*(BG)$ when G is modular or quasi-dihedral.

We consider the group

$$G_{p^{m+2}} = \left\langle a, b \;\middle|\; a^{p^{m+1}} = b^p = 1,\ bab^{-1} = a^{p^m+1} \right\rangle, \quad m > 1.$$

This group is called the modular group $M_{p^{m+2}}$ if $p \geq 3$ and the quasi-dihedral group $QD_{2^{m+2}}$ if $p = 2$ and $m \geq 3$.

In [36, 40], it was shown that $K(s)^*(BG)$ is generated as a $K(s)^*(pt)$-module by Chern classes of complex vector bundles. The multiplicative structure has been determined only modulo certain indeterminacy. In order to obtain explicit ring structure, it was suggested in [12, 35] to use some artificial generators not equal to Chern classes.

Our aim here is to apply the formulas for the transferred Chern classes derived in [5] and determine $K^*(BG)$ for the modular and quasi-dihedral groups completely in terms of Chern classes. The dihedral, semi-dihedral, and generalized quaternion groups are considered in [6].

The group $G_{p^{m+2}}$ is the semidirect product $Z/p^{m+1} \rtimes Z/p$ and there is the exact sequence

$$1 \to Z/p^{m+1} \to G_{p^{m+2}} \to Z/p \to 1,$$

where

$$Z/p^{m+1} = \langle a \rangle, \quad Z/p = \langle b \rangle.$$

For the canonical complex line bundle $\xi \to BZ/p^{m+1}$ and its first Chern class u, we have

$$b(\xi) = \xi^{\otimes 1+p^m}, \quad b(u) = [1+p^m](u).$$

Let

$$\xi_\pi \to BG_{p^{m+2}}$$

be the p-plane bundle transferred from ξ (see [1, 2]),

$$c_i = c_i(\xi_\pi), \quad i = 1, \dots, p,$$

be the Chern classes, and let

$$c = c_1(\theta)$$

be the Chern class of the line complex bundle, the pullback by the projection $G_{p^{m+2}} \to Z/p$. Then it is proved in [36] that

$$K(s)^*(BG_{p^{m+2}}) = K(s)^*[c, c_1, \dots, c_p]/\text{relations} \tag{8.1}$$

and the relations are given modulo some ideal.

In [4, 5] we introduced and calculated the following polynomials A_i in two variables:

$$A_i(z^{p-1}, Z) \in K(s)^*[z, Z]/[p]_F(z), \quad s > 1, \quad i = 1, \dots, p-1,$$

uniquely determined by the equations

$$\sigma_i = A_i(z^{p-1}, \sigma_p) + p^{-1}\binom{p}{i} v_s y^i z^{p^s-1},$$

where
$$\sigma_k = \sigma_k(y, F(y,z), \dots, F(y,(p-1)z)), \quad k = 1, \dots, p,$$
is ith elementary symmetric function and F is the formal group law.

For $p = 2$, we have
$$A_1(z, Z) = z + v_s \sum_{i=1}^{s-1} z^{2^s - 2^i} Z^{2^{i-1}}.$$

Our main result is the following theorem.

Theorem 8.1. *We have*
$$K(s)^*(BG_{p^{m+2}}) \cong K(s)^*[c, c_1, \dots, c_p]/R$$
and the ideal R is generated by
$$c^{p^s},\ x^{p^s},\ cc_i^*,\ xc_i^{**},\ c_i^* c_j^{**},\ i + j \neq p - 1, \quad c_i^* c_{p-1-i}^{**} - p^{-2}\binom{p}{i}\binom{p}{i+1} v_s x^{p^s-1} c^{p-1},$$
where
$$x = v_s^{(p^{ms}-1)/(p^s-1)} c_p^{p^{ms-1}}, \quad c_i^* = c_i - A_i(c^{p-1}, c_p), \quad c_i^{**} = c_i - A_i(x^{p-1}, c_p).$$

It is natural to compare our description of $K(s)^*(BG_{p^{m+2}})$ and the one given in [12], in terms of alternative generating set. Note that we have a smaller number of generating relations, even if less explicit.

Corollary 8.2. (i) *We have*
$$K(s)^*(BQD_{2^{m+2}}) \cong K(s)^*[c, c_1, c_2]/\left(c^{2^s},\ x^{2^s},\ cc_1^*,\ xc_1^{**},\ c_1^* c_1^{**}\right),$$
where
$$x = v_s^{(2^{ms}-1)/(2^s-1)} c_2^{2^{ms-1}}$$
and
$$c_1^* = c_1 + c + v_s \sum_{i=1}^{s-1} c^{2^s - 2^i} c_2^{2^{i-1}}, \quad c_1^{**} = c_1 + x + v_s \sum_{i=1}^{s-1} x^{2^s - 2^i} c_2^{2^{i-1}}.$$

(ii) $c^2 x = c x^2$, $c_1^2 = c^2 + x^2 + cx + v_s^2 (cxc_2)^{2^{s-1}}$.

More precisely, the following assertion holds.

Proposition 8.3. (i) *A $K(s)^*$-basis for $K(s)^*(BG_{p^{m+2}})$ is*

$c_1, \dots, c_{p-1}$;
c_p^i, $\quad i = 1, \dots, p^{(m+1)s-1} - 1$;
$c_k c_p^j$, $\quad k = 1, \dots, p-1,\ j = 1, \dots, p^{ms-1} - 1$;
$c^l c_p^m$, $\quad l = 1, \dots, p^s - 1,\ m = 0, \dots, p^{ms-1} - 1$;
$c^n c_p^q$, $\quad n = 1, \dots, p-1,\ q = p^{ms-1}, \dots, p^{(m+1)s-1} - 1$.

(ii) *The relations*
$$cc_{p-1}^* = 0, \quad xc_{p-1}^{**} = 0$$
imply
$$c^p x = c x^p.$$
Then for $i, j = 1, \dots, p-1$, the relations
$$cc_i^* = 0, \quad xc_i^{**} = 0,$$

$$c_i^* c_{p-1-i}^{**} = p^{-2}\binom{p}{i}\binom{p}{p-1-i} v_s x^{p^s-1} c^{p-1},$$
$$c_i^* c_j^{**} = 0, \quad j \neq p-1-i,$$

give basic expressions for cc_i, $c_i c_p^{p^{ms-1}}$, *and* $c_i c_j$, *respectively.*

Proposition 8.3 agrees with the rank of $K(s)^*(BG_{2^{m+2}})$ as a free $K(s)^*$ module. This number is $p^{ms-1}(p^{s+1}+p^s-1)$ and is given by the generalized character theory [23]. In particular, $K(s)^{\text{odd}}(BG_{p^{m+2}}) = 0$ by (39). Hence

$$\operatorname{rank}_{K(s)} K(s)^{\text{even}}(BG_{p^{m+2}}) = \chi_{s,p}(G_{p^{m+2}})$$

and the rank follows from the formula for $\chi_{s,p}(G_{p^{m+2}})$ (see [12, Lemma 2.2]).

Now we prove the relations of Theorem 8.1 and show that they give the decompositions of Proposition 8.3. Then by the rank argument this will imply that the system of relations is complete.

We recall some results from [5] (see also [4]).

Let $X \to X/\pi$ be a regular cyclic covering of prime index p defined by a free action of cyclic group π on X and let

$$\operatorname{Tr}_\pi^* : K(s)^*(X) \to K(s)^*(X/\pi)$$

be its associated transfer homomorphism [17, 27]. Let $\eta \to X$ be a complex line bundle and $\eta_\pi \to X/\pi$ be the transferred η.

In [5] we proved that modulo $\operatorname{Im}\operatorname{Tr}_\pi^*$, the Chern classes $c_i(\eta_\pi)$, $1 \le i < p$, can be written in terms of the polynomials A_i from Theorem 8.1. Namely, we have constructed certain classes $\omega_i(\eta)$ such that

$$c_i(\eta_\pi) = A_i((c_1\psi)^{p-1}, \quad c_p(\eta_\pi)) + \operatorname{Tr}_\pi^*(\omega_i(\eta)), \tag{8.2}$$

where $\psi \to X/\pi$ is the complex line bundle associated with the covering $X \to X/\pi$. We will need the following two consequences.

Let

$$\xi \to BZ/p^{m+1}, \quad \xi_\pi \to BG_{p^{m+2}}, \quad \theta \to BG_{p^{m+2}}$$

be the bundles of the Introduction and let $c = c_1(\theta)$, $u = c_1(\xi)$, and $c_k = c_k(\xi_\pi)$, $k = 1, \dots, p$ be their corresponding Chern classes.

Lemma 8.4. $c_i = A_i(c^{p-1}, c_p) + \operatorname{Tr}^*(\omega_i(\xi))$, $i = 1, \dots, p-1$.

In the notation of Theorem 8.1, $c_i^* = \operatorname{Tr}^*(\omega_i(\xi))$. For instance, $c_1^* = \operatorname{Tr}^*(u)$. The values $\operatorname{Tr}^*(u^k)$ for $2 \le k \le p-1$ can be calculated as follows.

Lemma 8.5. (i) *We have*

$$\operatorname{Tr}_\pi^*(u^k) = \sum_{i=1}^{k-1}(-1)^{i+1}\operatorname{Tr}^*(u^{k-i})c_i + (-1)^{k+1}kc_k^*.$$

(ii) *Modulo the ideal in* $K(s)^*(BG_{p^{m+2}})$ *generated by* $c^p, c_1, \dots, c_{p-1}$, *we have*

$$\operatorname{Tr}^*(u^k) = 0, \quad 2 \le k < p-1, \qquad \operatorname{Tr}^*(u^{p-1}) = -c^{p-1}.$$

Proof. (i) As in the Introduction, let u, $b(u), \dots, b^{p-1}(u)$ be the Chern classes of the bundles ξ, $b(\xi), \dots, b^{p-1}(\xi)$. Then

$$u^k = \sum_{i=1}^{k-1}(-1)^{i+1}u^{k-i}\sigma_i(u, b(u), \dots, b^{p-1}(u)) + (-1)^{k+1}u\sigma_{k-1}(b(u), \dots, b^{p-1}(u)).$$

It follows from the definition of ω_i (see [4]) that $u\sigma_{k-1}(b(u), \dots, b^{p-1}(u))$ contains k summands, each with transfer value $\operatorname{Tr}^*(\omega_k) = c_k^*$.

(ii) This follows from (i) and the fact that modulo c^p, we have

$$A_k(c^{p-1}, c_p) = 0, \quad 2 \le k < p-1, \qquad A_{p-1}(c^{p-1}, c_p) = -c^{p-1}.$$

The lemma is proved. $\square$

We will need also the following simple facts which follow immediately from the definitions.

Lemma 8.6. *Let $Z/p^m = \langle a^p \rangle$ and $Z/p^{m+1} = \langle a \rangle$ be subgroups of $G_{p^{m+2}}$. Let $\rho : BZ/p^m \to BZ/p^{m+1}$ be the standard p-covering and Tr_ρ be the corresponding transfer map. Then the following assertions hold.*

(i) *The line bundle associated with ρ is $\xi^{\otimes p^m}$. The transferred trivial bundle 1_ρ is*

$$1 + \xi^{\otimes p^m} + \dots + \xi^{\otimes (p-1)p^m}.$$

(ii) *The bundle*

$$\pi^*(\xi_\pi) = \xi \otimes (1 + \xi^{\otimes p^m} + \dots + \xi^{\otimes (p-1)p^m})$$

is the transferred bundle of $\rho^*(\xi)$. Further,

$$c_1(\xi^{\otimes p^m}) = v_s^{(p^{ms}-1)/(p^s-1)} u^{p^{ms}} = \pi^*(x), \quad \mathrm{Tr}^*_\rho(1) = -v_s \pi^*(x)^{p^s-1}.$$

(iii) *We have*

$$c_i(\pi^*(\xi_\pi)) = A_i\Big((\pi^*(x))^{p-1},\ c_p(\pi^*(\xi_\pi))\Big) + \mathrm{Tr}^*_\rho(\omega_i(\rho^*(\xi)))$$

and since the bundle $\rho^*(\xi^{p^m})$ is trivial,

$$\omega_i(\rho^*(\xi)) = p^{-1}\binom{p}{i}\rho^*(u^i).$$

The proof of Theorem 8.1 is organized as follows. In a sequence of lemmas below, we prove all relations of Theorem 8.1. The relations given by Lemma 8.7(ii) and Lemma 8.10 are not yet the decompositions of cc_i and c_ic_j, $i, j = 1, \dots, p-1$, in the basis of Proposition 8.3: there will arise terms divisible by the factor $c^p x$, but to eliminate these we just have to apply $c^p x = cx^p$ of Lemma 8.9. Lemma 8.8(ii) gives the proper decomposition of xc_i. Together with $c^{p^s} = x^{p^s} = 0$ (Lemma 8.7(i) and Lemma 8.8(i)) and $x = v_s^{(p^{ms}-1)/(p^s-1)} c_p^{p^{ms-1}}$ (Lemma 8.11(ii)), these relations give the proper decompositions of all other monomials in $c, c_1, \dots, c_p$. This system of relations is complete.

Lemma 8.7. (i) $c^{p^s} = 0$;

(ii) $cc_i = cA_i(c^{p-1}, c_p)$, $i = 1, \dots, p-1$.

Proof. (i) We have $\theta^{\otimes p} = 1$. Hence $c_1(\theta^{\otimes p}) = c^{p^s} = 0$.

The relations (ii) follow from Lemma 8.4 since $\pi^*(c) = 0$ and by the Frobenius reciprocity of the transfer

$$c\,\mathrm{Tr}^*(\omega_i(\xi)) = \mathrm{Tr}^*(\pi^*(c)\omega_i(\xi)) = 0.$$

On the other hand, we have the bundle relation

$$\theta \otimes \xi_\pi = \xi_\pi. \tag{8.3}$$

Equating the Chern classes in (8.3) and applying the splitting principle we get the same relations after simplifying them. This was proved for the universal example in [4]. $\square$

Lemma 8.8. *Let $x = c_1((\det \xi_\pi)^{\otimes p^{m-1}})$. Then*

(i) $x^{p^s} = 0$;

(ii) $xc_i = xA_i(x^{p-1}, c_p)$, $i = 1, \dots, p-1$.

Proof. Assertion (i) immediately follows from Lemma 8.7(i) and the definition of x.

(ii) We have the bundle relation

$$(\det \xi_\pi)^{\otimes p^{m-1}} \otimes \xi_\pi = \xi_\pi \tag{8.4}$$

which comes from the Frobenius reciprocity:

$$(\det \xi_\pi)^{\otimes p^{m-1}} \otimes \xi_\pi = (\pi^*((\det \xi_\pi)^{\otimes p^{m-1}}) \otimes \xi)_\pi = (\xi^{\otimes p^m+1})_\pi = (b^*(\xi))_\pi = \xi_\pi.$$

For the last equality, we recall from the Introduction the action of $b \in Z/p$ in BZ/p^{m+1} and that transfer is constant on the orbit elements.

Now we can equate the Chern classes in (8.3) and proceed as in the proof of Lemma 8.7. $\square$

Lemma 8.9. (i) $c^p x = cx^p$;
(ii) $c_i x^{p^s-1} = 0$, $i < p$;
(iii) $c^p x^{p^s-1} = 0$.

Proof. (i) Multiplying the relations of Lemmas 8.7 and 8.8 by x and c, respectively, for $i = p-1$ and then equating the right-hand sides, we have

$$cxA_{p-1}(c^{p-1}, c_p) = cxA_{p-1}(x^{p-1}, c_p).$$

Then note that, up to an invertible factor, the difference $A_{p-1}(c^{p-1}, c_p) - A_{p-1}(x^{p-1}, c_p)$ coincides with $c^{p-1} - x^{p-1}$. Thus, $cx(c^{p-1} - x^{p-1}) = 0$.

(ii) By Lemma 8.8

$$c_i x^{p^s-1} = c_i x x^{p^s-2} = xA_i(x^{p-1}, c_p)x^{p^s-2}.$$

By definition, $A_i(0, c_p) = 0$. Thus, the right-hand side contains the factor x^{p^s} hence is trivial again by Lemma 8.8.

(iii) Since $c^p x^{p^s-1} = c^p x^{p^s-2} x$ divisible by x^{p^s}, it is zero by Lemma 8.8. $\square$

Lemma 8.10. *We have $c_i^* c_j^{**} = 0$, $j \neq p-1-i$, and*

$$c_i^* c_{p-1-i}^{**} = p^{-2}\binom{p}{i}\binom{p}{p-1-i} v_s x^{p^s-1} c^{p-1}.$$

Proof. We use the Frobenius reciprocity of the transfer and formulas for transferred classes. Let Tr_ρ be the transfer map of the covering

$$\rho : BZ/p^m \to BZ/p^{m+1}$$

and let $\mathrm{Tr}_\pi = \mathrm{Tr}$ be the transfer map of π as above. Then $(c_i - A_i(c^{p-1}, c_p))(c_j - A_j(x^{p-1}, c_p))$ is equal to

$$\begin{aligned}
&\mathrm{Tr}_\pi^*(\omega_i)\big(c_j - A_j(x^{p-1}, c_p)\big) && \text{by Lemma 8.4}\\
&= \mathrm{Tr}_\pi^*\Big(\omega_i(\xi)\pi^*(c_j - A_j(x^{p-1}, c_p))\Big) && \text{by the Frobenius reciprocity}\\
&= \mathrm{Tr}_\pi^*\Big(\omega_i(\xi)\,\mathrm{Tr}_\rho^*\,\omega_j(\rho^*(\xi))\Big) && \text{by Lemma 8.6(iii)}\\
&= \mathrm{Tr}_\pi^*\Big(\mathrm{Tr}_\rho^*\big(\rho^*(\omega_i(\xi))(\omega_j(\rho^*(\xi)))\big)\Big) && \text{by the Frobenius reciprocity}\\
&= \mathrm{Tr}_\pi^*\Big(\mathrm{Tr}_\rho^*\big(p^{-2}\binom{p}{i}\binom{p}{j}\rho^*(u^{i+j})\big)\Big) && \text{by Lemma 8.6(iii)}\\
&= p^{-2}\binom{p}{i}\binom{p}{j}\mathrm{Tr}_\pi^*\Big(\mathrm{Tr}_\rho^*(1)(u^{i+j})\Big) && \text{by the Frobenius reciprocity}\\
&= -p^{-2}\binom{p}{i}\binom{p}{j}\mathrm{Tr}_\pi^*\Big(v_s\pi^*(x)^{p^s-1}(u^{i+j})\Big) && \text{by Lemma 8.6(ii)}
\end{aligned}$$

$$= -p^{-2}\binom{p}{i}\binom{p}{j} v_s x^{p^s-1}\operatorname{Tr}^*_\pi(u^{i+j}) \qquad \text{by the Frobenius reciprocity.}$$

Then by Lemma 8.5(ii), $\operatorname{Tr}^*(u^{i+j})$ is in the ideal $(c^p, c_1, \ldots, c_{p-1})$ if $0 \le i+j \le p-1$ and hence annihilates x^{p^s-1} by Lemma 8.9. If $j = p-1-i$, then

$$\operatorname{Tr}^*(u^{i+j}) = -c^{p-1} \mod (c^p, c_1, \ldots, c_{p-1})$$

by Lemma 8.5(ii) and the result follows.

Finally, let $i+j = p+q$, $q = 0, \ldots, p-2$. Note that

$$\rho^*(\pi^*(c_p)) = \rho^*(u^p).$$

Then

$$\begin{aligned}\operatorname{Tr}^*_\pi\left(\operatorname{Tr}^*_\rho\left(\rho^*(u^{i+j})\right)\right) &= \operatorname{Tr}^*_\pi\left(\operatorname{Tr}^*_\rho\left(\rho^*(\pi^*(c_p))\rho^*(u^q)\right)\right)\\ &= \operatorname{Tr}^*_\pi\left(\operatorname{Tr}^*_\rho(1)\pi^*(c_p)u^q\right) = \operatorname{Tr}^*_\pi\left(\pi^*(x^{p^s-1})\pi^*(c_p)u^q\right) = x^{p^s-1}c_p\operatorname{Tr}^*_\pi(u^q) = 0.\end{aligned}$$

Here, as above, $\operatorname{Tr}^*_\pi(u^q)$ is in the ideal $(c^p, c_1, \ldots, c_{p-1})$ and hence annihilates x^{p^s-1}. □

Now let us evaluate x in Theorem 8.1.

Lemma 8.11. (i) *$c_i^{p^s} = 0$ for $i = 1, \ldots, p-1$ and odd p and $c_1^{2^s+1} = 0$, for $p = 2$. $c_p^{p^s}$ restricts to $u^{p^{s+1}}$ in $K(s)^*(BZ/p^{m+1})$.*

(ii) *We have*

$$x = v_s^{(p^{ms}-1)/(p^s-1)} c_p^{p^{ms-1}}.$$

Proof. First, let p be odd.

(i) Lemma 8.4 and Lemma 8.5(i) imply that, modulo c, the classes $c_1, \ldots, c_{p-1}$ can be written in $\operatorname{Tr}^*(u), \ldots, \operatorname{Tr}^*(u^{p-1})$ and vice versa. Then by Lemma 8.7, $c^{p^s} = 0$; hence it suffices to prove that

$$\operatorname{Tr}^*(u^k)^{p^s} = 0.$$

c_i restricts to the ith elementary symmetric function in the variables $u, [1+p^m](u), \ldots, [1+(p-1)p^m](u)$ in $K(s)^*(BZ/p^{m+1})$.

Note that

$$([p^m](u))^{p^s} = u^{p^{(m+1)s}} = 0$$

and recall (see [4, 31]) that

$$[1+ip^m](u) = u + i[p^m](u) + v_s w_1(u, i[p^m](u))^{p^{s-1}},$$

where w_1 is a homogeneous polynomial:

$$w_1(u,v) = -\sum_{0\le j\le p} p^{-1}\binom{p}{j} u^j v^{p-j}.$$

Thus, c_i and $\operatorname{Tr}^*(u^i)$, $i = 1, \ldots, p-2$, restrict trivially in $K(s)^*(BZ/p^{m+1})$ modulo $([p^m](u))^{p^{s-1}}$ and c_{p-1} and $\operatorname{Tr}^*(u^{p-1})$ restrict trivially modulo $([p^m](u))^{p-1}$.

Similarly, c_p restricts to u^p modulo $([p^m](u))^{p^{s-1}}$. Then

$$(p^{s-1})(p^s-1), (p-1)(p^s-1) > p^s.$$

Hence $\operatorname{Tr}(u^k)^{p^s-1}$ restricts trivially in $K(s)^*(BZ/p^{m+1})$ and by the Frobenius reciprocity

$$\operatorname{Tr}^*(u^k)(\operatorname{Tr}^*(u^k))^{p^s-1} = \operatorname{Tr}^*(u^k\cdot 0) = 0.$$

(ii) Let $\det\xi_\pi$ be the determinant. The Chern class $\gamma = c_1(\det\xi_\pi)$ can be written in terms of the classes $c_1, \ldots, c_p$ in the standard way.

After raising to the power $p^{ms-s} \geq p^s$, we have that (i) implies

$$\gamma^{p^{ms-s}} = v_s^{p^{ms-s}} c_p^{p^{ms-1}}. \tag{8.5}$$

Then $v_s^{(p^{(m-1)s}-1)/(p^s-1)} \gamma^{p^{ms-s}}$ is the Chern class of the bundle

$$(\det(\xi_\pi))^{\otimes p^{m-1}}$$

and (ii) follows.

For $p = 2$, note that for the Chern class of $\det \xi_\pi$, as the standard series in c_1 and c_2, we have modulo c_1

$$\gamma = v_s c_2^{2^{s-1}}.$$

Again

$$x = v_s^{(2^{(m-1)s}-1)/(2^s-1)} \gamma^{2^{ms-s}}$$

and it suffices to prove that $c_1^{2^s+1} = 0$. By the definition of quasi-dihedral group $m > 2$, $2^{ms-s} > 2^s + 1$ and we can proceed as in case of odd p.

Recall that $c^{2^s} = x^{2^s} = 0$ by Lemma 8.7(i) and Lemma 8.8(i). Also by Lemma 8.10, $c^2 x = cx^2$ hence we have for $i + j \geq 2^s + 1$

$$c^i x^j = 0. \tag{8.6}$$

Now the formulas for c_1^* and c_1^{**} simplify $c_1^* c_1^{**} = 0$ as

$$c_1^2 = c_1 c + c_1 x + cx = c^2 + x^2 + cx + v_s^2 (cxc_2)^{2^{s-1}}. \tag{8.7}$$

Hence after raising (8.3) to the power 2^{s-1}, we have

$$c_1^{2^s} = c^{2^{s-1}} x^{2^{s-1}}. \tag{8.8}$$

Thus,

$$c_1^{2^s+1} = x^{2^{s-1}} c^{2^{s-1}} c_1 = x^{2^{s-1}} c^{2^{s-1}-1} cc_1 = x^{2^{s-1}} c^{2^{s-1}-1} c^2 = 0$$

by (8.8), $cc_1 = 0$ modulo c^2, and (8.6) the result follows. □

9. Dihedral, Semidihedral, and Generalized Quaternion Groups

Now let

$$G = \left\langle a, b \;\middle|\; a^{2^{m+1}} = 1,\ b^2 = a^e,\ bab^{-1} = a^r \right\rangle, \quad m \geq 1,$$

and either

(i) $e = 0$ and $r = -1$ (the dihedral group $D_{2^{m+2}}$ of order 2^{m+2}), or
(ii) $e = 2^m$ and $r = -1$ (the generalized quaternion group $Q_{2^{m+2}}$), or
(iii) $m \geq 2$, $e = 0$, and $r = 2^m - 1$ (the semidihedral group $SD_{2^{m+2}}$).

Consider the following Chern classes c, x, c_1, and c_2 of dimensions $|c| = |x| = |c_1| = 2$ and $|c_2| = 4$:

$$c = c_1(\eta_1), \quad \eta_1 : G/\langle a \rangle \cong \mathbf{Z}/2 \to \mathbb{C}^*, \quad b \mapsto -1,$$
$$x = c_1(\eta_2), \quad \eta_2 : G/\langle a^2, b \rangle \cong \mathbf{Z}/2 \to \mathbb{C}^*, \quad a \mapsto -1,$$

and $c_i = c_i(\xi_{\pi_1})$, where

$$\xi_{\pi_1} \to B\langle a, b \rangle$$

is the plane bundle transferred from the canonical line bundle

$$\xi \to B\langle a \rangle,$$

for the double covering

$$\pi_1 : B\langle a \rangle \to B\langle a, b \rangle$$

corresponding to η_1.

Theorem 9.1. (i) *$K(s)^*(BG) = K(s)^*[c, x, c_2]/R$ and the relations R are determined by*

$$c^{2^s} = x^{2^s} = 0, \tag{9.1}$$

$$v_s c c_2^{2^{s-1}} = v_s \sum_{i=1}^{s-1} c^{2^s - 2^i + 1} c_2^{2^{i-1}} + \begin{cases} 0 & \text{if } G \text{ is dihedral,} \\ c^2 & \text{if } G \text{ is quaternion,} \\ cx & \text{if } G \text{ is semidihedral;} \end{cases} \tag{9.2}$$

$$v_s^2 c_2^{2^s} = \begin{cases} cx + x^2 & \text{if } G = D_8, \\ c^2 + cx + x^2 & \text{if } G = Q_8 \end{cases} \tag{9.3}$$

and for $m > 1$

$$v_s^{2\kappa(m)} c_2^{2^{ms}} = cx + x^2 \tag{9.4}$$

for G of all three types;

$$v_s x c_2^{2^{s-1}} = v_s \sum_{i=1}^{s-1} x^{2^s - 2^i + 1} c_2^{2^{i-1}} + \begin{cases} cx + x^2 & \text{if } G = D_8, \\ x^2 & \text{if } G = Q_8; \end{cases} \tag{9.5}$$

for $m > 1$,

$$v_s x c_2^{2^{s-1}} = v_s x \sum_{i=1}^{s-1} c^{2^s - 2^i} c_2^{2^{i-1}} + \sum_{i=1}^{ms} v_s^{1+\kappa(m)+2^{ms}-2^i} c_2^{(2^{ms}+1)2^{s-1} - (2^s-1)2^{i-1}}$$

$$+ \begin{cases} 0 & \text{if } G \text{ is dihedral,} \\ cx & \text{if } G \text{ is quaternion} \\ & \text{or semidihedral,} \end{cases}$$

where

$$\kappa(m) = \frac{2^{ms} - 1}{2^s - 1}.$$

(ii) *$c^2 x = cx^2$, $c_1^{2^{ms}+1} = 0$, and $c_2^{(2^{ms}+1)2^{s-1}} = 0$.*

Together with the covering π_1, we can consider the covering

$$\pi_2 : B\langle a^2, b\rangle \to B\langle a, b\rangle$$

corresponding to η_2. Then let

$$\eta_{\pi_2} \to BG$$

be the transferred line bundle associated with double covering

$$\langle a^4, b\rangle \to \langle a^2, b\rangle.$$

The bundles ξ_{π_1} and η_{π_2} coincide if $m = 1$, but if $m > 1$, then

$$\eta_{\pi_2} = (\xi^{\otimes 2^{m-1}})_{\pi_1}. \tag{9.6}$$

The following bundle relations hold.

Lemma 9.2. (i) *$\eta_i^{\otimes 2} = \mathbb{C}$, $\eta_1 \otimes \xi_{\pi_1} = \xi_{\pi_1}$;*

(ii) *$\eta_i \otimes \eta_{\pi_2} = \eta_{\pi_2}$;*

(iii) *$\eta_{\pi_2}^{\otimes 2} = \mathbb{C} \oplus \eta_1 \oplus \eta_2 \oplus \eta_1 \otimes \eta_2$;*

(iv) *$\det \xi_{\pi_1}$ is η_1 if G is dihedral, the trivial bundle $\mathbb{C}$ if G is quaternion, and $\eta_1 \otimes \eta_2$ if G is semidihedral, and for $m > 1$, we have $\det \eta_{\pi_2} = \eta_1$ in all three cases;*

(v) *we have*

$$((\xi^{\otimes 2^i})_{\pi_1})^{\otimes 2} = (\xi^{\otimes 2^{i+1}})_{\pi_1} \oplus \mathbb{C} \oplus \eta_1$$

for $1 \leq i < m-1$*. The bundle*

$$\xi_{\pi_1} \otimes \xi_{\pi_1} = (\xi^{\otimes 2})_{\pi_1} \oplus (\xi \otimes \xi^{\otimes r})_{\pi_1}$$

is

$$(\xi^{\otimes 2})_{\pi_1} \oplus \mathbb{C} \oplus \eta_1$$

if G *is dihedral or quaternion and is*

$$(\xi^{\otimes 2})_{\pi_1} \oplus \eta_1 \otimes \eta_2 \oplus \eta_2$$

if G *is semidihedral.*

Proof. These relations are the consequences of the Frobenius reciprocity of the transfer in complex K-theory. For example,

$$\begin{aligned}\eta_{\pi_2}^{\otimes 2} = (\xi^{2^{m-1}})_{\pi_1}^{\otimes 2} = (\xi^{2^{m-1}} \otimes \pi_1^*((\xi^{2^{m-1}})_{\pi_1}))_{\pi_1} = (\xi^{2^m} \oplus \mathbb{C})_{\pi_1} \\ = \eta_2 \otimes (\mathbb{C})_{\pi_1} \oplus \mathbb{C} \oplus \eta_1 = \eta_2 \otimes (\mathbb{C} \oplus \eta_1) \oplus \mathbb{C} \oplus \eta_1.\end{aligned}$$

The lemma is proved. □

We recall the transfer formula from [5] (see also [4]).

Let $X \to X/\pi$ be a regular double covering defined by a free involution on X, $\xi \to X$ be a complex line bundle, ξ_π the transferred bundle, and let

$$\mathrm{Tr}_\pi^* : K(s)^*(X) \to K(s)^*(X/\pi)$$

be the associated transfer homomorphism [17, 27]. Then

$$c_1(\xi_\pi) = c_1(\psi) + v_s \sum_{i=1}^{s-1} c_1(\psi)^{2^s - 2^i} c_2(\xi_\pi)^{2^{i-1}} + \mathrm{Tr}_\pi^*(c_1(\xi)), \tag{9.7}$$

where $\psi \to X/\pi$ is the complex line bundle associated to the covering $X \to X/\pi$.

The following lemma is an easy consequence of the recursive formula for the FGL given in [31, 4.3.9] (see also [4, Lemma 5.3]).

Lemma 9.3. (i) *For the Honda formal group law at* $p = 2$*,* $s > 1$*, we have*

$$F(y, z) = y + z + v_s (yz)^{2^{s-1}}$$

modulo $y^{2^{2(s-1)}}$ *(or modulo* $z^{2^{2(s-1)}}$*).*

(ii) *We have*

$$F(y, z) = y + z + v_s \Phi(v_s, y, z)^{2^{s-1}},$$

where

$$\Phi(v_s, y, z) = yz + v_s (yz)^{2^{s-1}} (y + z)$$

modulo $(yz)^{2^{s-1}}(y+z)^{2^{s-1}}$*.*

For two line bundles with the Chern classes y and z, respectively, $\Phi(v_s, y, z)$ can be regarded as the $K(s)^*$ orientation class of their sum.

Lemma 9.4. *Let* $m > 1$ *and either* $r = -1$ *or* $r = 2^m - 1$*. We have in* $K(s)^*[u]/(u^{2^{(m+1)s}})$

$$u^{2^{ms}} = \sum_{i=1}^{ms} v_s^{2^{ms} - 2^i} (u[r](u))^{2^{(m+1)s-1} - (2^s-1)2^{i-1}} + [r](u)(u + [r](u))^{2^{ms}-1}.$$

Proof. The obvious decomposition in $\mathbb{F}_2[y,z]$

$$y^{2^k} = \sum_{i=1}^{k} (y+z)^{2^k-2^i}(yz)^{2^{i-1}} + y(y+z)^{2^k-1}$$

for $y=u$, $z=[r](u)$, and $k=ms$ implies

$$u^{2^{ms}} = \sum_{i=1}^{ms} (u+[r](u))^{2^{ms}-2^i}(u[r](u))^{2^{i-1}} + u(u+[r](u))^{2^{ms}-1}.$$

We equate the monomials

$$(u+[r](u))^{2^{ms}-2^i} = (u+[r](u))^{2^i+\cdots+2^{ms-1}}$$

to the monomials

$$v_s^{2^{ms}-2^i}(u[r](u))^{(2^{ms}-2^i)2^{s-1}} = v_s^{2^i+\cdots+2^{ms-1}}(u[r](u))^{(2^i+\cdots+2^{ms-1})2^{s-1}}$$

by the equation

$$(u+[r](u))^2 = v_s^2(u[r](u))^{2^s}$$

modulo some irrelevant factor as follows.

The nilpotence degree for u is $2^{(m+1)s}$, hence is $2^{(m+1)s-1}$ for $u[r](u)$. Since it is 2^s for $F(u,[2^m-1])$ (whereas $F(u,[-1](u))=0$), the nilpotence degree for $u+[r](u)$ is 2^{ms} by Lemma 9.3(ii).

Therefore, it suffices to show

$$(u+[r](u))^2 = v_s^2(u[r](u))^{2^s} \mod (u+[r](u))^4.$$

Lemma 9.3(ii) implies

$$(u+[r](u))^2 = v_s^2(u[r](u))^{2^s} + F(u,[r](u))^2 \mod (u+[r](u))^{2^s}$$

and the dihedral and quaternion cases follow.

For the semidihedral group, we have

$$F(u,[2^m-1](u)) = v_s^{\kappa(m)}u^{2^{ms}}.$$

Also,

$$u^{2^{ms+1}} = (u[r]u)^{2^{ms}}$$

as $u^{2^{ms}} = ([r](u))^{2^{ms}}$. Therefore, we obtain, modulo $(u+[r](u))^{2^s}$ ignoring powers of v_s,

$$(u[r](u))^{2^s} = (u+[r](u))^2 + (u[r](u))^{2^{ms}};$$

moreover,

$$F(u,[r](u))^2 = (u[r](u))^{2^{ms}} = ((u+[r](u))^2 + (u[r](u))^{2^{ms}})^{2^{ms-s}} = 0$$

as $ms-s+1>s$. The lemma is proved. □

As mentioned in Sec. 7, it was proved in [36] that as a $K(s)^*(pt)$-module, $K(s)^*$ of the spaces we consider is generated by the Chern classes c, x, and c_2 defined above. Let $\tilde{c}_1$ and $\tilde{c}_2$ be the Chern classes of the bundle η_{π_2}.

Lemma 9.2(i) implies $c^{2^s}=0$ and $x^{2^s}=0$ as $[2](c)=v_sc^{2^s}=0$ and similarly for x.

Let

$$c_1^* = c_1 + c + v_s\sum_{i=1}^{s-1} c^{2^s-2^i}c_2^{2^{i-1}}, \quad c_1^{**} = \tilde{c}_1 + x + v_s\sum_{i=1}^{s-1} x^{2^s-2^i}\tilde{c}_2^{2^{i-1}}. \tag{9.8}$$

By (9.7),

$$c_1^* \in \operatorname{Im}\operatorname{Tr}^*_{\pi_1}, \quad c_1^{**} \in \operatorname{Im}\operatorname{Tr}^*_{\pi_2};$$

hence

$$c_1^{2^s} \in \operatorname{Im} \operatorname{Tr}^*_{\pi_1}, \quad \tilde{c}_1^{2^s} \in \operatorname{Im} \operatorname{Tr}^*_{\pi_2}$$

as $c^{2^s} = x^{2^s} = 0$.

By the Frobenius reciprocity, $cc_1^* = 0$; hence by (9.7)

$$c^{2^{s-1}} c_1^{2^{s-1}} = 0, \quad x^{2^{s-1}} c_1^{2^{s-1}} = c^{2^{s-1}} x^{2^{s-1}}$$

modulo $\operatorname{Tr}^*_{\pi_1}(u)$, $u = c_1(\xi)$. From (8.8) we obtain

$$c_1^{2^{s-1}} = c^{2^{s-1}} \mod \operatorname{Tr}^*_{\pi_1}(u).$$

Hence

$$c_1(\det \xi_{\pi_1}) = c_1 + v_s c_2^{2^{s-1}} \mod \operatorname{Tr}^*_{\pi_1}(u). \tag{9.9}$$

Then note that

$$F(c, x) = c + x + v_s c^{2^{s-1}} x^{2^{s-1}};$$

hence combining (9.8) and (9.9) we obtain, modulo $\operatorname{Tr}^*_{\pi_1}(u)$ and $c^{2^{s-1}} x^{2^{s-1}}$,

$$v_s c_2^{2^{s-1}} + v_s \sum_{i=1}^{s-1} c^{2^s - 2^i} c_2^{2^{i-1}} = \begin{cases} 0 & \text{if } G \text{ is dihedral,} \\ c & \text{if } G \text{ is quaternion,} \\ x & \text{if } G \text{ is semidihedral.} \end{cases} \tag{9.10}$$

Also we have

$$c_1(\det \eta_{\pi_2}) = \tilde{c}_1 + v_s \tilde{c}_2^{2^{s-1}} + v_s \tilde{c}_1^{2^s}. \tag{9.11}$$

To prove (9.11) we need the relations (ii) of Theorem 9.1. These are consequences of the following relations (9.12)–(9.15).

Lemma 9.2(v) and (9.6) implies that modulo c and x, the Chern classes $\tilde{c}_1$ and $\tilde{c}_2$ coincide with the first and second Chern classes of $(\xi_{\pi_1})^{\otimes 2^{m-1}}$, respectively:

$$\tilde{c}_1 = v_s^{\frac{2^{(m-1)s} - 1}{2^s - 1}} c_1^{2^{(m-1)s}}, \quad \tilde{c}_2 = v_s^{\frac{2(2^{(m-1)s} - 1)}{2^s - 1}} c_2^{2^{(m-1)s}}. \tag{9.12}$$

On the other hand, consecutively equating Chern classes of both sides of the equation in Lemma 9.2(iii), we obtain, respectively,

$$\tilde{c}_1^{2^s} = c^{2^{s-1}} x^{2^{s-1}} \tag{9.13}$$

for $m = 1$,

$$v_2^2 c_2^{2^s} = \begin{cases} cx + x^2 & \text{if } G = D_8, \\ c^2 + cx + x^2 & \text{if } G = Q_8 \end{cases} \tag{9.14}$$

for $m > 1$,

$$v_2^2 \tilde{c}_2^{2^s} = cx + x^2$$

in all cases, and

$$c^2 x + c x^2 = 0, \tag{9.15}$$

and we obtain formula (9.3) from Theorem 9.1 and relations (ii). Here we use the splitting principle and write formally

$$\eta_{\pi_2} = \lambda_1 \oplus \lambda_2, \quad \eta_{\pi_2}^{\otimes 2} = \lambda_1^{\otimes 2} \oplus \lambda_2^{\otimes 2} \oplus 2\lambda_1 \otimes \lambda_2.$$

Also we take into account that the determinant $\lambda_1 \otimes \lambda_2$ is known by Lemma 9.2(iv). Let $m > 1$ and $\lambda_1 \otimes \lambda_2 = \eta_1$. Then by the first equation for the Chern classes

$$v_s \tilde{c}_1^{2^s} = c + x + c + x + v_s c^{2^{s-1}} x^{2^{s-1}} \;\Rightarrow\; (9.13).$$

By (9.12) and formula (9.1) of Theorem 9.1,

$$\tilde{c}_1^{2^s} \in \operatorname{Im} \operatorname{Tr}^*_{\pi_1}.$$

Also as before,

$$\tilde{c}_1^{2^s} \in \operatorname{Im} \operatorname{Tr}^*_{\pi_2}.$$

Hence

$$c\tilde{c}_1^{2^s} = x\tilde{c}_1^{2^s} = 0$$

and, by (9.13),

$$c^i x^j = 0 \quad \text{for } i+j > 2^s.$$

Multiplying (9.8) by $c^{2^{s-1}} x^{2^{s-1}}$ we obtain $c_1^{2^{ms}+1} = 0$; therefore, Lemma 9.3 implies

$$c_2^{(2^{ms}+1)2^{s-1}} = 0.$$

Then the second equation yields

$$v_s^2 \tilde{c}_2^{2^s} + c^2 = cx + (c+x)(c+x+v_s c^{2^{s-1}} x^{2^{s-1}}) \ \Rightarrow \ (9.14).$$

The third equation yields

$$0 = v_s \tilde{c}_1^{2^s} c^2 = cx(c+x+v_s c^{2^{s-1}} x^{2^{s-1}}) = cx(c+x)$$

and (9.15) follows. Similarly, for $m = 1$, but $\tilde{c}_i = c_i$, and for $G = Q_8$, the determinant $\lambda_1 \otimes \lambda_2$ is trivial.

To prove formula (9.2) of Theorem 9.1 for $m > 1$, we raise (9.9) to the power $2^{ms-s} > 2^s$. We obtain

$$c_2^{2^{ms-1}} = 0 \mod \operatorname{Tr}^*_{\pi_1}(u). \tag{9.16}$$

By the Frobenius reciprocity of the transfer, (9.16) implies

$$cc_2^{2^{ms-1}} = 0. \tag{9.17}$$

Then, as above,

$$c^i x^j = 0 \quad \text{for } i+j \geq 2^s+1.$$

Multiplying (9.10) by c, we obtain formula (9.2) of Theorem 9.1.

Now let $m = 1$. Then $\tilde{c}_i = c_i$ and $cc_1^* = 0$ by (9.7). Hence multiplying (9.11) by c, we obtain formula (9.2) of Theorem 9.1.

Proof of formula (9.5) *of Theorem* 9.1. Let $m > 1$. By the above definitions, we have

$$\pi_1^*(\xi_{\pi_1}) = \xi \oplus \xi^{\otimes r}, \quad \pi^*(\eta_2) = \xi^{\otimes 2^m}.$$

Then Lemma 9.3 implies

$$v_s^{-\kappa(m)} \pi_1^*(x) = u^{2^{ms}} = \pi_1^*\left(\sum_{i=1}^{ms} v_s^{2^{ms}-2^i} c_2^{2^{(m+1)s-1}-(2^s-1)2^{i-1}}\right) + [r](u)\pi_1^*(c_1^{2^{ms}-1}). \tag{9.18}$$

We apply transfer to (9.18) after multiplying by u. By (9.7), $c_1^{2^{ms}-1} \in \operatorname{Im}\operatorname{Tr}^*_{\pi_1}$ is in the annihilator of c, hence

$$\operatorname{Tr}^*_{\pi_1}(u\ [r](u))c_1^{2^{ms}-1} = \operatorname{Tr}^*_{\pi_1}(1)c_2 c_1^{2^{ms}-1} = v_s c^{2^s-1} c_2 c_1^{2^{ms}-1} = 0$$

and we obtain

$$\operatorname{Tr}^*_{\pi_1}(u)\left(x + \sum_{i=1}^{ms} v_s^{\kappa(m)+2^{ms}-2^i} c_2^{2^{(m+1)s-1}-(2^s-1)2^{i-1}}\right) = 0. \tag{9.19}$$

Now multiplying (9.10) by

$$x + \sum_{i=1}^{ms} v_s^{\kappa(m)+2^{ms}-2^i} c_2^{2^{(m+1)s-1}-(2^s-1)2^{i-1}}$$

and using (9.17) and (9.19), we obtain the dihedral and quaternion cases. For the semidihedral group, it remains to show that

$$x\left(x+\sum_{i=1}^{ms} v_s^{\kappa(m)+2^{ms}-2^i} c_2^{2^{(m+1)s-1}-(2^s-1)2^{i-1}}\right) = cx. \tag{9.20}$$

For simplicity, we introduce the notation

$$\Sigma = \sum_{i=1}^{ms} v_s^{\kappa(m)+2^{ms}-2^i} c_2^{2^{(m+1)s-1}-(2^s-1)2^{i-1}}.$$

Then

$$\Sigma(x+\Sigma) = 0$$

since (9.16) implies $\Sigma = 0$ modulo $\mathrm{Tr}^*_{\pi_1}(u)$ and

$$\Sigma^2 = v_s^{2\kappa(m)} c_2^{2^{ms}}$$

since the nilpotence degree of c_2 is $2^{(m+1)s-1} + 2^{s-1}$.

Thus,

$$x\Sigma = v_s^{2\kappa(m)} c_2^{2^{ms}}$$

and (9.20) follows from formula (9.3) of Theorem 9.1.

Now let $m = 1$. Then

$$\tilde{c}_i = c_i, \quad cc_1^* = 0, \quad xc_1^{**} = 0$$

by (9.7); $\det \eta_{\pi_2}$ is η_1 (for $G = D_8$) or the trivial bundle (for $G = Q_8$). Hence multiplying (9.11) by x, we obtain formula (9.5) of Theorem 9.1. □

It remains to show that the given relations give a ring of correct rank, which is

$$2^{(m+1)s-1} + 2^{2s} - 2^{s-1}$$

according to the generalized character theory [23]. This follows by counting the obvious explicit bases of these rings according to Theorem 9.1:

(i) for $G = D_8$ or Q_8,

$$\left\{c^i c_2^j,\ x^i c_2^j,\ c^i x c_2^j,\ c_2^k \;\middle|\; 1 \le i < 2^s,\ 0 \le j < 2^{s-1},\ 0 \le k < 2^s\right\};$$

(ii) for $m > 1$ and all three cases,

$$\left\{c^i c_2^j,\ c^i x c_2^j,\ c_2^k \;\middle|\; 1 \le i < 2^s,\ 0 \le j < 2^{s-1},\ 0 \le k < (2^{ms}+1)2^{s-1}\right\}.$$

Of course, there are alternative bases: for example, if we consider cx as the decomposable in Theorem 9.1, then for $m > 1$, the $K(s)^*$-base for $K(s)^*(BG)$ is:

(i) for $G = D_{2^{m+2}}$,

$$\left\{c^i c_2^j,\ x^i c_2^j,\ c_2^k \;\middle|\; 1 \le i < 2^s - j,\ 0 \le j < 2^s - 1,\ 0 \le k < (2^{ms}+1)2^{s-1}\right\};$$

(ii) for $G = Q_{2^{m+2}}$,

$$\left\{cc_2^i,\ xc_2^i,\ c_2^j \;\middle|\; 0 \le i < (2^s-1)2^{s-1},\ 0 \le j < (2^{ms}+1)2^{s-1}\right\};$$

(iii) and for $G = SD_{2^{m+2}}$,

$$\left\{c^i c_2^j,\ xc_2^k,\ c_2^l \;\middle|\; 1 \le i < 2^s - j,\ 0 \le j < 2^s - 1,\ 0 \le k < (2^s-1)2^{s-1},\ 0 \le l < (2^{ms}+1)2^{s-1}\right\}.$$

A natural question arises concerning the relationship between our calculations and those of [35, 36], in terms of an alternative generating set.

Example 9.5 ($K(2)^*(BD_8)$)**.** This example shows that the ring structures given in [35] must be corrected. For D_8, they are correct modulo the minimal (one-dimensional) ideal lying in the kernel of the restriction maps corresponding to all proper subgroups.

Let A be the version of $K(2)^*(BD_8)$ of Theorem 9.1 and B be its version from [35]. Then

$$A = \mathbb{F}_2[v_2^{\pm 1}][c,\ x,\ c_2]/\big(c^4,\ x^4,\ c^3c_2 + cc_2^2,\ v_2x^3c_2 + v_2xc_2^2 + cx + x^2,\ v_2^2c_2^4 + cx + x^2\big),$$
$$B = \mathbb{F}_2[v_2^{\pm 1}][y_1, y_2, \hat{c}_2]/\big(y_1^4,\ y_2^4,\ v_2^2\hat{c}_2^4 + v_2y_1\hat{c}_2^2,\ v_2y_1\hat{c}_2^2 + v_2y_2\hat{c}_2^2,\ v_2y_2\hat{c}_2^2 + y_1y_2\big).$$

Choose the following basis in A over $\mathbb{F}_2[v_2^{\pm 1}]$:

$$\Big\langle 1,\ c,\ x,\ c^2,\ cx,\ x^2,\ c_2,\ c^3,\ x^3,\ cc_2,\ xc_2,\ cx^2,\ cx^2c_2,$$
$$c^2c_2,\ x^2c_2,\ cxc_2,\ c_2^2,\ cc_2^2,\ xc_2^2,\ cxc_2^2,\ c_2^3,\ xc_2^3\Big\rangle$$

and assume that there is a graded isomorphism $f : B \to A$. Then by dimension considerations,

$$f(y_1) = \epsilon_{11}c + \epsilon_{12}x + \epsilon_{13}v_2c_2^2 + \epsilon_{14}v_2x^2c_2 + \epsilon_{15}v_2cxc_2 + \epsilon_{16}v_2c^2c_2 + \epsilon_{17}v_2^2xc_2^3,$$
$$f(y_2) = \epsilon_{21}c + \epsilon_{22}x + \epsilon_{23}v_2c_2^2 + \epsilon_{24}v_2x^2c_2 + \epsilon_{25}v_2cxc_2 + \epsilon_{26}v_2c^2c_2 + \epsilon_{27}v_2^2xc_2^3,$$
$$f(\hat{c}_2) = \alpha_1c_2 + \alpha_2c^2 + \alpha_3cx + \alpha_4x^2 + \alpha_5v_2xc_2^2 + \alpha_6v_2cc_2^2 + \alpha_7v_2cx^2c_2,$$

where ϵ_{ij}, $\alpha_k \in \mathbb{F}_2$.

Then $y_1^4 = 0$ implies

$$\Big(\epsilon_{11}c + \epsilon_{12}x + \epsilon_{13}v_2c_2^2 + \epsilon_{14}v_2x^2c_2 + \epsilon_{15}v_2cxc_2 + \epsilon_{16}v_2c^2c_2 + \epsilon_{17}v_2^2xc_2^3\Big)^4$$
$$= \epsilon_{13}c_2^8 = \epsilon_{13}x^2c_2 + \epsilon_{13}cxc_2 + \epsilon_{13}v_2xc_2^3 = 0,$$

hence $\epsilon_{13} = 0$.

Similarly, $y_2^4 = 0$ implies $\epsilon_{23} = 0$.

Next,

$$f((y_1 - y_2)\hat{c}_2^2)c_2^5 = 0$$

implies

$$(\epsilon_{12} + \epsilon_{22})\alpha_1cxc_2^2 = 0.$$

Necessarily $\alpha_1 \neq 0$, since otherwise c_2 would not be in the image of f. Thus we have

$$\epsilon_{12} = \epsilon_{22}.$$

Moreover, these are not zero as otherwise x would not be in the image of f. Thus we have

$$f(y_1) = \epsilon_{11}c + x + \epsilon_{14}v_2x^2c_2 + \epsilon_{15}v_2cxc_2 + \epsilon_{16}v_2c^2c_2 + \epsilon_{17}v_2^2xc_2^3,$$
$$f(y_2) = \epsilon_{21}c + x + \epsilon_{24}v_2x^2c_2 + \epsilon_{25}v_2cxc_2 + \epsilon_{26}v_2c^2c_2 + \epsilon_{27}v_2^2xc_2^3,$$
$$f(\hat{c}_2) = c_2 + \alpha_2c^2 + \alpha_3cx + \alpha_4x^2 + \alpha_5v_2xc_2^2 + \alpha_6v_2cc_2^2 + \alpha_7v_2cx^2c_2.$$

Taking this into account, we see that

$$f(y_1\hat{c}_2^2 - y_1y_2)c_2^2 = 0$$

implies

$$\epsilon_{11} + \epsilon_{21} + \epsilon_{24} + \epsilon_{27} = 0$$

and

$$f(y_2\hat{c}_2^2 - y_1y_2)c_2^2 = 0$$

implies

$$\epsilon_{11} + \epsilon_{21} + \epsilon_{14} + \epsilon_{17} = 0,$$

whereas

$$f(y_1y_2-\hat{c}_2^4)c_2^2=0$$

implies

$$\epsilon_{11}+\epsilon_{21}+\epsilon_{14}+\epsilon_{24}+\epsilon_{17}+\epsilon_{27}=1.$$

Hence

$$\epsilon_{11}+\epsilon_{21}=\epsilon_{14}+\epsilon_{17}=\epsilon_{24}+\epsilon_{27}=1.$$

But these relations imply that

$$(f(y_1)f(\hat{c}_2)^2-f(y_2)f(\hat{c}_2)^2)x=cxc_2^2,$$

which should actually be zero as $(y_1-y_2)\hat{c}_2^2=0$.

10. Symplectic Cobordism

Here the decompositions of the products of Ray elements and low-dimensional free generators of the symplectic cobordism ring are obtained. In particular, it is stated that most of the $4n$-dimensional generators with n small belong after multiplication on Ray elements ϕ_i, $i\ge 0$, to the ideal spanned by low-dimensional Ray elements.

Let $\nu\to B\mathbb{Z}/2$ be the canonical real line bundle and $\zeta\to \mathrm{BSp}(1)$ be the canonical symplectic line bundle. Since there is an additive isomorphism

$$\mathrm{MSp}^*(B\mathbb{Z}/2\wedge \mathrm{BSp}(1))\approx \mathrm{MSp}^*(B\mathbb{Z}/2)[x],$$

where $x=e(\zeta)$ is the Euler class of ζ, we see that the Euler class of the symplectic virtual bundle $((\nu-1)\otimes_R(\zeta-4))$ has the form

$$e((\nu-1)\otimes_R(\zeta-4))=\sum_{i\ge1}\alpha_i x^i$$

for some elements $\alpha_i\in \mathrm{MSp}^{4-4i}(B\mathbb{Z}/2)$.

For the restrictions of α_i to the symplectic cobordism ring of the n-dimensional real projective space RP^n, the notation $\theta_i(n)$ is used. These elements α_i and $\theta_i(n)$ have been studied by Buchstaber in [14].

Since $RP^1=S^1$ and $\mathrm{MSp}^{\tilde{1}}(S^1)\approx Z$, the restriction of $\theta_i(n)$ has the form $\theta_i(1)=s_1\theta_i$ for the generator $s_1\in \mathrm{MSp}^{\tilde{1}}(S^1)$ and for some coefficients $\theta_i\in \mathrm{MSp}^{3-4i}(pt)$. These elements θ_i are called Ray elements. The elements θ_1 and θ_{2i}, $i\ge1$, are indecomposable and have order 2 (see [31]) and $\theta_{2i+1}=0$ (see [34]). Let us write $\phi_0=\theta_1$ and $\phi_i=\theta_{2i}$.

Let ζ_i be the pullback bundle by the projection of $\mathrm{BSp}(1)^{\times3}$ on the ith factor.

Then

$$\zeta_1\otimes_C\zeta_2\otimes_C\zeta_3=(\zeta_1\otimes_H\zeta_2)\otimes_R\zeta_3$$

is the symplectic bundle over $\mathrm{BSp}(1)^{\times3}$.

By the calculation with the Hurevicz homomorphism, we see that in terms of the coefficients a_{klm} of the first Conner–Floyd symplectic Pontryagin class

$$pf_1(\zeta_1\otimes_C\zeta_2\otimes_C\zeta_3)=\sum_{k+l+m\ge1}a_{klm}pf_1^k(\zeta_1)pf_1^l(\zeta_2)pf_1^m(\zeta_3)$$

the structure of MSp_{4k}, $n\le4$, from [32, 37] can be interpreted as follows:

k	MSp_{4k}	Generators
1	Z	$a_{011}\cong 2x_1$
2	$Z+Z$	$a_{012}\cong 2x_2$, $a_{111}\cong x_1^2$
3	$Z+Z+Z$	$a_{022}\cong 2x_3$, $a_{011}a_{111}\cong 2x_1^3$, $a_{211}\cong x_1x_2$,
4	$\bigoplus^5 Z$	$a_{014}\cong 2x_4$, $a_{011}a_{211}\cong a_{012}a_{111}\cong 2x_1^2x_2$, $a_{122}\cong x_2^2$, $a_{111}^2\cong x_1^4$, $2y$

where $\cong$ is the equality mod $2\,\mathrm{MSp}_*$.

In the following two subsections, we calculate the Hurevicz homomorphism and the Becker–Gottlieb transfer and prove the following theorem.

Theorem 10.1. *For all $i \geq 0$ and $j = 1, 2, 3, 4$, we have*:

(a) $\phi_i(2x_j) = 0$;
(b) $\phi_i(x_1^2)$ *and* $\phi_i(x_2^2)$ *belong to the ideal* $\phi_0\,\mathrm{MSp}^*$;
(c) $\phi_i(x_1x_2)$ *belong to the ideal* $\phi_0\,\mathrm{MSp}^* + \phi_1\,\mathrm{MSp}^*$.

10.1. The Hurevicz homomorphism. Let

$$h : \pi_*(\mathrm{MSp}) \to H_*(\mathrm{MSp}) = Z[q_1, q_2, \dots]$$

be the Hurevicz homomorphism. Since $\pi_{4n}(\mathrm{MSp})$ is torsion free for small n (see [32, 37]), the Hurevicz homomorphism is monomorphism in these dimensions. So in low dimensions $4n$ the Hurevicz homomorphism determines all relations. Our aim here is to express the coefficients a_{klm} from the Introduction by the generators x's.

The Hurevicz homomorphism for these a_{klm} are calculated in [26]. In low dimensions we have

$$h(a_{100}) = h(a_{010}) = h(a_{001}) = 4, \quad h(a_{200}) = h(a_{020}) = h(a_{002}) = 0,$$
$$h(a_{110}) = h(a_{101}) = h(a_{011}) = 24q_1, \quad h(a_{111}) = 360q_2,$$
$$h(a_{210}) = \dots = h(a_{012}) = 60q_2 - 24q_1^2, \quad h(a_{300}) = \dots = h(a_{003}) = 0,$$
$$h(a_{220}) = \dots = h(a_{022}) = 280q_3 - 120q_1q_2 + 24q_1^3,$$
$$h(a_{310}) = \dots = h(a_{013}) = 112q_3 - 96q_1q_2 + 48q_1^3,$$
$$h(a_{211}) = \dots = h(a_{112}) = 1680q_3 - 360q_1q_2,$$
$$h(a_{122}) = \dots = h(a_{122}) = 75600q_4 - 3360q_1q_3 + 360q_1^2q_2,$$
$$h(a_{410}) = \dots = h(a_{140}) = 180q_4 - 360q_1q_3 + 420q_1^2q_2 - 120q_2^2 - 120q_1^4.$$

Then the Hurevicz images of generators x's from the Introduction are calculated in [33]. Namely,

$$h(2x_1) = 24q_1, \quad h(2x_2) = 20q_2 - 8q_1^2, \quad h(x_1^2) = 144q_1^2,$$
$$h(2x_3) = 56q_3 - 72q_1q_2 + 24q_1^3, \quad h(x_1x_2) = 120q_1q_2 - 48q_1^3;$$
$$h(x_1^3) = 3456q_1^3, \quad h(2x_4) = 12q_4 - 24q_1q_3 - 8q_2^2 + 28q_1^2q_2 - 8q_1^4,$$
$$\frac{1}{2}(x_2^2 + x_1x_3) = 50q_2^2 + 168q_1q_3 - 256q_1^2q_2 + 80q_1^4,$$
$$h(x_2^2) = 100q_2^2 - 80q_1^2q_2 + 16q_1^4, \quad h(2x_1^2x_2) = 2880q_1^2q_2 - 1152q_1^4, \quad h(x_1^4) = 20736q_1^4.$$

We are interested of mod 2 nonzero coefficients a_{011}, a_{111}, a_{022}, a_{122}, a_{112}, a_{120}, and a_{140}.

By the above equalities we have

$$a_{011} = 2x_1, \quad a_{111} = 18(2x_2) + x_1^2, \quad a_{120} = 3(2x_2),$$
$$a_{022} = 5(2x_3) + 2(x_1x_2), \quad a_{211} = 30(2x_3) + 15(x_1x_2),$$
$$a_{122} = 1050(2x_4) + 130\left(\frac{1}{2}\left(\frac{1}{4}x_1x_3 + \frac{1}{4}x_2^2\right)\right) + 19(x_2^2) + 2(2x_1^2), \quad a_{140} = 15(2x_4).$$

So we have the following assertion.

Proposition 10.2. *The following relations hold* $\bmod 2\,\mathrm{MSp}_*$:

$$a_{011} = 2x_1, \quad a_{111} = x_1^2, \quad a_{022} = 2x_3, \quad a_{122} = x_2^2,$$

$$a_{112} = x_1 x_2, \quad a_{120} = 2x_2, \quad a_{140} = 2x_4.$$

10.2. Calculations with transfer. Let ξ be a universal $U(1)$ bundle and Λ be a universal Spin(3) bundle. Then the bundle $\pi : BU(1) \to \mathrm{BSp}(1)$ is the sphere bundle of Λ and

$$\pi^*(\zeta) = \xi + \bar{\xi}, \quad \pi^*(\Lambda) = \xi^2 + R, \quad \zeta \otimes_H \zeta = \Lambda + R,$$

where ζ is a universal Sp(1) bundle as above. Let N be the normalizer of the torus $U(1)$ in Sp(1). The classifying space BN coincides with the orbit space of complex projective space $CP(\infty)$ under free involution I, which acts by

$$I : [z_0, z_1, \dots] \to [-\bar{z}_1, \bar{z}_0]$$

on homogeneous coordinates.

The bundle $p : BN \to \mathrm{BSp}(1)$ coincides with the projective bundle of Λ, hence we have the canonical splitting

$$p^*(\Lambda) = \mu + \nu,$$

where μ and ν are a plane and linear real bundles. Of course, for the double covering $q : BU(1) \to BN$ we have $q^*(\mu) = \xi^2$ and $q^*(\nu) = R$.

Let $\tau(\pi)$ and $\tau(p)$ be the transfer maps of the bundles π and p. The following lemma follows from [18].

Lemma 10.3.

$$\pi^* \tau(\pi)^* = 1 + I^*, \quad \pi^*(p)^* = q^*.$$

The following lemma follows from the definitions.

Lemma 10.4. *For the Atiyah transfer* ! *of the double covering*

$$1 \times q : BU(1) \times BU(1) \to BU(1) \times BN$$

we have

$$\left(\xi_1 \xi_2^2 + \bar{\xi_1} \bar{\xi}_2^2\right)_! = (\xi_1 + \bar{\xi_1}) \otimes_R \mu.$$

Let f be the map $f : BN \to BZ_2$ induced by projection of N on the Weyl group Z_2 and let $\tau^*(1 \times q)$ be the Becker–Gottlieb transfer homomorphism for the above double covering $1 \times q$.

Lemma 10.5. *For some elements γ_i from $\mathrm{MSp}^*(RP(\infty))$, the following formula holds*:

$$\tau^*(1 \times q)\Big(pf_1\big(\xi_1 \xi_2^2 + \bar{\xi}_1 \bar{\xi}_2^2\big)\Big) = pf_1\big(r\xi_1 \otimes_R p^*(\zeta_2)\big) + \sum_{i \geq 0} f^*(\gamma_i) pf_2^i\big(r\xi_1 \otimes_R \zeta_2\big).$$

Proof. Taking into account Lemma 3.2, we show that the proof follows from the following formula [34]. Let Q be the double covering $Q : X \to B$, $\eta \to X$ be the symplectic line bundle, $\eta_! \to B$ be the Atiyah transfer image of η, $\tau(Q)$ be the Becker–Gottlieb transfer map for Q, and $F : X \to RP(\infty)$ be the classifying map of the real line bundle associated with the double covering Q. Then for some elements γ_i from $\mathrm{MSp}^*(RP(\infty))$, the following formula holds:

$$\tau(Q)^*(pf_1(\eta)) = pf_1(\eta_!) + \sum_{i \geq 0} f^*(\gamma_i) pf_2^i(\eta_!).$$

The lemma is proved. □

Lemma 10.6.

$$\phi_j \tau^*(\pi)(a) = 0 \quad \forall a \in \mathrm{MSp}^*(BU(1)).$$

Proof. Let $\delta(\pi)$ be the Boardman map [11]. Then we know from [9] that

$$\tau^*(\pi)(a) = \delta(\pi)(ae(\xi_2^2)),$$

where $e(\xi_2^2)$ is the Euler class of the bundle ξ_2^2, which is the bundle of tangents along the fibers. Then from [19, 20] we have

$$\theta_j e(\xi_2^2) = 0;$$

this proves Lemma 10.6. □

Recall from [20, 29] that the bundle Λ is MSp-orientable and the corresponding Euler class has the form

$$e(\Lambda) = \sum_{j\geq 1} \theta_j pf_1(\zeta_1)^j.$$

We denote the restrictions of π and p to symplectic projective space $HP(4)$ by the same symbols. The total spaces of these bundles coincides with complex projective space $CP(9)$ and orbit space $CP(9)/I$ under free involution I which acts by

$$[z_0, z_1, \dots, z_8, z_9] \to [-\bar{z}_1, \bar{z}_0, \dots, -\bar{z}_9, \bar{z}_8]$$

in homogeneous coordinates.

Proposition 10.7. *For the bundle* $1\times\pi : BU(1)\times BU(1) \to BU(1)\times \mathrm{BSp}(1)$*, we have*

$$\phi_j\tau^*(1\times\pi)(pf_i(r\xi_1 \otimes_R \zeta_2) = 0$$

for all $j \geq 0$ *and* $i = 1, 2$.

Proof. In $\mathrm{MSp}^*(BU(1)\times \mathrm{BSp}(1)) = \mathrm{MSp}^*(BU(1))[[pf_1(\zeta_2)]]$ we have

$$pf_i(r\xi_1 \otimes_R \zeta_2) = \sum_{k\geq 0} \omega_k^{(i)} pf_1^k(\zeta_2).$$

Then

$$\phi_j\tau^*\left(\sum_{k\geq 0} \omega_k^{(i)} pf_1^k(\zeta_2)\right) = \sum_{k\geq 0} \phi_k\tau^*(\omega_k^{(i)} pf_1^k(\zeta_2)) = 0.$$

The last equation follows from Lemma 10.6. □

Proposition 10.8. *In* $\mathrm{MSp}^*(HP(4)\times HP(4))$*, the following relations hold for all* $j \geq 0$*:*

(a) $\phi_0 pf_1(\zeta_1) + \phi_1 pf_1^2(\zeta_1) + \phi_2 pf_1^4(\zeta_1)$ *divides* $\phi_j pf_1(\zeta_1 \otimes_C \zeta_2^2)$;
(b) $\phi_j pf_1(\zeta_1 \otimes_R \zeta_2) = 0$.

Proof. (a) The bundle

$$CP(9)\times HP(4) \to HP(4)\times HP(4)$$

coincides with the sphere bundle of the pullback of

$$\Lambda \to HP(4)$$

by the projection on the first factor

$$HP(4)\times HP(4) \to HP(4).$$

Thus, we must prove that $\phi_j pf_1(\zeta_1 \otimes_C \zeta_2^2)$ goes to zero in $\mathrm{MSp}^*(CP(9)\times HP(4))$ by the homomorphism $\pi\times 1$.

The transfer

$$\tau = \tau(1\times\pi)$$

of the bundle

$$CP(9) \times CP(9) \to CP(9) \times HP(4)$$

is the composition of two transfers

$$\tau_1 = \tau(1 \times q), \quad \tau_2 = \tau(1 \times p)$$

corresponding to

$$CP(9) \times CP(9) \to CP(9) \times CP(9)/I$$

and

$$CP(9) \times CP(9)/I \to CP(9) \times HP(4),$$

respectively.

By the definitions, we have

$$pf_1\Big((\xi_1 + \bar{\xi_1}) \otimes_C \zeta_2^2\Big) = pf_1\Big((\xi_1 + \bar{\xi_1}) \otimes_R \Lambda + R\Big) = pf_1\Big((\xi_1 + \bar{\xi_1}) \otimes_R \Lambda\Big) + pf_1\Big(\xi_1 + \bar{\xi_1}\Big).$$

Applying Lemmas 10.4 and 10.5, we have

$$\tau_1^*\Big(pf_1\Big(\xi_1\xi_2^2 + \bar{\xi_1}\bar{\xi_2^2}\Big)\Big) = pf_1\Big((\xi_1 + \bar{\xi_1}) \otimes_R \mu\Big) + \sum_{i \geq 0} f^*(\gamma_i) pf_2^i\Big((\xi_1 + \bar{\xi_1}) \otimes_R \mu\Big).$$

Then by definitions

$$pf_1\Big((\xi_1 + \bar{\xi_1}) \otimes_R \mu\Big) = (1 \times p)^* pf_1\Big((\xi_1 + \bar{\xi_1}) \otimes_R \Lambda\Big) - pf_1\Big((\xi_1 + \bar{\xi_1}) \otimes_R \nu\Big),$$

$$\tau^*\Big(pf_1\Big(\xi_1\xi_2^2 + \bar{\xi_1}\bar{\xi_2^2}\Big)\Big) = \tau_2^*\Big((pf_1(\xi_1 + \bar{\xi_1}) \otimes_R \mu)\Big) + \tau_2^*\left(\sum_{i \geq 0} f^*(\gamma_i) pf_2^i\Big((\xi_1 + \bar{\xi_1}) \otimes_R \mu\Big)\right),$$

and

$$\begin{aligned}\tau_2^*\Big(\Big(pf_1(\xi_1 + \bar{\xi_1}) \otimes_R \mu\Big)\Big) &= \tau_2^*(1 \times p)^*\Big(pf_1\Big((\xi_1 + \bar{\xi_1}) \otimes_R \Lambda\Big)\Big) - \tau_2^*\Big(pf_1(\xi_1 + \bar{\xi_1}) \otimes_R \nu\Big) \\ &= pf_1\Big((\xi_1 + \bar{\xi_1}) \otimes_R \Lambda\Big)\tau_2^*(1 \times p^*)(1) - \tau_2^*\Big(pf_1(\xi_1 + \bar{\xi_1}) \otimes_R \nu\Big).\end{aligned}$$

Now we must prove that

$$\tau_2^*(1 \times p^*)(1) = 1, \quad \tau_2^*\Big(pf_1(\xi_1 + \bar{\xi_1}) \otimes_R \nu\Big) = pf_1(\zeta_1),$$

$$\tau_2^*\left(\sum_{i \geq 0} f^*(\gamma_i) pf_2^i\Big(\xi_1 + \bar{\xi_1}) \otimes_R \mu\right) = 0.$$

Then since $(\xi_1 + \bar{\xi_1}) \otimes_R \nu)$ is the pullback of the bundle

$$\zeta \times \eta \to \mathrm{BSp}(1) \times BZ_2$$

by the map $\pi \times f$, we have

$$pf_1\Big((\xi_1 + \bar{\xi_1}) \otimes_R \nu\Big) = pf_1\Big(\xi_1 + \bar{\xi_1}\Big) + \sum_{i \geq 0} f^*(\delta_i) pf_1^i(\xi_1 + \bar{\xi_1}).$$

Similarly,

$$(\xi_1 + \bar{\xi_1}) \otimes_R \mu$$

is the pullback of the bundle

$$\zeta \times \eta(2) \to \mathrm{BSp}(1) \times BO(2)$$

and hence

$$pf_2\Big((\xi_1+\bar{\xi_1})\otimes_R\mu\Big)=pf_2(2\mu)+\sum_{i\geq 0}\sigma_i pf_1^i(\xi_1+\bar{\xi_1}).$$

So we have

$$\begin{aligned}\tau^*\Big(pf_1\Big(\xi_1\xi_2^2+\bar{\xi_1}\bar{\xi_2^2}\Big)\Big)&=pf_1\Big((\xi_1+\bar{\xi_1})\otimes_R\Lambda\Big)\tau_2^*(1\times p^*)(1)\\&\quad-\tau_2^*(pf_1(\zeta_1))+\tau_2^*\Big(pf_1(\xi_1+\bar{\xi_1})\otimes_R\nu\Big)+\tau_2^*\left(\sum_{i\geq 0}(\alpha_i pf_1(\zeta_1))\right)\end{aligned}$$

for some $\alpha_i\in \mathrm{MSp}^*(CP(9)/I)$.

It is known from [37] that up to dimension 32, MSp^{4n} is torsion-free. Therefore, $\mathrm{MSp}^{4n}(HP(4))$ is torsion-free for $4n\geq -12$. Now Lemma 10.3 asserts that for any element a from

$$\ker q^*\,\mathrm{MSp}^{4n}(CP(9)/I)\to \mathrm{MSp}^{4n}(CP(9)),\quad 4n\geq -12,$$

which comes from $\mathrm{MSp}^*(BN)$, we have

$$\tau^*(p)(a)=0.$$

Also

$$\tau^*(p)(1)=1,\quad \tau^*(\pi)(1)=2.$$

But the minimal degree of the elements α_i is -12. This proves Proposition 10.8(a).

The proof of (b) follows from Lemma 10.3. For the bundle $\pi\times 1$ from Proposition 10.7 we have

$$\begin{aligned}(\pi\times 1)^*\tau^*(\pi\times 1)\big(pf_1(r\xi_1\otimes\zeta_2)\big)&=(1+I)^*\Big(pf_1\Big((\xi_1+\bar{\xi_1})\otimes_C\zeta_2\Big)\Big)\\&=2pf_1\Big((\xi_1+\bar{\xi_1})\otimes_C\zeta_2\Big)=(\pi\times 1)^*(\zeta_1\otimes_R\zeta_2).\end{aligned}$$

The proof now follows from Proposition 10.7. □

Proof of Theorem 10.1. From Propositions 10.2 and 10.8(a) we have in $\mathrm{MSp}^*(HP(4)\times HP(4)$ the relation of the form

$$\theta_j\Big((2x_1)v^2+(x_1^2)uv^2+(2x_3)v^4+(x_1x_2)u^2v^2+(x_2^2)uv^4+\cdots\Big)=\sum_{i\geq 1}(\theta_i v^i)\left(\sum_{k,l\geq 0}b_{kl}^{(j)}u^kv^l\right)$$

for $u=pf_1(\zeta_1)$, $v=pf_1(\zeta_2)$, $j\geq 1$, and some elements

$$b^{(j)}=\sum_{k,l\geq 0}b_{kl}^{(j)}u^kv^l)\in \mathrm{MSp}*(\mathrm{BSp}(1)^2).$$

Since $\theta_{2i+1}=0$ (see [34]) by the equality of the coefficients at the monomials uv^2, u^2v^2, and uv^4, we obtain (b) and (c).

Similarly from Propositions 10.2 and 10.8(b) we have

$$\theta_j(a_{110})v^2+a_{120}uv^2+a_{220}u^2v^2+a_{140}uv^4+\dots)=0,$$

i.e.,

$$\theta_j((2x_1)v^2+(2x_2)uv^2+(2x_3)v^4+(2x_4)uv^4+\dots)=0,$$

and we have (a). □

10.3. On symplectic cobordisms of real projective plane. Let r be the generator of $\mathrm{MSp}^2(RP^2) = \mathbb{Z}/2$. Recall two formulas in $\mathrm{MSp}^*(RP^2)$ from [20]:

$$2\theta_i(2) = \theta_1\theta_i r$$

and

$$\theta_i(2)\theta_i(2) = \theta_i\theta_j r.$$

As above, let $\phi_0 = \theta_1$ and $\phi_i = \theta_{2i}$.

Theorem 10.9. *In* $\mathrm{MSp}^*(RP^2)$*, the following relations hold*:

$$\theta_{8n+7}(2) = 0, \quad \theta_{8n+3}(2) = \phi_{2n+1}^2 r, \quad \theta_{4k+1}(2) = \sum_{i=0}^{k-1} \phi_{2i+1}\phi_{2(k-i)} r$$

for $n \geq 0$ *and* $k \geq 1$.

Proof. We use the Becker–Gottlieb transfer map. Namely, we need two formulas. First, as was proved by Buchstaber [20] for the transfer map $\tau(\pi)$ of the double covering $\pi : S^\infty \to RP^\infty$, we have

$$\tau^*(\pi)(1) = 2 + \sum_{k=1}^{\infty} a_k y^{k-1}, \tag{10.1}$$

where y is the Euler class of the bundle ν from the definition of the elements $\theta_i(n)$. So for the covering $\pi : S^2 \to RP^2$, this implies

$$\tau^*(\pi)(1) = 2 + \theta_1(2). \tag{10.2}$$

The second formula we need is the following [34].

Let p be a double covering $p : X \to B$, η be the symplectic line bundle $\eta \to X$, η_p be the transferred η, $\tau(p)$ be the transfer map of p, and $f : B \to RP^\infty$ be the classifying map of the real bundle ν over B associated with the covering p. Then for some elements $\gamma_i \in \mathrm{MSp}^{4-8i}(RP^\infty)$ the following formula holds in $\mathrm{MSp}^*(B)$:

$$\tau(p)^*(P_1(\eta)) = P_1(\eta_p) + \sum_{i\geq 0} f^*(\gamma_i) P_2(\eta_p), \tag{10.3}$$

where P_i are symplectic Pontryagin classes. Applying (10.3) to the transfer map $\tau = \tau(\pi \times 1)$ for the double covering

$$\pi \times 1 : S^2 \times HP^\infty \to RP^2 \times HP^\infty$$

and taking into account the fact that for the transferred bundle $\zeta_p = \zeta + \nu \otimes_R \zeta$ we have

$$P_1(\zeta_p) = x + x + \sum_{i\geq 1} \theta_i(2)x^i, \quad p_2(\zeta_p) = x(x + \sum_{i\geq 1} \theta_i(2)x^i),$$

we obtain

$$\tau^*(x) = x + x + \sum_{i\geq 1} \theta_i(2)x^i + \sum_{j\geq 1} f^*(\gamma_i) \left(1 + \sum_{i\geq 1} \theta_i(2)x^{i-1}\right)^j x^{2j}.$$

On the other hand, by (10.1) we have

$$\tau^*(x) = (2 + \theta_i(2)))x.$$

We obtain

$$\sum_{i\geq 2} \theta_i(2)x^i = -\sum_{j\geq 1} f^*(\gamma_i) \left(1 + \sum_{i\geq 1} \theta_i(2)x^{i-1}\right)^j x^{2j}. \tag{10.4}$$

The theorem is proved. $\square$

The diagonal of $RP^2 \wedge RP^2$ coincides with $RP^1 \wedge RP^1$, i.e., with S^2, and the diagonal map $RP^2 \to RP^2 \wedge RP^2$ factors as composition of the projection $RP^2 \to S^2$ onto top cell with the inclusion of the bottom cell. Then the triple diagonal map $RP^2 \to RP^2 \wedge RP^2 \wedge RP^2$ is null-homotopic. This means that for any $\alpha, \beta \in \mathrm{MSp}^*(RP^2)$, we have $\alpha\beta = \alpha_1\beta_1 r$, where $s_1\alpha_1$ and $s_1\beta_1$ are restrictions of α and β to $RP^1 = S^1$. In particular, all triple products in $\mathrm{MSp}^*(RP^2)$ are zero.

After remarks on double and triple products, the proof is completed by (10.3) and induction on i.

11. Some Examples and Tables

First, recall from [31] that generators for

$$\begin{array}{ccc} \pi_* BP & \subset & H_* BP \\ \updownarrow & & \updownarrow \\ Z_{(p)}[v_1, v_2, \dots] & \subset & Z_{(p)}[m_1, m_2, \dots] \end{array}$$

where

$$|v_n| = 2(p^n - 1) = |m_n|,$$

are given by

$$v_n = pm_{n-} \sum_{i=1}^{n-1} m_i v_{n-i}^{p^i}.$$

We use the above formula with the following well-known formulas

$$F(x,y) = \exp(\log x + \log y), \quad \log x = \sum_{n \geq 0} m_n x^{n+1}$$

for calculating $\alpha_{ij} = \alpha_{ij}^{BP}$.

11.1. Coefficients of formal group law in BP theory.

$$\begin{aligned}
\alpha_{11} &= -v_1, \\
\alpha_{12} &= v_1^2, \\
\alpha_{13} &= -2v_1^3 - 2v_2, \\
\alpha_{22} &= -4v_1^3 - 3v_2, \\
\alpha_{14} &= 3v_1^4 + 4v_1v_2, \\
\alpha_{23} &= 10v_1^4 + 11v_1v_2, \\
\alpha_{15} &= -4v_1^5 + 6v_1^2v_2, \\
\alpha_{24} &= -21v_1^5 - 28v_1^2v_2, \\
\alpha_{33} &= -34v_1^5 - 43v_1^2v_2, \\
\alpha_{16} &= 6v_1^6 + 12v_1^3v_2 + 4v_2^2, \\
\alpha_{25} &= 43v_1^6 + 75v_1^3v_2 + 18v_2^2, \\
\alpha_{34} &= 101v_1^6 + 164v_1^3v_2 + 34v_2^2, \\
\alpha_{17} &= -10v_1^7 - 24v_1^4v_2 - 14v_1v_2^2 - 4v_3, \\
\alpha_{26} &= -88v_1^7 - 190v_1^4v_2 - 89v_1v_2^2 - 14v_3, \\
\alpha_{35} &= -275v_1^7 - 551v_1^4v_2 - 226v_1v_2^2 - 28v_3,
\end{aligned}$$

$$\alpha_{44} = -394v_1^7 - 765v_1^4v_2 - 302v_1v_2^2 - 35v_3,$$
$$\alpha_{18} = 15v_1^8 + 40v_1^5v_2 + 28v_1^2v_2^2 + 8v_1v_3,$$
$$\alpha_{27} = 169v_1^8 + 420v_1^5v_2 + 257v_1^2v_2^2 + 46v_1v_3,$$
$$\alpha_{36} = 680v_1^8 + 1586v_1^5v_2 + 879v_1^2v_2^2 + 126v_1v_3,$$
$$\alpha_{45} = 1303v_1^8 + 2933v_1^5v_2 + 1543v_1^2v_2^2 + 203v_1v_3,$$
$$\alpha_{19} = -22v_1^9 - 66v_1^6v_2 - 58v_1^3v_2^2 - 12v_1^2v_3 - v_2^3,$$
$$\alpha_{28} = -312v_1^9 - 880v_1^6v_2 - 688v_1^3v_2^2 - 104v_1^2v_3 - 72v_2^3,$$
$$\alpha_{37} = -1573v_1^9 - 4192v_1^6v_2 - 3001v_1^3v_2^2 - 382v_1^2v_3 - 260v_2^3,$$
$$\alpha_{46} = -3861v_1^9 - 9900v_1^6v_2 - 6707v_1^3v_2^2 - 791v_1^2v_3 - 523v_2^3,$$
$$\alpha_{55} = -5156v_1^9 - 13042v_1^6v_2 - 8671v_1^3v_2^2 - 1001v_1^2v_3 - 654v_2^3.$$

11.2. Example $n = 2$. Let
$$\alpha_{ij} = \alpha_{ij}^{G(2)}.$$
Then using the map
$$BP \to G(n),$$
where
$$BP^* = Z_{(p)}[v_1, v_2, \dots], \quad G(n)^* = Z_{(p)}[v_n.v_n^{-1}],$$
which satisfies $v_r \to v_r$ if $r = n$ and $v_r \to 0$ otherwise, we have
$$\alpha_{13} = -2v_2, \quad \alpha_{22} = -3v_2, \quad \alpha_{16} = 4v_2^2, \quad \alpha_{25} = 18v_2^2, \quad \alpha_{34} = 34v_2^2,$$
and otherwise $\alpha_{ij} = 0$ for $i + j = 8$.

11.3. Initial segments of the formal group law in Morava theory $K(s)^*$. For $p = 2$ and $s = 2$:

$$\begin{aligned}
&x + y + v_2x^2y^2 \\
&\quad + {v_2}^3\left(x^6y^4 + x^4y^6\right) \\
&\quad + {v_2}^5\left(y^{12}x^4 + y^4x^{12}\right) \\
&\quad + {v_2}^7\left(y^{10}x^{12} + y^{12}x^{10} + y^{14}x^8 + y^8x^{14}\right) \\
&\quad + {v_2}^9\left(y^{20}x^8 + y^8x^{20}\right) \\
&\quad + {v_2}^{11}\left(y^8x^{26} + y^{10}x^{24} + y^{24}x^{10} + y^{26}x^8\right) \\
&\quad + {v_2}^{13}\left(y^{12}x^{28} + y^{28}x^{12}\right) \\
&\quad + {v_2}^{15}\left(y^{30}x^{16} + y^{16}x^{30} + y^{18}x^{28} + y^{20}x^{26} + y^{22}x^{24} + y^{24}x^{22} + y^{26}x^{20} + y^{28}x^{18}\right) \\
&\quad + {v_2}^{17}\left(y^{20}x^{32} + y^{32}x^{20}\right) \\
&\quad + {v_2}^{19}\left(y^{16}x^{42} + y^{18}x^{40} + y^{40}x^{18} + y^{42}x^{16}\right) \\
&\quad + {v_2}^{21}\left(y^8x^{56} + y^{20}x^{44} + y^{40}x^{24} + y^{56}x^8 + y^{24}x^{40} + y^{44}x^{20}\right) \\
&\quad + {v_2}^{23}\left(y^{16}x^{54} + y^{18}x^{52} + y^{20}x^{50} + y^{22}x^{48} + y^{48}x^{22} + y^{50}x^{20} + y^{52}x^{18} + y^{54}x^{16}\right) \\
&\quad + {v_2}^{25}\left(y^{28}x^{48} + y^{16}x^{60} + y^{48}x^{28} + y^{60}x^{16}\right) \\
&\quad + {v_2}^{27}\left(y^{24}x^{58} + y^{26}x^{56} + y^{56}x^{26} + y^{58}x^{24}\right) \\
&\quad + {v_2}^{29}\left(y^{16}x^{72} + y^{28}x^{60} + y^{32}x^{56} + y^{40}x^{48} + y^{48}x^{40} + y^{60}x^{28} + y^{56}x^{32} + y^{72}x^{16}\right)
\end{aligned}$$

$$+ {v_2}^{31}\left(y^{32}x^{62} + y^{34}x^{60} + y^{38}x^{56} + y^{36}x^{58} + y^{40}x^{54} + y^{42}x^{52} + y^{44}x^{50} + y^{48}x^{46}\right.$$
$$\left. + y^{46}x^{48} + y^{50}x^{44} + y^{52}x^{42} + y^{54}x^{40} + y^{56}x^{38} + y^{58}x^{36} + y^{60}x^{34} + y^{62}x^{32}\right) + \dots.$$

For $p = 2$ and $s = 3$:

$$\begin{aligned} & x + y + v_3x^4y^4 \\ & \quad + {v_3}^5\left(y^{20}x^{16} + y^{16}x^{20}\right) \\ & \quad + {v_3}^9\left(y^{16}x^{48} + y^{48}x^{16}\right) \\ & \quad + {v_3}^{21}\left(y^{80}x^{68} + y^{84}x^{64} + y^{64}x^{84} + y^{68}x^{80}\right) \\ & \quad + {v_3}^{25}\left(y^{112}x^{64} + y^{64}x^{112}\right) + \dots. \end{aligned}$$

For $p = 2$ and $s = 4$:

$$\begin{aligned} & x + y + v_4y^8x^8 \\ & \quad + {v_4}^9\left(y^{72}x^{64} + y^{64}x^{72}\right) \\ & \quad + {v_4}^{17}\left(y^{64}x^{192} + y^{192}x^{64}\right) \\ & \quad + {v_4}^{73}\left(y^{520}x^{576} + y^{512}x^{584} + y^{584}x^{512} + y^{576}x^{520}\right) + \dots. \end{aligned}$$

For $p = 2$ and $s = 5$:

$$\begin{aligned} & x + y + v_5y^{16}x^{16} \\ & \quad + {v_5}^{17}\left(y^{272}x^{256} + y^{256}x^{272}\right) \\ & \quad + {v_5}^{33}\left(y^{768}x^{256} + y^{256}x^{768}\right) + \dots. \end{aligned}$$

For $p = 3$ and $s = 2$:

$$\begin{aligned} & x + y + v_2\left(2\,x^6y^3 + 2\,x^3y^6\right) \\ & \quad + {v_2}^4\left(2\,y^{12}x^{21} + y^9x^{24} + y^{15}x^{18} + y^{18}x^{15} + y^{24}x^9 + 2\,y^{21}x^{12}\right) \\ & \quad + {v_2}^7\left(2\,y^{21}x^{36} + 2\,y^{18}x^{39} + 2\,y^{36}x^{21} + y^{30}x^{27} + y^{27}x^{30} + 2\,y^{39}x^{18}\right) \\ & \quad + {v_2}^{10}\left(2\,y^{45}x^{36} + 2\,y^{36}x^{45} + 2\,y^{18}x^{63} + 2\,y^9x^{72} + 2\,y^{72}x^9 + 2\,y^{63}x^{18}\right) \\ & \quad + {v_2}^{13}\left(y^{48}x^{57} + 2\,y^{27}x^{78} + y^{57}x^{48} + 2\,y^{33}x^{72} + y^{30}x^{75} + 2\,y^{60}x^{45} + y^{36}x^{69}\right. \\ & \qquad + 2\,y^{39}x^{66} + 2\,y^{66}x^{39} + 2\,y^{45}x^{60} + y^{42}x^{63} + 2\,y^{51}x^{54} + 2\,y^{54}x^{51} + y^{75}x^{30} \\ & \qquad \left. + \; 2\,y^{78}x^{27} + y^{63}x^{42} + y^{69}x^{36} + 2\,y^{72}x^{33}\right) \\ & \quad + {v_2}^{16}\left(2\,y^{90}x^{39} + 2\,y^{93}x^{36} + 2\,y^{39}x^{90} + 2\,y^{36}x^{93} + y^{63}x^{66} + y^{66}x^{63}\right) \\ & \quad + {v_2}^{19}\left(2\,y^{36}x^{117} + y^{27}x^{126} + y^{81}x^{72} + y^{72}x^{81} + 2\,y^{54}x^{99} + 2\,y^{99}x^{54} + y^{126}x^{27} + 2\,y^{117}x^{36}\right) \\ & \quad + {v_2}^{22}\left(y^{69}x^{108} + y^{63}x^{114} + 2\,y^{57}x^{120} + y^{60}x^{117} + 2\,y^{66}x^{111} + y^{54}x^{123} + y^{123}x^{54}\right. \\ & \qquad + 2\,y^{81}x^{96} + y^{108}x^{69} + 2\,y^{96}x^{81} + 2\,y^{111}x^{66} + y^{114}x^{63} + y^{117}x^{60} + 2\,y^{120}x^{57} \\ & \qquad \left. + \; y^{93}x^{84} + y^{84}x^{93} + 2\,y^{87}x^{90} + 2\,y^{90}x^{87}\right) \\ & \quad + {v_2}^{25}\left(2\,y^{57}x^{144} + 2\,y^{54}x^{147} + 2\,y^{75}x^{126} + 2\,y^{72}x^{129} + y^{84}x^{117} + y^{81}x^{120}\right. \\ & \qquad + y^{99}x^{102} + 2\,y^{93}x^{108} + 2\,y^{90}x^{111} + y^{102}x^{99} + 2\,y^{111}x^{90} + 2\,y^{108}x^{93} + y^{117}x^{84} \\ & \qquad \left. + \; y^{120}x^{81} + 2\,y^{129}x^{72} + 2\,y^{126}x^{75} + 2\,y^{144}x^{57} + 2\,y^{147}x^{54}\right) \\ & \quad + {v_2}^{28}\left(2\,y^{54}x^{171} + 2\,y^{90}x^{135} + 2\,y^{135}x^{90} + 2\,y^{171}x^{54}\right) + \dots. \end{aligned}$$

For $p=3$ and $s=3$:

$$
\begin{aligned}
&x+y+v_3\left(2\,x^9y^{18}+2\,x^{18}y^9\right)\\
&\quad+{v_3}^{10}\left(y^{81}x^{180}+2\,y^{90}x^{171}+y^{99}x^{162}+y^{162}x^{99}+2\,y^{171}x^{90}+y^{180}x^{81}\right)\\
&\quad+{v_3}^{19}\left(2\,y^{171}x^{324}+2\,y^{162}x^{333}+y^{243}x^{252}+2\,y^{333}x^{162}+y^{252}x^{243}+2\,y^{324}x^{171}\right)\\
&\quad+{v_3}^{28}\left(2\,y^{567}x^{162}+2\,y^{162}x^{567}+2\,y^{648}x^{81}+2\,y^{81}x^{648}+2\,y^{324}x^{405}+2\,y^{405}x^{324}\right)+\dots.
\end{aligned}
$$

For $p=7$ and $s=2$:

$$
\begin{aligned}
&x+y+v_2\left(4\,y^{35}x^{14}+6\,y^{42}x^{7}+4\,y^{14}x^{35}+2\,y^{28}x^{21}+2\,y^{21}x^{28}+6\,y^{7}x^{42}\right)\\
&\quad+{v_2}^{8}\big(4\,y^{105}x^{280}+3\,y^{98}x^{287}+3\,y^{126}x^{259}+4\,y^{119}x^{266}+2\,y^{154}x^{231}+3\,y^{140}x^{245}\\
&\qquad+5\,y^{147}x^{238}+4\,y^{133}x^{252}+y^{91}x^{294}+2\,y^{182}x^{203}+5\,y^{175}x^{210}+2\,y^{168}x^{217}\\
&\qquad+5\,y^{161}x^{224}+2\,y^{217}x^{168}+5\,y^{210}x^{175}+2\,y^{203}x^{182}+5\,y^{196}x^{189}+5\,y^{189}x^{196}\\
&\qquad+4\,y^{252}x^{133}+3\,y^{245}x^{140}+5\,y^{238}x^{147}+2\,y^{231}x^{154}+5\,y^{224}x^{161}\\
&\qquad+3\,y^{273}x^{112}+4\,y^{266}x^{119}+3\,y^{259}x^{126}+6\,y^{315}x^{70}+y^{308}x^{77}+6\,y^{301}x^{84}\\
&\qquad+y^{294}x^{91}+3\,y^{287}x^{98}+4\,y^{280}x^{105}+y^{336}x^{49}+6\,y^{329}x^{56}+y^{322}x^{63}\\
&\qquad+y^{49}x^{336}+y^{63}x^{322}+6\,y^{56}x^{329}+6\,y^{70}x^{315}+y^{77}x^{308}+6\,y^{84}x^{301}+3\,y^{112}x^{273}\big)\\
&\quad+{v_2}^{15}\big(4\,y^{413}x^{308}+6\,y^{427}x^{294}+2\,y^{420}x^{301}+2\,y^{448}x^{273}+6\,y^{441}x^{280}+4\,y^{98}x^{623}\\
&\qquad+4\,y^{462}x^{259}+4\,y^{455}x^{266}+4\,y^{308}x^{413}+6\,y^{490}x^{231}+5\,y^{119}x^{602}+6\,y^{476}x^{245}\\
&\qquad+4\,y^{315}x^{406}+6\,y^{126}x^{595}+2\,y^{469}x^{252}+2\,y^{322}x^{399}+2\,y^{518}x^{203}+4\,y^{511}x^{210}\\
&\qquad+4\,y^{504}x^{217}+2\,y^{497}x^{224}+2\,y^{553}x^{168}+y^{546}x^{175}+3\,y^{539}x^{182}+6\,y^{525}x^{196}\\
&\qquad+5\,y^{609}x^{112}+5\,y^{602}x^{119}+6\,y^{595}x^{126}+4\,y^{588}x^{133}+3\,y^{574}x^{147}+4\,y^{133}x^{588}\\
&\qquad+y^{567}x^{154}+2\,y^{560}x^{161}+3\,y^{147}x^{574}+4\,y^{623}x^{98}+6\,y^{616}x^{105}\\
&\qquad+6\,y^{329}x^{392}+y^{154}x^{567}+2\,y^{168}x^{553}+2\,y^{161}x^{560}+3\,y^{182}x^{539}+y^{175}x^{546}\\
&\qquad+6\,y^{196}x^{525}+4\,y^{210}x^{511}+2\,y^{203}x^{518}+4\,y^{217}x^{504}+6\,y^{231}x^{490}+2\,y^{224}x^{497}\\
&\qquad+6\,y^{392}x^{329}+2\,y^{252}x^{469}+4\,y^{259}x^{462}+6\,y^{245}x^{476}+4\,y^{406}x^{315}+2\,y^{399}x^{322}\\
&\qquad+2\,y^{273}x^{448}+2\,y^{301}x^{420}+4\,y^{266}x^{455}+6\,y^{280}x^{441}+6\,y^{294}x^{427}+6\,y^{105}x^{616}\\
&\qquad+5\,y^{112}x^{609}\big)+\dots.
\end{aligned}
$$

11.4. Examples of the polynomials A_k^1. Below are some $A_k^1(z,Z)$ from Sec. 5, $y=z^{p-1}$ and $\sigma_p=Z$.

For $p=3$ and $s=2$:

$$
\begin{aligned}
\sigma_1&=v_2y^3\sigma_3+v_2y^4x,\\
\sigma_2&=2\,{v_2}^2y^3{\sigma_3}^4+2\,v_2y^2{\sigma_3}^2+x^2v_2y^4+2\,y.
\end{aligned}
$$

For $p=5$ and $s=3$:

$$
\begin{aligned}
\sigma_1&=v_3y^{25}{\sigma_5}^5+v_3y^{30}\sigma_5+v_3y^{31}x,\\
\sigma_2&=4\,{v_3}^2y^{25}{\sigma_5}^{30}+4\,{v_3}^2y^{30}{\sigma_5}^{26}+3\,v_3y^{19}{\sigma_5}^{10}+v_3y^{24}{\sigma_5}^{6}+3\,v_3y^{29}{\sigma_5}^{2}+2\,v_3y^{31}x^2,\\
\sigma_3&=2\,{v_3}^3y^{25}{\sigma_5}^{55}+2\,{v_3}^3y^{30}{\sigma_5}^{51}+{v_3}^2y^{19}{\sigma_5}^{35}+2\,{v_3}^2y^{24}{\sigma_5}^{31}+{v_3}^2y^{29}{\sigma_5}^{27}\\
&\quad+2\,v_3y^{13}{\sigma_5}^{15}+v_3y^{18}{\sigma_5}^{11}+v_3y^{23}{\sigma_5}^{7}+2\,v_3y^{28}{\sigma_5}^{3}+2\,x^3v_3y^{31},
\end{aligned}
$$

$$\sigma_4 = 4\,{v_3}^4y^{25}{\sigma_5}^{80} + 4\,{v_3}^4y^{30}{\sigma_5}^{76} + 4\,{v_3}^3y^{19}{\sigma_5}^{60} + 3\,{v_3}^3y^{24}{\sigma_5}^{56}$$
$$+ 4\,{v_3}^3y^{29}{\sigma_5}^{52} + 4\,{v_3}^2y^{13}{\sigma_5}^{40} + 2\,{v_3}^2y^{18}{\sigma_5}^{36} + 2\,{v_3}^2y^{23}{\sigma_5}^{32} + 4\,{v_3}^2y^{28}{\sigma_5}^{28}$$
$$+ 4\,y^7v_3{\sigma_5}^{20} + y^{12}v_3{\sigma_5}^{16} + 4\,y^{17}v_3{\sigma_5}^{12} + y^{22}v_3{\sigma_5}^{8} + 4\,y^{27}v_3{\sigma_5}^{4} + x^4v_3y^{31} + 4\,y.$$

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M. Bakuradze
Razmadze Mathematical Institute, Tbilisi, Georgia
E-mail: maxo@rmi.acnet.ge