

Computing the Krichever genus

Malkhaz Bakuradze

Dedicated to Hvedri Inassaridze on his 80th birthday

Received: 11 March 2013 / Accepted: 23 July 2013 / Published online: 21 August 2013
© Tbilisi Centre for Mathematical Sciences 2013

Abstract Let ψ denote the genus that corresponds to the formal group law having invariant differential $\omega(t)$ equal to $\sqrt{1 + p_1t + p_2t^2 + p_3t^3 + p_4t^4}$ and let κ classify the formal group law strictly isomorphic to the universal formal group law under strict isomorphism $x\mathbf{CP}(x)$. We prove that on the rational complex bordism ring the Krichever-Höhn genus ϕ_{KH} is the composition $\psi \circ \kappa^{-1}$. We construct certain elements A_{ij} in the Lazard ring and give an alternative definition of the universal Krichever formal group law. We conclude that the coefficient ring of the universal Krichever formal group law is the quotient of the Lazard ring by the ideal generated by all A_{ij} , $i, j \geq 3$.

Keywords Complex cobordism · Formal group law · Elliptic genera

1 Rational Krichever-Höhn genus

For the current state of complex cobordism and formal group laws we refer the reader to the excellent survey [5].

A formal group law over a commutative ring with unit R is a power series $F(x, y) \in R[[x, y]]$ satisfying

- (i) $F(x, 0) = F(0, x) = x$,
- (ii) $F(x, y) = F(y, x)$,
- (iii) $F(x, F(y, z)) = F(F(x, y), z)$.

Communicated by Nick Inassaridze.

M. Bakuradze (✉)
Faculty of Exact and Natural Sciences,
Iv. Javakishvili Tbilisi State University,
Tbilisi, Georgia
e-mail: malkhaz.bakuradze@tsu.ge

Let F and G be formal group laws. A homomorphism from F to G is a power series $v(x) \in R[[x]]$ with constant term 0 such that

$$v(F(x, y)) = G(v(x), v(y)).$$

It is an isomorphism if $v'(0)$ (the coefficient at x) is a unit in R , and a strict isomorphism if the coefficient at x is 1.

If F is a formal group law over a commutative \mathbb{Q} -algebra R , then it is strictly isomorphic to the additive formal group law $x + y$. In other words, there is a strict isomorphism $l(x)$ from F to the additive formal group law, called the logarithm of F , so that $F(x, y) = l^{-1}(l(x) + l(y))$. The inverse to logarithm is called the exponential of F .

The logarithm $l(x) \in R \otimes \mathbb{Q}[[x]]$ of a formal group law F is given by

$$l(x) = \int_0^x \frac{dt}{\omega(t)}, \quad \omega(x) = \frac{\partial F(x, y)}{\partial y}(x, 0).$$

There is a ring L , called the universal Lazard ring, and a universal formal group law $F(x, y) = \sum a_{ij}x^i y^j$ defined over L . This means that for any formal group law G over any commutative ring with unit R there is a unique ring homomorphism $r : L \rightarrow R$ such that $G(x, y) = \sum r(a_{ij})x^i y^j$.

The formal group law of geometric cobordism was introduced in [12]. Following Quillen we will identify it with the universal Lazard formal group law as it is proved in [13] that the coefficient ring of complex cobordism $MU_* = \mathbb{Z}[x_1, x_2, \dots]$, $|x_i| = 2i$ is naturally isomorphic as a graded ring to the universal Lazard ring.

The Krichever-Höhn genus or the general four variable complex elliptic genus [7, 8]

$$\phi_{KH} : MU_* \otimes \mathbb{Q} \rightarrow \mathbb{Q}[q_1, \dots, q_4]$$

is a graded \mathbb{Q} -algebra homomorphism defined by the following property: if one denotes by $f_{Kr}(x)$ the exponential of the Krichever [6] universal formal group law \mathcal{F}_{Kr} , then the series

$$h(x) := \frac{f'_{Kr}(x)}{f_{Kr}(x)}$$

satisfies the differential equation

$$(h'(x))^2 = S(h(x)), \tag{1.1}$$

where $S(x) = x^4 + q_1x^3 + q_2x^2 + q_3x + q_4$, the generic monic polynomial of degree 4 with formal parameters q_i of weights $|q_i| = 2i$.

This genus is the universal genus on the rational bordism ring of SU -manifolds which is multiplicative in fiber bundles of SU -manifolds with compact connected structure group.

To generalize the Ochanine elliptic genus from $\Omega_*^{SO} \otimes \mathbb{Q}$ to $\mathbb{Q}[\mu, \epsilon]$ (see [9]), a new elliptic genus ψ is defined in [10],

$$\psi : MU_* \otimes \mathbb{Q} \rightarrow \mathbb{Q}[p_1, p_2, p_3, p_4],$$

to be the genus whose logarithm equals

$$\int_0^x \frac{dt}{\omega(t)}, \quad \omega(t) = \sqrt{1 + p_1t + p_2t^2 + p_3t^3 + p_4t^4}$$

and p_i are again formal parameters $|p_i| = 2i$.

It is proved in [10] that the ψ -genus is the universal genus on the rational complex bordism ring which is multiplicative in projectivizations $P(E)$ of complex vector bundles $E \rightarrow B$ over Calabi-Yau 3-folds B (i.e. B is a compact Kähler manifold with vanishing first Chern class).

Clearly to calculate the values of ψ on \mathbf{CP}_i , the generators of the rational complex bordism ring $MU_* \otimes \mathbb{Q} = \mathbb{Q}[\mathbf{CP}_1, \mathbf{CP}_2, \dots]$ we need only the Taylor expansion of $(1 + y)^{-1/2}$ as by above definition

$$(1 + p_1x + p_2x^2 + p_3x^3 + p_4x^4)^{-1/2} = \log'_\psi = \sum_{i \geq 1} \psi(\mathbf{CP}_i)x^i.$$

A straightforward calculation shows that ψ is surjective [10]:

$$\begin{aligned} \psi(\mathbf{CP}_1) &= -\frac{1}{2}p_1; \\ \psi(\mathbf{CP}_2) &= \frac{3}{8}p_1^2 - \frac{1}{2}p_2; \\ \psi(\mathbf{CP}_3) &= -\frac{5}{16}p_1^3 + \frac{3}{4}p_1p_2 - \frac{1}{2}p_3; \\ \psi(\mathbf{CP}_4) &= \frac{35}{128}p_1^4 - \frac{15}{16}p_1^2p_2 + \frac{3}{8}p_2^2 + \frac{3}{4}p_1p_3 - \frac{1}{2}p_4. \end{aligned}$$

It is natural to ask whether we can similarly calculate ϕ_{KH} in an elementary manner, different from that relying on the formulas in [3] and [7].

Let κ be the classifying map of the formal group law \tilde{F} over the rational Lazard ring defined as follows.

Let $\mathbf{CP}(x) = 1 + \sum_{i \geq 1} \mathbf{CP}_i x^i$, where \mathbf{CP}_i is the bordism class of the complex projective space CP^i and let

$$v(x) := x\mathbf{CP}(x)$$

be the strict isomorphism $F \rightarrow \tilde{F}$, where F is the universal formal group law, so that

$$v(F(x, y)) = \tilde{F}(v(x), v(y)).$$

Now let

$$\kappa : MU_* \otimes \mathbb{Q} \rightarrow MU_* \otimes \mathbb{Q}$$

be the classifying map of \tilde{F} and let $\log(x)$ and $\text{mog}(x)$ be the logarithm series of F and \tilde{F} respectively then by definition

$$\text{mog}(x) = \log(v^{-1}(x)), \quad \kappa(\log(x)) = \text{mog}(x).$$

Therefore the value $\kappa(\mathbf{CP}_i)$ is determined by equating the coefficients at x^i in

$$\sum_{i \geq 1} \frac{\kappa(\mathbf{CP}_i)}{i+1} x^{i+1} = \sum_{i \geq 1} \frac{\mathbf{CP}_i}{i+1} (v^{-1}(x))^{i+1}. \quad (1.2)$$

For instance

$$\begin{aligned} \kappa(\mathbf{CP}_1) &= -\mathbf{CP}_1; \\ \kappa(\mathbf{CP}_2) &= 3\mathbf{CP}_1^2 - 2\mathbf{CP}_2; \\ \kappa(\mathbf{CP}_3) &= -10\mathbf{CP}_1^3 + 12\mathbf{CP}_1\mathbf{CP}_2 - 3\mathbf{CP}_3; \\ \kappa(\mathbf{CP}_4) &= 35\mathbf{CP}_1^4 - 60\mathbf{CP}_1^2\mathbf{CP}_2 + 20\mathbf{CP}_1\mathbf{CP}_3 + 10\mathbf{CP}_2^2 - 4\mathbf{CP}_4. \end{aligned}$$

The following theorem shows how ϕ_{KH} is related to ψ .

Theorem 1.1 *Let F be the universal formal group law and \tilde{F} its strictly isomorphic formal group law under strict isomorphism $v(x) = x\mathbf{CP}(x)$. Let $t : \mathbb{Q}[p_1, \dots, p_4] \rightarrow \mathbb{Q}[q_1, \dots, q_4]$ be the ring isomorphism defined by $t(p_i) = q_i$ and $\kappa : MU_* \otimes \mathbb{Q} \rightarrow MU_* \otimes \mathbb{Q}$ the classifying map of \tilde{F} . Then*

- i) *the pair $(t, \sum_{i \geq 0} \phi_{KH}(\mathbf{CP}_i)x^{i+1})$ is the strict isomorphism from $\phi_{KH}(F)$ to $\psi(F)$, i.e., the series $\sum_{i \geq 0} \phi_{KH}(\mathbf{CP}_i)x^{i+1}$ is the strict isomorphism from $\phi_{KH}(F)$ to $t\psi(F)$ in the usual sense.*
- ii) *A method to compute ϕ_{KH} is given by the formula $\phi_{KH} = t \circ \psi \circ \kappa^{-1}$.*

To establish the theorem, we need the following two lemmas.

Lemma 1.2 *Let exp be the exponent of F . The series $\frac{1}{h(x)} = \text{exp}(x)/\text{exp}'(x)$ is invertible. Furthermore, the inverse $j(x)$ of this series coincides with then $\text{mog}(x)$, the logarithmic series of \tilde{F} .*

Proof Note that

$$j^{-1}(\log(x)) = \frac{\text{exp}(\log(x))}{\text{exp}'(\log(x))} = x\mathbf{CP}(x).$$

Hence by definition of $v(x)$ one has

$$j^{-1}(\text{mog}(x)) = j^{-1}(\log(v^{-1}(x))) = v^{-1}(x)\mathbf{CP}(v^{-1}(x)) = x,$$

as $x\mathbf{CP}(x) = v(x)$. □

Lemma 1.3 *Let $\tilde{\omega} = \frac{1}{\text{mog}'(x)}$ be the invariant differential form of the formal group law \tilde{F} above. Then the condition of Krichever-Höhn (1.1) is satisfied if and only if $\tilde{\omega}(x)^2$ is a polynomial of degree 4.*

Proof Since the series $j(x)$ is invertible, the condition (1.1) is equivalent to

$$(h'(j(x)))^2 = S(h(j(x))).$$

But $h(j(x)) = 1/x$, hence

$$(h \circ j)'(x) = h'(j(x))j'(x) = -1/x^2,$$

so that by Lemma 1.2

$$h'(j(x)) = -\tilde{\omega}(x)/x^2.$$

It follows that the condition (1.1) is equivalent to

$$\tilde{\omega}(x)^2 = x^4 S(1/x),$$

where on the right hand side one clearly has a degree four polynomial. □

Now let us prove Theorem 1.1. Lemma 1.3 implies that $\phi_{KH}(\tilde{F})$ is of type $\psi(F)$, that is corresponding invariant form $\omega(x)^2$ is a polynomial of degree 4. By definition the formal group law $\psi(F)$ is universal with this property. Therefore there is classifying map of $\phi_{KH}(\tilde{F})$, that is unique ring homomorphism $t : \mathbb{Q}[p_1, \dots, p_4] \rightarrow \mathbb{Q}[q_1, \dots, q_4]$ such that $t(\psi(F)) = \phi_{KH}(\tilde{F}) = \phi_{KH}(\kappa(F))$. Therefore $t \circ \psi = \phi_{KH} \circ \kappa$. This proves (i). For (ii) note that by definition κ is isomorphism and we get $t \circ \psi \circ \kappa^{-1} = \phi_{KH}$. Finally $t(p_i) = q_i$ as t is unique and it sends $1 + p_1t + p_2t^2 + p_3t^3 + p_4t^4$ to $1 + q_1t + q_2t^2 + q_3t^3 + p_4t^4$.

2 Integral Krichever genus

Now we turn to the universal Krichever formal group law F_{Kr} [8] and prove that it coincides with the universal formal group law by Buchstaber F_B (with a minor specialisation that does not affect the formal group law).

In [4] Buchstaber has given the analytical solution of a functional equation for the exponent of the formal group law of the form

$$\mathcal{F}_B(x, y) = \sum \alpha_{ij} x^i y^j = \frac{A(y)x^2 - A(x)y^2}{B(y)x - B(x)y}. \tag{2.1}$$

Note that if our series A and B have the form

$$A(t) = A_0 + A_1t + A_2t^2 + O(t^3),$$

$$B(t) = B_0 + B_1t + B_2t^2 + O(t^3),$$

then the coefficient $B_1 = B'(0)$ does not affect the formal group law.

Lemma 2.1 *Let $\omega(x) = \frac{\partial F(x,y)}{\partial y}(x, 0)$ be the invariant form of the universal formal group law F , and let \mathcal{F} be the formal group law of the form (2.1), with $B'(0) = A'(0)$. Then the invariant form of \mathcal{F} equals $B(x)$, i.e., $B(x)$ is the image of $\omega(x)$ under the ring homomorphism classifying the formal group law \mathcal{F} .*

Proof To see this note that

$$\mathcal{F}(x, y) = \frac{A_0}{B_0}(x + y) + O(xy).$$

So if $\mathcal{F}(x, y)$ is a formal group law we must have $A_0 = B_0$, and after dividing the numerator and denominator appropriately we may assume that $A_0 = B_0 = 1$. We then furthermore calculate

$$\mathcal{F}(x, y) = x + y + A_1xy + \sum_{i=2}^{\infty} B_i(x^i y + xy^i) + O(x^2y^2).$$

We thus have

$$\alpha_{1i} = B_i, \quad i \geq 2, \quad \alpha_{11} = A_1 = A'(0) = B'(0)$$

and

$$1 + \sum_{i \geq 1} \alpha_{1i}t^i$$

is restricted $(1 + \sum_{i \geq 1} [\mathbf{CP}^i]t^i)^{-1} = \omega(t)$. □

Let us now present a minor modification of the analysis of \mathcal{F}_B performed in [11] and as Lemma 2.1 suggests to introduce

$$A(x, y) = \sum A_{ij}x^i y^j = F(x, y)(x\omega(y) - y\omega(x)). \tag{2.2}$$

We define the universal Nadiradze formal group law \mathcal{F}_N by the obvious classifying map of the Lazard ring to its quotient ring by the ideal generated by all A_{ij} with $i, j \geq 3$.

Proposition 2.2 *Let L denote the Lazard ring.*

i) *In $L[[x]]$, the identity $\omega'(x) - \omega'(0) = 2x\hat{\omega}(x)$ holds, where $\hat{\omega}(x) = \sum_{i \geq 1} \omega_i x^{i-1}$.*

ii) *The formal series $A(x, y)$ satisfies the identity*

$$A(x, y) = (x\omega(y) + y\omega(x) - \omega'(0)xy) (x\omega(y) - y\omega(x)) + (\omega(x)\hat{\omega}(x) - \omega(y)\hat{\omega}(y)) x^2 y^2$$

in $L[[x, y]]/(xy)^3$.

Proof Let f and g be the exponent and logarithm of F , respectively. Hence $F(x, y) = f(g(x) + g(y))$ and

$$f'(x) = 1/g'(f(x)) = \omega(f(x)), \quad f'(g(x)) = \omega(f(g(x))) = \omega(x). \tag{2.3}$$

Let $\omega(x) = 1 + b_1x + b_2x^2 + \dots$. Since $g''(0) = -f''(0) = -\omega'(0) = -b_1$, we conclude

$$\frac{\partial^2 F}{\partial y^2}(x, 0) = f''(g(x)) + f'(g(x))g''(0) = \omega'(x)\omega(x) - \omega'(0)\omega(x). \tag{2.4}$$

This implies (i), since the left hand side of (2.4) has factor 2, and $\omega(x)$ is invertible. (ii) Because of antisymmetry we have modulo $(xy)^3$

$$A(x, y) = A(y)x^2 - A(x)y^2 = \sum (A_{i2}x^2y^i - A_{i2}x^i y^2).$$

We want to calculate $-\sum A_{i2}x^i$ in terms of $\omega(x)$.

Applying $\frac{\partial^2}{\partial y^2}(x, 0)$ to (2.2) and taking into account (2.3) and (2.4) we obtain

$$-2 \sum A_{i2}x^i = x\omega(x)\omega'(x) - x\omega'(0)\omega(x) + 2x\omega'(0)\omega(x) - 2\omega^2(x) + 2b_2x^2.$$

Since the coefficients are in the Lazard ring, this reasoning implies

$$-\sum A_{i2}x^i = x\omega(x) \frac{\omega'(x) - \omega'(0)}{2} + x\omega'(0)\omega(x) - \omega^2(x) + b_2x^2.$$

Consequently

$$\sum (A_{i2}x^2y^i - A_{i2}x^i y^2) = (x\omega(y) + y\omega(x))(x\omega(y) - y\omega(x)) - \omega'(0)xy(x\omega(y) - y\omega(x)) + \omega(x)\hat{\omega}(x)x^2y^2 - \omega(y)\hat{\omega}(y)x^2y^2. \quad \square$$

In order to compute the Krichever genus on the coefficients of the formal group law of geometric cobordism, in [6], the universal Krichever formal group law \mathcal{F}_{Kr} is defined as

$$\mathcal{F}_{Kr}(x, y) = xb(y) + yb(x) - b'(0)xy + \frac{b(x)\beta(x) - b(y)\beta(y)}{xb(y) - yb(x)}x^2y^2, \tag{2.5}$$

where $\beta(x) = \frac{b'(x)-b'(0)}{2x}$. In [6], it is moreover proved that

$$b(x) = \frac{\partial \mathcal{F}_{Kr}}{\partial y}(x, 0).$$

Lemma 2.1 and Proposition 2.2 (ii) imply that \mathcal{F}_{Kr} can alternatively be defined by the classifying map of \mathcal{F}_N if $B(x)$ is written as $b(x)$. Thus $\mathcal{F}_N = \mathcal{F}_{Kr}$.

Following [4], the authors of [3] consider the following formal group law corresponding to the Krichever genus

$$\mathcal{F}_b(u_1, u_2) = u_1c(u_2) + u_2c(u_1) - au_1u_2 - \frac{d(u_1) - d(u_2)}{u_1c(u_2) - u_2c(u_1)}u_1^2u_2^2.$$

It follows from Lemma 2.1 and Proposition 2.2 (ii) that, if we take $c'(0) = a$, the Krichever genus $\mathcal{F}_b(x, y)$ coincides with (2.5), that is $c(x) = b(x)$ and $d(x) = -b(x)\beta(x)$.

Thus we get the following

Theorem 2.3 *Let F be the universal formal group law, let $\omega(x) = \frac{\partial F(x,y)}{\partial y}(x, 0)$ be its invariant form, and let*

$$\sum A_{ij}x^i y^j = F(x, y)(x\omega(y) - y\omega(x)).$$

The Buchstaber, Krichever and Nadiradze formal group laws coincide, that is,

$$\mathcal{F}_b = \mathcal{F}_{Kr} = \mathcal{F}_N,$$

and the coefficient ring is the quotient of the Lazard ring by the ideal generated by all A_{ij} with $i, j \geq 3$.

In [1], we calculated the coefficient ring of the Nadiradze formal group law \mathcal{F}_N up to dimension 26. Namely, there is a set of polynomial generators z_1, z_2, \dots of the Lazard ring for which the low degree defining relations are

$$5z_5 = z_2z_3 + 2z_1z_4, \quad 2z_6 = 0, \quad z_1z_6 = 0, \quad z_3z_6 = 0, \quad z_{10} = 0, \quad z_5z_6 = 0, \quad z_{12} = 0,$$

and $7z_7, 2z_8, 3z_9, 11z_{11}$, and $13z_{13}$ are decomposable. For reasons of space, we omit here these long decompositions. We note that our calculations agree with the results in [6] on the structure of the coefficient ring of \mathcal{F}_{Kr} obtained in terms of the associativity equation. The new information here concerning the Krichever group and hence the Krichever genus is that, in dimensions 20 and 24, there are no indecomposable elements, because in these dimensions $z_{10} = 0$ and $z_{12} = 0$.

The question arises in which dimensions any element is multiplicatively decomposable.

Acknowledgments The author was supported by Volkswagen Foundation, Ref.: I/84 328 and by Rustaveli NSF, DI/16/5-103/12

References

1. Bakuradze, M., Jibladze, M.: On the coefficient ring of rational formal group law. *Proc. Razmadze Math. Inst.* **159**, 1–9 (2012)
2. Bakuradze, M.: Formal group laws by Buchstaber, Krichever and Nadiradze coincide. *Uspekhi Mat. Nauk* **68**(3), 189–190 (2013)
3. Buchstaber, V.M., Panov, T., Ray, N.: Toric genera. *Intern. Math. Res. Notices* **16**, 3207–3262 (2010)
4. Buchstaber, V.M.: Functional equations associated with addition theorems for elliptic functions and two-valued algebraic groups. *Uspekhi Mat. Nauk* **45**(3(273)), 185–186 (1990)
5. Buchstaber, V.M.: Complex cobordism and formal groups. *Russian Math. Surv.* **67**(5), 891–950 (2012)
6. Buchstaber, V.M., Bunkova, E.Y.: Krichever Formal groups. *Funct. Anal. appl.* **45**(2), 23–44 (2011)
7. Höhn, G.: Komplexe elliptische Geschlechter und S^1 -äquivariante Kobordismustheorie, Bonn und Valendar, Available as [arXiv:math/0405232](https://arxiv.org/abs/math/0405232) (1991)
8. Krichever, I.: Generalized elliptic genera and Baker-Akhiezer functions. *Math. Notes* **47**, 132–142 (1990)
9. Ochanine, S.: Sur les genres multiplicatifs définis par des integrales elliptiques. *Topology* **26**(2), 143–151 (1987)
10. Schreieder, S.: Dualization invariance and a new complex elliptic genus, *J. reine angew. Math.* to appear, [arXiv:1109.5394v3 \[math.AT\]](https://arxiv.org/abs/1109.5394v3) (2012). doi:[10.1515/crelle-2012-0085](https://doi.org/10.1515/crelle-2012-0085)
11. Nadiradze, R.: Formal group and cohomology theories. Dissertation, Doctor of Science Degree, Tbilisi (1995)
12. Novikov, S.P.: The methods of algebraic topology from the viewpoint of cobordism theory. *Math. USSR-Izv.* **1**(4), 827–913 (1967)
13. Quillen, D.: On the formal group laws of unoriented and complex cobordism theory. *Bull. Amer. Math. Soc.* **75**, 1293–1298 (1969)