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THE BOUNDARY CONTACT PROBLEM FOR HEMITROPIC ELASTIC SOLIDS WITH FRICTION ARISING ALONG THE NORMAL

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Abstract. We investigate boundary-contact problem of statics of the theory of elasticity for homogeneous hemitropic elastic medium with regard friction. We consider two cases, the so-called coercive case(when elastic media is fixed along some parts of the boundary), and the semi-coercive case (the boundary is not fixed anywhere). The problem is equivalently reduced to a spatial variational inequality. Based on variational inequality approach,we prove existence and uniqueness theorems for weak solutions. We prove that the solutions continuously depend on the data of the original problem. In the semi-coercive case, the necessary condition of solvability of the corresponding contact problem is written explicitly. This condition under some certain restrictions is sufficient as well.

რეზიუმე. ნაშრომში შესწავლილია დრეკადობის თეორიის სტატიკის სასაზღვრო-საკონტაქტო ამოცანა ერთგვაროვანი ჰემიტროპული სხეულებისათვის ხახუნის გათვალისწინებით. ამ შემთხვევაში ხახუნის ძალა წარმოიქმნება სხეულის ნორმალის მიმართულებით გადაადგილებისას, ნაცვლად მხების მიმართულებისა. განხილულია როგორც კოერციტიული (როდესაც სხეული საზღვრის გარკვეული დადებითი ზომის ნაწილით ჩამაგრებულია), ისე არაკოერციტიული შემთხვევა. ამოცანის სივრცით ვარიაციულ უტოლობაზე ეკვივალენტურად დაყვანის მეშვეობით შესწავლილია სუსტი ამონახსნის არსებობის, ერთადერთობის და ამონახსნის მონაცემებზე უწყვეტად დამოკიდებულების საკითხი. არაკორციტიულ შემთხვევაში ცხადი სახით ამოწერილია ამონახსნის არსებობის აუცილებელი და საკმარისი პირობა. ეს პირობა გარკვეულ დამატებით შეზღუდვებში წარმოადგენს საკმარის პირობასაც.

1. Introduction

In the present work we investigate unilateral contact problems for homogeneous hemitropic elastic solids with friction. We consider the case when the friction forces arise not under tangential but under normal displacement

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(see [5]). The model of the theory of elasticity is presented in which, unlike the classical theory, an elementary particle of a body along with displacements undergoes rotation, and hence the motion of an elementary particle is characterized by the independent displacement vector and micro-rotation vector.

The origin of the rational theories of polar continua goes back to brothers E. and F. Cosserat [3], [4], who gave a development of the mechanics of continuous media in which each material point has the six degrees of freedom defined by 3 displacement components and 3 microrotation components (for the history of the problem see [6], [19], [24], [28], and the references therein).

A micropolar solid which is not isotropic with respect to inversion is called *hemitropic, noncentrosymmetric,* or *chiral*.

Refined mathematical models describing the hemitropic properties of elastic materials have been proposed by Aero and Kuvshinski [1], [2]. In the mathematical theory of hemitropic elasticity there are introduced the asymmetric force stress tensor and moment stress tensor, which are kinematically related with the asymmetric strain tensor and torsion (curvature) tensor via the constitutive equations. All these quantities are expressed in terms of the components of the displacement and microrotation vectors. In turn, the displacement and microrotation vectors satisfy a coupled complex system of second order partial differential equations. We note that the governing equations in this model become very involved and generate 6×6 matrix partial differential operator of second order. Evidently, the corresponding 6×6 matrix boundary differential operators describing the force stress and couple stress vectors have also involved structure in comparison with the classical case. Particular problems of the elasticity theory of hemitropic continuum have been considered in [20], [21], [28], [29]. The frictionless contact problems of statics for hemitropic solids have been studied in [11], [12], [15], while the contact problems of statics with friction are analyzed in the references [10], [13], [14]. The basic boundary value and transmission problems of hemitropic elasticity have been studied by the potential method for smooth and non-smooth Lipschitz domains in [25]. Similar unilateral problems of the classical linear elasticity theory with various modifications have been considered in many monographs and papers, [5], [7], [8], [9],[17], [18], [30] (see also the references therein).

The paper is organized as follows. First we collect the basic field equations of statics of the theory of elasticity for hemitropic media in vector and matrix forms, introduce the generalized stress operator and potential energy quadratic form. Then we present a reasonable mathematical model of the boundary conditions that apply to hemitropic solids in contact with friction. We first consider the case when some part of the boundary is mechanically fixed. Here we show how the original problem can be reformulated as a coercive spatial variational inequality. We present a detailed analysis of these inequalities and investigate existence and uniqueness of a weak solution of the original contact problem. Further, we treat a more complicated case when only traction-contact conditions are considered on the whole boundary. In this case, the corresponding mathematical problem is not solvable, in general. We derive the necessary conditions of solvability and formulate also some sufficient conditions of solvability in explicit form.

2. Field Equations and Green's Formulas

2.1. **Basic Equations.** Let $\Omega \subset \mathbb{R}^3$ be a bounded, simply connected domain with C^∞ smooth boundary $S := \partial\Omega$, $\overline{\Omega} = \Omega \cup S$. Throughout the paper $n(x) = (n_1(x), n_2(x), n_3(x))$ denotes the outward unit normal vector at the point $x \in S$.

We assume that Ω is occupied by a homogeneous hemitropic elastic material. Denote by $u = (u_1, u_2, u_3)^\top$ and $\omega = (\omega_1, \omega_2, \omega_3)^\top$ the *displacement vector* and the *micro-rotation vector*, respectively; here and in what follows the symbol $(\cdot)^\top$ denotes transposition.

In the hemitropic elasticity theory we have the following constitutive equations for the *force stress tensor* $\{\tau_{pq}\}$ and the *couple stress tensor* $\{\mu_{pq}\}$:

$$\begin{aligned}\tau_{pq} = \tau_{pq}(U) := (\mu+\alpha)\,\partial_p u_q + (\mu-\alpha)\,\partial_q u_p + \lambda\delta_{pq}\operatorname{div} u + \delta\,\delta_{pq}\operatorname{div}\omega +\\ + (\varkappa+\nu)\,\partial_p\omega_q + (\varkappa-\nu)\,\partial_q\omega_p - 2\alpha\sum_{k=1}^{3}\varepsilon_{pqk}\,\omega_k,\end{aligned} \tag{2.1}$$

$$\begin{aligned}\mu_{pq} = \mu_{pq}(U) := \delta\,\delta_{pq}\operatorname{div} u + (\varkappa+\nu)\Big[\partial_p u_q - \sum_{k=1}^{3}\varepsilon_{pqk}\,\omega_k\Big] + \beta\,\delta_{pq}\operatorname{div}\omega +\\ + (\varkappa-\nu)\Big[\partial_q u_p - \sum_{k=1}^{3}\varepsilon_{qpk}\,\omega_k\Big] + (\gamma+\varepsilon)\,\partial_p\omega_q + (\gamma-\varepsilon)\,\partial_q\omega_p,\end{aligned} \tag{2.2}$$

where $U = (u, \omega)^\top$, δ_{pq} is the Kronecker delta, $\partial = (\partial_1, \partial_2, \partial_3)$ with $\partial_j = \partial/\partial x_j$, ε_{pqk} is the permutation (Levi-Civitá) symbol, and α, β, γ, δ, λ, μ, ν, $\varkappa$ and ε are the material constants [1], [26].

The components of the force stress vector $\tau^{(n)} = (\tau_1^{(n)}, \tau_2^{(n)}, \tau_3^{(n)})^\top$ and the couple stress vector $\mu^{(n)} = (\mu_1^{(n)}, \mu_2^{(n)}, \mu_3^{(n)})^\top$, acting on a surface element with a normal vector $n = (n_1, n_2, n_3)$, read as

$$\tau_q^{(n)} = \sum_{p=1}^{3}\tau_{pq}\,n_p, \quad \mu_q^{(n)} = \sum_{p=1}^{3}\mu_{pq}\,n_p, \quad q = 1, 2, 3. \tag{2.3}$$

Denote by $T(\partial,n)$ the generalized 6×6 matrix differential stress operator [26]

$$T(\partial,n)=\begin{bmatrix} T^{(1)}(\partial,n) & T^{(2)}(\partial,n)\\ T^{(3)}(\partial,n) & T^{(4)}(\partial,n)\end{bmatrix}_{6\times 6}, \quad T^{(j)}=\Big[T^{(j)}_{pq}\Big]_{3\times 3}, \quad j=\overline{1,4},$$

where

$$\begin{aligned}
&T^{(1)}_{pq}(\partial,n)=(\mu+\alpha)\,\delta_{pq}\,\partial_n+(\mu-\alpha)\,n_q\,\partial_p+\lambda\,n_p\,\partial_q,\\
&T^{(2)}_{pq}(\partial,n)=(\varkappa+\nu)\,\delta_{pq}\,\partial_n+(\varkappa-\nu)\,n_q\,\partial_p+\delta\,n_p\,\partial_q-2\,\alpha\sum_{k=1}^{3}\varepsilon_{pqk}\,n_k,\\
&T^{(3)}_{pq}(\partial,n)=(\varkappa+\nu)\,\delta_{pq}\,\partial_n+(\varkappa-\nu)\,n_q\,\partial_p+\delta\,n_p\,\partial_q,\\
&T^{(4)}_{pq}(\partial,n)=(\gamma+\varepsilon)\,\delta_{pq}\,\partial_n+(\gamma-\varepsilon)\,n_q\,\partial_p+\beta\,n_p\,\partial_q-2\,\nu\sum_{k=1}^{3}\varepsilon_{pqk}\,n_k.
\end{aligned}\tag{2.4}$$

Here $\partial_n=\partial/\partial n$ denotes the directional derivative along the vector n (normal derivative).

From formulas (2.1), (2.2) and (2.3) it can be easily checked that

$$\Big(\tau^{(n)},\,\mu^{(n)}\Big)^\top=T(\partial,n)\,U.$$

The equilibrium equations of statics in the theory of hemitropic elasticity read as [26]

$$\begin{aligned}
&\sum_{p=1}^{3}\partial_p\,\tau_{pq}(x)+\varrho\,F_q(x)=0,\\
&\sum_{p=1}^{3}\partial_p\,\mu_{pq}(x)+\sum_{l,r=1}^{3}\varepsilon_{qlr}\,\tau_{lr}(x)+\varrho\,\Psi_q(x)=0,\quad q=1,2,3,
\end{aligned}$$

where $F=(F_1,F_2,F_3)^\top$ and $\Psi=(\Psi_1,\Psi_2,\Psi_3)^\top$ are the body force and body couple vectors per unit mass and ϱ is the mass density of the elastic material. Using the constitutive equations (2.1) and (2.2) we can rewrite the equilibrium equations in terms of the displacement and micro-rotation vectors,

$$\begin{aligned}
&(\mu+\alpha)\,\Delta\,u(x)+(\lambda+\mu-\alpha)\,\mathrm{grad}\,\mathrm{div}\,u(x)+(\varkappa+\nu)\,\Delta\,\omega(x)+\\
&\quad+(\delta+\varkappa-\nu)\,\mathrm{grad}\,\mathrm{div}\,\omega(x)+2\alpha\,\mathrm{curl}\,\omega(x)+\varrho\,F(x)=0,\\
&(\varkappa+\nu)\,\Delta\,u(x)+(\delta+\varkappa-\nu)\,\mathrm{grad}\,\mathrm{div}\,u(x)+2\alpha\,\mathrm{curl}\,u(x)+\\
&\quad+(\gamma+\varepsilon)\,\Delta\,\omega(x)+(\beta+\gamma-\varepsilon)\,\mathrm{grad}\,\mathrm{div}\,\omega(x)+4\nu\,\mathrm{curl}\,\omega(x)-\\
&\quad-4\alpha\,\omega(x)+\varrho\,\Psi(x)=0,
\end{aligned}\tag{2.5}$$

where $\Delta=\partial_1^2+\partial_2^2+\partial_3^2$ is the Laplace operator.

Let us introduce the matrix differential operator generated by the left hand side expressions of the system (2.5):

$$L(\partial) := \begin{bmatrix} L^{(1)}(\partial) & L^{(2)}(\partial) \\ L^{(3)}(\partial) & L^{(4)}(\partial) \end{bmatrix}_{6\times 6},$$

where

$$L^{(1)}(\partial) := (\mu+\alpha)\,\Delta\, I_3 + (\lambda+\mu-\alpha)\, Q(\partial),$$
$$L^{(2)}(\partial) = L^{(3)}(\partial) := (\varkappa+\nu)\,\Delta\, I_3 + (\delta+\varkappa-\nu)\, Q(\partial) + 2\,\alpha\, R(\partial),$$
$$L^{(4)}(\partial) := [(\gamma+\varepsilon)\,\Delta - 4\,\alpha]\, I_3 + (\beta+\gamma-\varepsilon)\, Q(\partial) + 4\,\nu\, R(\partial).$$

Here and in the sequel I_k stands for the $k\times k$ unit matrix and

$$Q(\partial) := [\,\partial_k\partial_j\,]_{3\times 3}, \quad R(\partial) := \begin{bmatrix} 0 & -\partial_3 & \partial_2 \\ \partial_3 & 0 & -\partial_1 \\ -\partial_2 & \partial_1 & 0 \end{bmatrix}_{3\times 3}.$$

Equations (2.5) can be written in matrix form as

$$L(\partial)U(x) + \mathcal{G}(x) = 0, \ \ x \in \Omega,$$

where $U = (u,\omega)^\top$ and $\mathcal{G} = (\varrho F,\ \varrho\Psi)^\top$.

2.2. **Green's formulas.** For real-valued vector functions $U = (u,\omega)^\top$ and $U' = (u',\omega')^\top$ from the class $[C^2(\overline{\Omega})]^6$ the following Green formula holds [26]

$$\int\limits_{\Omega} \big[L(\partial)U \cdot U' + E(U,U')\big]\, dx = \int\limits_{S} \big\{T(\partial,n)U\big\}^+ \cdot \big\{U'\big\}^+ \, dS, \tag{2.6}$$

where $\{\,\cdot\,\}^+$ denotes the trace operator on S from Ω, while $E(\cdot\,,\,\cdot)$ is the bilinear form defined by the equality:

$$\begin{aligned} E(U,U') = E(U',U) = \sum_{p,q=1}^{3} \Big\{ & (\mu+\alpha)u'_{pq}u_{pq} + (\mu-\alpha)u'_{pq}u_{qp} + \\ & + (\varkappa+\nu)(u'_{pq}\omega_{pq} + \omega'_{pq}u_{pq}) + (\varkappa-\nu)(u'_{pq}\omega_{qp} + \omega'_{pq}u_{qp}) + \\ & + (\gamma+\varepsilon)\omega'_{pq}\omega_{pq} + (\gamma-\varepsilon)\omega'_{pq}\omega_{qp} + \\ & + \delta(u'_{pp}\omega_{qq} + \omega'_{qq}u_{pp}) + \lambda u'_{pp}u_{qq} + \beta\omega'_{pp}\omega_{qq}\Big\}, \end{aligned} \tag{2.7}$$

where u_{pq} and ω_{pq} are the so called *strain* and *torsion* (*curvature*) tensors for hemitropic bodies,

$$u_{pq} = u_{pq}(U) = \partial_p u_q - \sum_{k=1}^{3} \varepsilon_{pqk}\omega_k, \ \ \omega_{pq} = \omega_{pq}(U) = \partial_p\omega_q, \tag{2.8}$$
$$p,q = 1,2,3.$$

Here and in what follows $a \cdot b$ denotes the usual scalar product of two vectors $a, b \in \mathbb{R}^m: \ a \cdot b = \sum_{j=1}^m a_j\, b_j$.

From formulas (2.7) and (2.8) we get

$$
\begin{aligned}
E(U,U') = {} & \frac{3\lambda+2\mu}{3}\Big(\operatorname{div} u + \frac{3\delta+2\varkappa}{3\lambda+2\mu}\operatorname{div}\omega\Big)\Big(\operatorname{div} u' + \frac{3\delta+2\varkappa}{3\lambda+2\mu}\operatorname{div}\omega'\Big) + \\
& + \frac{1}{3}\Big(3\beta+2\gamma - \frac{(3\delta+2\varkappa)^2}{3\lambda+2\mu}\Big)(\operatorname{div}\omega)(\operatorname{div}\omega') + \\
& + \Big(\varepsilon - \frac{\nu^2}{\alpha}\Big)\operatorname{curl}\omega \cdot \operatorname{curl}\omega' + \\
& + \frac{\mu}{2}\sum_{k,j=1,\, k\neq j}^{3}\Big[\frac{\partial u_k}{\partial x_j} + \frac{\partial u_j}{\partial x_k} + \frac{\varkappa}{\mu}\Big(\frac{\partial \omega_k}{\partial x_j} + \frac{\partial \omega_j}{\partial x_k}\Big)\Big]\times \\
& \quad \times \Big[\frac{\partial u'_k}{\partial x_j} + \frac{\partial u'_j}{\partial x_k} + \frac{\varkappa}{\mu}\Big(\frac{\partial \omega'_k}{\partial x_j} + \frac{\partial \omega'_j}{\partial x_k}\Big)\Big] + \\
& + \frac{\mu}{3}\sum_{k,j=1}^{3}\Big[\frac{\partial u_k}{\partial x_k} - \frac{\partial u_j}{\partial x_j} + \frac{\varkappa}{\mu}\Big(\frac{\partial \omega_k}{\partial x_k} - \frac{\partial \omega_j}{\partial x_j}\Big)\Big]\times \\
& \quad \times \Big[\frac{\partial u'_k}{\partial x_k} - \frac{\partial u'_j}{\partial x_j} + \frac{\varkappa}{\mu}\Big(\frac{\partial \omega'_k}{\partial x_k} - \frac{\partial \omega'_j}{\partial x_j}\Big)\Big] + \\
& + \Big(\gamma - \frac{\varkappa^2}{\mu}\Big)\sum_{k,j=1,\, k\neq j}^{3}\Big[\frac{1}{2}\Big(\frac{\partial \omega_k}{\partial x_j} + \frac{\partial \omega_j}{\partial x_k}\Big)\Big(\frac{\partial \omega'_k}{\partial x_j} + \frac{\partial \omega'_j}{\partial x_k}\Big) + \\
& \quad + \frac{1}{3}\Big(\frac{\partial \omega_k}{\partial x_k} - \frac{\partial \omega_j}{\partial x_j}\Big)\Big(\frac{\partial \omega'_k}{\partial x_k} - \frac{\partial \omega'_j}{\partial x_j}\Big)\Big] + \\
& + \alpha\Big(\operatorname{curl} u + \frac{\nu}{\alpha}\operatorname{curl}\omega - 2\,\omega\Big)\cdot\Big(\operatorname{curl} u' + \frac{\nu}{\alpha}\operatorname{curl}\omega' - 2\,\omega'\Big).
\end{aligned}
$$

The potential energy density function $E(U,U)$ is a positive definite quadratic form with respect to variables $u_{pq}(U)$ and $\omega_{pq}(U)$, i.e., there exists a positive number $c_0 > 0$ depending only on the material constants, such that

$$
E(U,U) \geq c_0 \sum_{p,q=1}^{3}\big[u_{pq}^2 + \omega_{pq}^2\big].
$$

The necessary and sufficient conditions for the quadratic form $E(U,U)$ to be positive definite are the following inequalities (see [2], [6], [15])

$$
\begin{aligned}
& \mu>0,\ \ \alpha>0,\ \ \gamma>0,\ \ \varepsilon>0,\ \ \lambda+2\mu>0,\ \ \mu\,\gamma-\varkappa^2>0,\ \ \alpha\,\varepsilon-\nu^2>0,\\
& (\lambda+\mu)(\beta+\gamma)-(\delta+\varkappa)^2>0,\ \ (3\lambda+2\mu)(3\beta+2\gamma)-(3\delta+2\varkappa)^2>0,\\
& d_1 := (\mu+\alpha)(\gamma+\varepsilon)-(\varkappa+\nu)^2>0,\\
& d_2 := (\lambda+2\mu)(\beta+2\gamma)-(\delta+2\varkappa)^2>0,
\end{aligned}
$$

$$\mu\left[(\lambda+\mu)(\beta+\gamma)-(\delta+\varkappa)^2\right]+(\lambda+\mu)(\mu\gamma-\varkappa^2)>0,$$
$$\mu\left[(3\lambda+2\mu)(3\beta+2\gamma)-(3\delta+2\varkappa)^2\right]+(3\lambda+2\mu)(\mu\gamma-\varkappa^2)>0.$$

The following assertion describes the null space of the energy quadratic form $E(U,U)$ (see [26]).

Lemma 2.1. *Let $U=(u,\omega)^\top\in[C^1(\overline{\Omega})]^6$ and $E(U,U)=0$ in Ω. Then*

$$u(x)=[a\times x]+b,\quad \omega(x)=a,\quad x\in\Omega,$$

where a and b are arbitrary three-dimensional constant vectors and symbol $[\cdot\times\cdot]$ denotes the cross product of two vectors.

Vectors of type $([a\times x]+b,a)$ are call *generalized rigid displacement vectors*. Observe that a generalized rigid displacement vector vanishes, i.e. $a=b=0$, if it is zero at a single point.

Throughout the paper $L_p(\Omega)$ $(1\le p\le\infty)$, $L_2(\Omega)=H^0(\Omega)$ and $H^s(\Omega)=H^s_2(\Omega)$, $s\in\mathbb{R}$, denote the Lebesgue and Bessel potential spaces (see, e.g., [22], [31]). The corresponding norms we denote by symbols $\|\cdot\|_{L_p(\Omega)}$ and $\|\cdot\|_{H^s(\Omega)}$. Denote by $D(\Omega)$ the class of $C^\infty(\Omega)$ functions with support in the domain Ω. If M is an open proper part of the manifold $\partial\Omega$, i.e., $M\subset\partial\Omega$, $M\neq\partial\Omega$, then by $H^s(M)$ we denote the restriction of the space $H^s(\partial\Omega)$ on M,

$$H^s(M):=\left\{r_{_M}\varphi:\ \varphi\in H^s(\partial\Omega)\right\},$$

where $r_{_M}$ stands for the restriction operator on the set M. Further, let

$$\widetilde{H}^s(M):=\left\{\varphi\in H^s(\partial\Omega):\ \operatorname{supp}\varphi\subset\overline{M}\,\right\}.$$

From the positive definiteness of the energy form $E(\cdot,\cdot)$ with respect to the variables (2.8) it follows that

$$B(U,U):=\int_\Omega E(U,U)dx\ge 0. \tag{2.9}$$

Moreover, there exist positive constants C_1 and C_2, depending only on the material parameters, such that the following Korn's type inequality (cf. [8] Part I, §12.)

$$B(U,U)\ge C_1\left\|U\right\|^2_{[H^1(\Omega)]^6}-C_2\left\|U\right\|^2_{[H^0(\Omega)]^6} \tag{2.10}$$

holds for an arbitrary real-valued vector function $U\in[H^1(\Omega)]^6$.

Remark 2.2. If $U\in[H^1(\Omega)]^6$ and on some open part $S^*\subset\partial\Omega$ the trace $\{U\}^+$ vanishes, i.e., $r_{_{S^*}}\{U\}^+=0$, then we have the strict Korn's inequality

$$B(U,U)\ge c\left\|U\right\|^2_{[H^1(\Omega)]^6}$$

with some positive constant $c>0$ which does not depend on the vector U. This follows from (2.10) and the fact that in this case $B(U,U)>0$ for $U\neq 0$ (see, e.g., [27], [23], Ch. 2, Exercise 2.17).

Remark 2.3. By standard limiting arguments Green's formula (2.6) can be extended to Lipschitz domains and to vector functions $U \in [H^1(\Omega)]^6$ with $L(\partial)U \in [L_2(\Omega)]^6$ and $U' \in [H^1(\Omega)]^6$ (see, [27], [22]),

$$\int\limits_{\Omega} \big[L(\partial)U \cdot U' + E(U,U')\big]\,dx = \big\langle \{T(\partial,n)U\}^+ \,,\, \{U'\}^+ \big\rangle_{\partial\Omega}, \tag{2.11}$$

where $\langle \cdot\,,\,\cdot\rangle_{\partial\Omega}$ denotes the duality between the spaces $[H^{-1/2}(\partial\Omega)]^6$ and $[H^{1/2}(\partial\Omega)]^6$, which generalizes the usual inner product in the space $[L_2(\partial\Omega)]^6$. By this relation the generalized trace of the stress operator $\{T(\partial,n)U\}^+ \in [H^{-1/2}(\partial\Omega)]^6$ is correctly determined.

3. Contact Problems with Friction

3.1. **Pointwise and Variational Formulation of the Contact Problem.** Let the boundary S of the domain Ω be divided into two open, connected and non-overlapping parts S_1 and S_2 of positive measure, $S = \overline{S_1} \cup \overline{S_2}$, $S_1 \cap S_2 = \varnothing$. Assume that the hemitropic elastic body occupying the domain Ω is in contact with another rigid body along the subsurface S_2.

Definition 3.1. The vector-function $U = (u,\omega)^\top \in [H^1(\Omega)]^6$ is said to be a weak solution of equation

$$L(\partial)U + \mathcal{G} = 0, \quad \mathcal{G} \in [L_2(\Omega)]^6, \tag{3.1}$$

in the domain Ω if

$$B(U,\Phi) = \int\limits_{\Omega} \mathcal{G} \cdot \Phi \,dx \quad \forall\, \Phi \in [\mathcal{D}(\Omega)]^6,$$

where B is given by formula (2.9).

In the sequel, for the force stress and couple stress vectors we use the following notation:

$$\mathcal{T}U := T^{(1)}u + T^{(2)}\omega, \qquad \mathcal{M}U := T^{(3)}u + T^{(4)}\omega,$$

where $T^{(j)}$, $j = \overline{1,4}$, is defined by formula (2.4).

For the normal and tangential components of the force stress vector we will use, respectively, the following notation

$$(\mathcal{T}U)_n := \mathcal{T}U \cdot n, \quad (\mathcal{T}U)_s := \mathcal{T}U - n(\mathcal{T}U)_n.$$

Further, let

$$\mathcal{G} = (\varrho F, \varrho\Psi)^\top \in [L_2(\Omega)]^6, \quad \varphi \in [H^{-1/2}(S_2)]^3, \quad g_i \in L_\infty(S_2), \quad i = 1,2$$

and $\quad g_1 \le 0 \le g_2$.

Consider the following contact problem of statics with a friction (cf. [5], ch. 3, sect. 5.4.1).

Problem (A). Find a weak solution $U = (u,\omega)^\top \in [H^1(\Omega)]^6$ of the equation (3.1) satisfying the following conditions:

$$r_{S_1}\{U\}^+ = 0 \quad \text{on} \quad S_1; \tag{3.2}$$

$$r_{S_2}\{\mathcal{M}U\}^+ = \varphi \quad \text{on} \quad S_2; \tag{3.3}$$

$$r_{S_2}\{(\mathcal{T}U)_s\}^+ = 0 \quad \text{on} \quad S_2; \tag{3.4}$$

$$\text{if} \quad g_1 < r_{S_2}\{(\mathcal{T}U)_n\}^+ < g_2, \qquad \text{then} \quad r_{S_2}\{u_n\}^+ = 0, \quad \text{on} \quad S_2, \tag{3.5}$$

$$\text{if} \quad g_1 = r_{S_2}\{(\mathcal{T}U)_n\}^+, \qquad \text{then} \quad r_{S_2}\{u_n\}^+ \geq 0, \quad \text{on} \quad S_2 \tag{3.6}$$

and

$$\text{if} \quad g_2 = r_{S_2}\{(\mathcal{T}U)_s\}^+, \qquad \text{then} \quad r_{S_2}\{u_n\}^+ \leq 0, \quad \text{on} \quad S_2, \tag{3.7}$$

where $\{\cdot\}^+$ denotes the trace operator on $S_i (i = 1, 2)$ from Ω.

This problem can be reformulated as a variational inequality. To this end, we introduce the following continuous convex functional on the space $[H^1(\Omega)]^3$

$$j(v) = \int\limits_{S_2} \big(- g_1[\{v_n\}^+]^+ + g_2[\{v_n\}^+]^- \big) dS \quad \forall v \in [H^1(\Omega)]^3, \tag{3.8}$$

where $[f]^+ := \max(f, 0)$ and $[f]^- := \max(-f, 0)$ for $f \in H^{1/2}(S_2)$, and a closed convex subset of $[H^1(\Omega)]^6$

$$\mathcal{K} := \big\{ V = (v, w)^\top \in [H^1(\Omega)]^6 : r_{S_1}\{V\}^+ = 0 \big\}.$$

Consider the following variational inequality: *Find* $U = (u,\omega)^\top \in \mathcal{K}$ *such that the spatial variational inequality*

$$B(U, V - U) + j(v) - j(u) \geq (\mathcal{G}, V - U) + \langle \varphi, r_{S_2}\{w - \omega\}^+ \rangle_{S_2} \tag{3.9}$$

holds for all $V = (v, w)^\top \in \mathcal{K}$.

Here and in what follows, the symbol $\langle \,\cdot\,, \,\cdot\, \rangle$ denotes the duality relation between the corresponding dual pairs $X^*(M)$ and $X(M)$. In particular, $\langle \,\cdot\,, \,\cdot\, \rangle_{S_2}$ in (3.9) denotes the duality relation between the spaces $[H^{-1/2}(S_2)]^3$ and $[\widetilde{H}^{1/2}(S_2)]^3$. In addition, the symbol $(\,\cdot\,, \,\cdot\,)$ will denote the inner product in the space $L_2(\Omega)$.

3.2. **Equivalence theorem.** Here we prove the following equivalence result.

Theorem 3.2. *If* $U = (u,\omega)^\top \in [H^1(\Omega)]^6$ *is a solution of problem* (A), *then* U *is a solution of the variational inequality* (3.9), *and vice versa.*

Proof. Let $U = (u,\omega)^\top \in [H^1(\Omega)]^6$ be a solution of problem (A), and $V = (v, w)^\top \in \mathcal{K}$. By virtue of the interior regularity theorems (see [8]), we have $U \in [H^2(\Omega')]^6$ for every domain $\overline{\Omega'} \subset \Omega$. Hence, by means of Green's formula (2.11), we get

$$\big\langle \{T(\partial,n)U\}^+, \{V-U\}^+ \big\rangle_S + (\mathcal{G}, V-U) = B(U, V-U) \quad \forall V = (v,w)^\top \in \mathcal{K}.$$

Since $r_{S_1}\{V-U\}^+ = 0$, we have

$$\big\langle r_{S_2}\{T(\partial,n)U\}^+, r_{S_2}\{V-U\}^+ \big\rangle_{S_2} = B(U, V-U) - (\mathcal{G}, V-U).$$

Taking into account the boundary conditions (3.3) and (3.4) this equation can be rewritten as

$$\begin{aligned} &\langle r_{S_2}\{(\mathcal{T}U)_n\}^+, r_{S_2}\{v_n - u_n\}^+\rangle_{S_2} + j(v) - j(u) = \\ &= B(U, V-U) - (\mathcal{G}, V-U) - \big\langle \varphi, r_{S_2}\{w-\omega\}^+ \big\rangle_{S_2} + j(v) - j(u). \end{aligned} \tag{3.10}$$

Let us show that, if conditions (3.5), (3.6) and (3.7) are fulfilled, then the left-hand side of the above equality is nonnegative. Indeed, let condition (3.5) be fulfilled. Then the left-hand side of the equality (3.10) can be rewritten as

$$\begin{aligned} &\big\langle r_{S_2}\{(\mathcal{T}U)_n\}^+, r_{S_2}\{v_n\}^+ \big\rangle_{S_2} + \big\langle -g_1, r_{S_2}[\{v_n\}^+]^+ \big\rangle_{S_2} + \\ &\quad + \big\langle g_2, r_{S_2}[\{v_n\}^+]^- \big\rangle_{S_2} = \\ &= \big\langle r_{S_2}\{(\mathcal{T}U)_n\}^+, r_{S_2}[\{v_n\}^+]^+ - r_{S_2}[\{v_n\}^+]^- \big\rangle_{S_2} + \\ &\quad + \big\langle -g_1, r_{S_2}[\{v_n\}^+]^+ \big\rangle_{S_2} + \big\langle g_2, r_{S_2}[\{v_n\}^+]^- \big\rangle_{S_2} = \\ &= \big\langle r_{S_2}\{(\mathcal{T}U)_n\}^+ - g_1, r_{S_2}[\{v_n\}^+]^+ \big\rangle_{S_2} + \\ &\quad + \big\langle g_2 - r_{S_2}\{(\mathcal{T}U)_n\}^+, r_{S_2}[\{v_n\}^+]^- \big\rangle_{S_2} \geq 0. \end{aligned}$$

Assume now that condition (3.6) is fulfilled. Then $r_{S_2}[\{u_n\}^+]^+ = r_{S_2}\{u_n\}^+$ and $r_{S_2}[\{u_n\}^+]^- = 0$. Therefore

$$\begin{aligned} &\big\langle r_{S_2}\{(\mathcal{T}U)_n\}^+, r_{S_2}[\{v_n\}^+]^+ - r_{S_2}[\{v_n\}^+]^- - r_{S_2}\{u_n\}^+ \big\rangle_{S_2} + \\ &\quad + \big\langle -g_1, r_{S_2}[\{v_n\}^+]^+ - r_{S_2}\{u_n\}^+ \big\rangle_{S_2} + \big\langle g_2, r_{S_2}[\{v_n\}^+]^- \big\rangle_{S_2} = \\ &= \big\langle r_{S_2}\{(\mathcal{T}U)_n\}^+ - g_1, r_{S_2}[\{v_n\}^+]^+ - \\ &\quad - r_{S_2}\{u_n\}^+ \big\rangle_{S_2} + \big\langle g_2 - r_{S_2}\{(\mathcal{T}U)_n\}^+, r_{S_2}[\{u_n\}^+]^- \big\rangle_{S_2} \geq 0. \end{aligned}$$

Finally, let the condition (3.7) be fulfilled. Then $r_{S_2}[\{u_n\}^+]^+ = 0$ and $r_{S_2}[\{u_n\}^+]^- = -r_{S_2}\{u_n\}^+$. Therefore

$$\begin{aligned} &\big\langle r_{S_2}\{(\mathcal{T}U)_n\}^+, r_{S_2}[\{v_n\}^+]^+ - r_{S_2}[\{v_n\}^+]^- - r_{S_2}\{u_n\}^+ \big\rangle_{S_2} + \\ &\quad + \big\langle -g_1, r_{S_2}[\{v_n\}^+]^+ \big\rangle_{S_2} + \big\langle g_2, r_{S_2}[\{v_n\}^+]^- + r_{S_2}\{u_n\}^+ \big\rangle_{S_2} = \\ &= \big\langle g_2 - r_{S_2}\{(\mathcal{T}U)_n\}^+, r_{S_2}[\{v_n\}^+]^- + r_{S_2}\{u_n\}^+ \big\rangle_{S_2} + \\ &\quad + \big\langle g_2 - g_1, r_{S_2}[\{v_n\}^+]^+ \big\rangle_{S_2} \geq 0. \end{aligned}$$

Taking into account the above relations, we obtain

$$B(U,V-U)+j(v)-j(u)\ge(\mathcal{G},V-U)+\langle\varphi,r_{S_2}\{w-\omega\}^+\rangle_{S_2}$$
$$\forall\,V=(v,w)^\top\in\mathcal{K}.$$

The first part of Theorem 3.1 is proved.

Now, let $U=(u,\omega)^\top\in\mathcal{K}$ be a solution of the variational inequality (3.9). Substitute $U\pm\Phi$ for V in (3.9) with an arbitrary $\Phi\in[\mathcal{D}(\Omega)]^6$. We obtain

$$B(U,\Phi)=(\mathcal{G},\Phi)\quad\forall\Phi\in[\mathcal{D}(\Omega)]^6,$$

which implies that U is a weak solution of equation (3.1). Again, by virtue of the interior regularity theorems (see [8]), almost everywhere in the domain Ω the equation (3.1) is satisfied. Thus taking into account the fact that $r_{S_1}\{V-U\}^+=0$ for all $V=(v,w)^\top\in\mathcal{K}$, Green's formula (2.11) yields

$$\big\langle r_{S_2}\{T(\partial,n)U\}^+,r_{S_2}\{V-U\}^+\big\rangle_{S_2}+j(v)-j(u)\ge \tag{3.11}$$
$$\ge\big\langle\varphi,r_{S_2}\{w-\omega\}^+\big\rangle_{S_2}\quad\forall\,V=(v,w)^\top\in\mathcal{K}.$$

Choose $V=(v,w)^\top\in\mathcal{K}$ such that $r_{S_2}\{v\}^+=r_{S_2}\{u\}^+$ and $r_{S_2}\{w\}^+=r_{S_2}\{\omega\}^+\pm r_{S_2}\psi$, where $\psi\in[\widetilde{H}^{1/2}(S_2)]^3$ is an arbitrary vector function. Then (3.11) yields

$$\big\langle r_{S_2}\{\mathcal{M}U\}^+-\varphi,r_{S_2}\psi\big\rangle_{S_2}=0\quad\forall\,\psi\in[\widetilde{H}^{1/2}(S_2)]^3.$$

Hence on S_2

$$r_{S_2}\{\mathcal{M}U\}^+=\varphi$$

and therefore (3.3) holds. Condition (3.2) is satisfied automatically, since $U\in\mathcal{K}$. Further, choose $V=(v,w)^\top\in\mathcal{K}$ such that

$$r_{S_2}\{w\}^+=r_{S_2}\{\omega\}^+,\;\;r_{S_2}\{v_n\}^+=r_{S_2}\{u_n\}^+,\;\;\text{and}\;\;r_{S_2}\{v_s\}^+=r_{S_2}\{u_s\}^+\pm\psi,$$

where $\psi\in[\widetilde{H}^{1/2}(S_2)]^3$ is an arbitrary vector function. Then from (3.11) we obtain

$$\big\langle r_{S_2}\{(\mathcal{T}U)_s\}^+,r_{S_2}\psi\big\rangle_{S_2}=0\quad\forall\,\psi\in[\widetilde{H}^{1/2}(S_2)]^3,$$

which proves (3.4). It remains to prove conditions (3.5), (3.6) and (3.7). Taking into account the above obtained relations, from (3.11) it follows that for all $V=(v,w)^\top\in\mathcal{K}$

$$\big\langle r_{S_2}\{(\mathcal{T}U)_n\}^+,r_{S_2}\{v_n-u_n\}^+\big\rangle_{S_2}+j(v)-j(u)\ge0.$$

Hence we have

$$\big\langle r_{S_2}\{(\mathcal{T}U)_n\}^+-g_1,r_{S_2}[\{v_n\}^+]^+\big\rangle_{S_2}+$$
$$+\big\langle g_2-r_{S_2}\{(\mathcal{T}U)_n\}^+,r_{S_2}[\{v_n\}^+]^-\big\rangle_{S_2}-$$
$$-\big\langle r_{S_2}\{(\mathcal{T}U)_n\}^+-g_1,r_{S_2}[\{u_n\}^+]^+\big\rangle_{S_2}-$$

$$- \left\langle g_2 - r_{S_2}\{(\mathcal{T}U)_n\}^+, r_{S_2}[\{u_n\}^+]^- \right\rangle_{S_2} \geq 0 \quad \forall\, V = (v,w)^\top \in \mathcal{K}. \quad (3.12)$$

Further, let $\lambda > 0$ be an arbitrary number and take $\pm\lambda\psi$ for $\{v_n\}^+$ in (3.12), where $\psi \in \widetilde{H}^{1/2}(S_2)$ is an arbitrary scalar function and $\psi \geq 0$. Then, from (3.12) we easily derive

$$g_1 \leq r_{S_2}\{(\mathcal{T}U)_n\}^+ \leq g_2 \quad (3.13)$$

and

$$\begin{aligned} &\left\langle r_{S_2}\{(\mathcal{T}U)_n\}^+ - g_1, r_{S_2}[\{u_n\}^+]^+ \right\rangle_{S_2} + \\ &+ \left\langle g_2 - r_{S_2}\{(\mathcal{T}U)_n\}^+, r_{S_2}[\{u_n\}^+]^- \right\rangle_{S_2} \leq 0. \end{aligned} \quad (3.14)$$

Taking into account inequality (3.13), from (3.14) we obtain

$$\left\langle r_{S_2}\{(\mathcal{T}U)_n\}^+ - g_1, r_{S_2}[\{u_n\}^+]^+ \right\rangle_{S_2} = 0$$

and

$$\left\langle g_2 - r_{S_2}\{(\mathcal{T}U)_n\}^+, r_{S_2}[\{u_n\}^+]^- \right\rangle_{S_2} = 0.$$

These equalities show that conditions (3.5), (3.6) and (3.7) are fulfilled, which completes the proof. □

3.3. **Existence and uniqueness theorems.** Here we investigate the so-called coercive case where the measure of the Dirichlet part of the boundary is of positive measure, i, e., meas $S_1 > 0$.

We prove the following uniqueness theorem.

Theorem 3.3. *The variational inequality* (3.9) *has at most one solution.*

Proof. Let $U = (u,\omega)^\top \in \mathcal{K}$ and $\widetilde{U} = (\widetilde{u},\widetilde{\omega})^\top \in \mathcal{K}$ be two solutions of the variational inequality (3.9). Then

$$B(U,\widetilde{U}-U) + j(\widetilde{u}) - j(u) \geq (\mathcal{G},\widetilde{U}-U) + \left\langle \varphi, r_{S_2}\{\widetilde{\omega}-\omega\} \right\rangle_{S_2}$$

and

$$B(\widetilde{U},U-\widetilde{U}) - j(\widetilde{u}) + j(u) \geq (\mathcal{G},U-\widetilde{U}) + \left\langle \varphi, r_{S_2}\{\omega-\widetilde{\omega}\} \right\rangle_{S_2}.$$

By summing these inequalities and applying property (2.9) we easily derive

$$B(U-\widetilde{U},U-\widetilde{U}) = 0.$$

Therefore, $U - \widetilde{U} = ([a\times x]+b, a)$ in Ω, where $a,b \in \mathbb{R}^3$ are arbitrary constant vectors (see Lemma 2.1). Since $r_{S_1}\{U-\widetilde{U}\}^+ = 0$, we conclude $a = b = 0$, i.e., $U = \widetilde{U}$ in Ω. □

To prove the existence result let us introduce the following functional on the set $\mathcal{K}$:

$$\begin{gathered} \mathcal{J}(V) = \frac{1}{2}B(V,V) + j(v) - (\mathcal{G},V) - \left\langle \varphi, r_{S_2}\{w\}^+ \right\rangle_{S_2} \\ \forall\, V = (v,w)^\top \in \mathcal{K}. \end{gathered} \quad (3.15)$$

Due to the symmetry property of the form $B(U,V)$ it is easy to show that the variational inequality (3.9) is equivalent to the minimization problem for functional (3.15) on the closed convex set $\mathcal{K}$, i.e., the variational inequality (3.9) is equivalent to the following minimizing problem:

Find $V_0 \in \mathcal{K}$ such that

$$\mathcal{J}(V_0) = \inf_{V\in\mathcal{K}} \mathcal{J}(V).$$

According to the general theory of variational inequalities (see [5], [8], [16]), the solvability of the minimization problem immediately follows from the coercivity of the functional $\mathcal{J}$, i.e., from the property

$$\mathcal{J}(V) \longrightarrow \infty, \quad \text{when} \quad \|V\|_{[H^1(\Omega)]^6} \longrightarrow \infty, \quad V \in \mathcal{K}.$$

Since the bilinear form $B(U,V)$ is coercive on the set $\mathcal{K}$ (see Remark 2.2) and the inequality $j(v) \geq 0$ holds, using the trace theorem it is easy to see that

$$\mathcal{J}(V) \geq c_1\|V\|^2_{[H^1(\Omega)]^6} - c_2\|V\|_{[H^1(\Omega)]^6} \quad \forall\, V \in \mathcal{K},$$

where c_1 and c_2 are some positive constants independent of V. This inequality shows that functional (3.15) is coercive on $\mathcal{K}$ in the above mentioned sense. Therefore, from the equivalence Theorem 3.1 we have the following existence result for the Problem (A).

Theorem 3.4. *Let meas* $S_1 > 0$, $\mathcal{G} \in [L_2(\Omega)]^6$, $\varphi \in [H^{-1/2}(S_2)]^3$, $g_i \in L_\infty(S_2)$ $(i = 1,2)$ *and* $g_1 \leq 0 \leq g_2$. *Then the Problem* (A) *has a unique solution in* $[H^1(\Omega)]^6$.

3.4. **Lipschitz continuous dependence on the problem data.** Let $U = (u,\omega)^\top \in [H^1(\Omega)]^6$ and $\widetilde{U} = (\widetilde{u},\widetilde{\omega})^\top \in [H^1(\Omega)]^6$ be the two solutions of Problem (A) corresponding to the data $\mathcal{G},\varphi,g_i$ and $\widetilde{\mathcal{G}},\widetilde{\varphi},\widetilde{g}_i$ $(i = 1,2)$, respectively. Thus we have two variational inequalities of type (3.9), the first one for U and the second one for $\widetilde{U}$. Substitute $V = \widetilde{U}$ in the first one and $V = U$ in the second one, and take thier sum to obtain

$$\begin{aligned} -B(U-\widetilde{U},U-\widetilde{U}) + j(\widetilde{u}) - \widetilde{j}(\widetilde{u}) + \widetilde{j}(u) - j(u) \geq \\ \geq -(\mathcal{G}-\widetilde{\mathcal{G}},U-\widetilde{U}) - \left\langle \varphi-\widetilde{\varphi}, r_{S_2}\{\omega-\widetilde{\omega}\}^+\right\rangle_{S_2}, \end{aligned}$$

where

$$\begin{aligned} j(\widetilde{u}) - \widetilde{j}(\widetilde{u}) + \widetilde{j}(u) - j(u) = \int\limits_{S_2} \big(g_1-\widetilde{g_1}\big)\big([\{u_n\}^+]^+ - [\{\widetilde{u_n}\}^+]^+\big)dS \,+ \\ + \int\limits_{S_2} \big(g_2-\widetilde{g_2}\big)\big([\{\widetilde{u_n}\}^+]^- - [\{u_n\}^+]^-\big)dS. \end{aligned}$$

Taking into account this inequality, the inclusions $U \in \mathcal{K}, \widetilde{U} \in \mathcal{K}$ and the strong Korn's inequality (see Remark 2.2), from the preceding inequality we obtain

$$\|U-\widetilde{U}\|_{[H^1(\Omega)]^6} \le C\big(\|\mathcal{G}-\widetilde{\mathcal{G}}\|_{[L_2(\Omega)]^6}+\|\varphi-\widetilde{\varphi}\|_{[H^{-1/2}(S_2)]^3}+ \\ +\|g_1-\widetilde{g}_1\|_{L_2(S_2)}+\|g_2-\widetilde{g}_2\|_{L_2(S_2)}\big),$$

where positive constant C does not depend on U and $\widetilde{U}$, and on the data of the problems under consideration. This estimate shows the desired Lipschitz continuous dependence of solutions on the data of the problem.

4. The Semicoercive Case

Let $S_1=\emptyset$, $S_2=S$, $\mathcal{G}\in[L_2(\Omega)]^6$, $\varphi\in[H^{-1/2}(S)]^3$, $g_i\in L_\infty(S)$ $(i=1,2)$ and $g_1\le 0\le g_2$. Consider the boundary contact problem.

Problem (B). Find a vector-function $U=(u,\omega)^\top\in[H^1(\Omega)]^6$, which is a weak solution of equation (3.1) satisfying the following boundary conditions:

$$\{\mathcal{M}U\}^+=\varphi, \quad \text{on} \quad S;$$
$$\{(\mathcal{T}U)_s\}^+=0, \quad \text{on} \quad S;$$
$$if \;\; g_1<\{(\mathcal{T}U)_n\}^+<g_2, \quad \text{then} \quad \{u_n\}^+=0, \quad \text{on} \quad S;$$
$$if \;\; g_1=\{(\mathcal{T}U)_n\}^+, \quad \text{then} \quad \{u_n\}^+\ge 0, \quad \text{on} \quad S$$

and

$$if \;\; g_2=\{(\mathcal{T}U)_n\}^+, \quad \text{then} \quad \{u_n\}^+\le 0, \quad \text{on} \quad S.$$

This problem can be reformulated as a variational inequality.

Find vector-function $U=(u,\omega)^\top\in[H^1(\Omega)]^6$, such that the variational inequality

$$B(U,V-U)+j(v)-j(u)\ge(\mathcal{G},V-U)+\big\langle\varphi,\{w-\omega\}^+\big\rangle_S \tag{4.1}$$

holds for all $V=(v,w)^\top\in[H^1(\Omega)]^6$, where

$$j(v)=\int\limits_S\big(-g_1[\{v_n\}^+]^+ + g_2[\{v_n\}^+]^-\big)\,dS.$$

Let $U=(u,\omega)^\top\in[H^1(\Omega)]^6$ be a solution of the variational inequality (4.1). Substitute first $V=0$ and afterwards $V=2U$ in inequality (4.1) to obtain

$$B(U,U)+j(u)=(\mathcal{G},U)+\big\langle\varphi,\{\omega\}^+\big\rangle_S \tag{4.2}$$

With the help of (4.2) we derive from (4.1)

$$B(U,V)+j(v)\ge(\mathcal{G},V)+\big\langle\varphi,\{w\}^+\big\rangle_S \quad \forall\, V=(v,w)^\top\in[H^1(\Omega)]^6. \tag{4.3}$$

Denote by $\mathcal{R}$ the set of solutions of the equation $B(U,U)=0$ in the space $[H^1(\Omega)]^6$ (see Lemma 2.1),

$$\mathcal{R}:=\big\{\xi=(\varrho,a)^\top\in[H^1(\Omega)]^6:\varrho=([a\times x]+b),\;\; a,b\in\mathbb{R}^3\big\}.$$

Substitute $\pm\xi \in \mathcal{R}$ for V in (4.3)

$$-\int\limits_S \left(-g_1[\{\varrho_n\}^+]^- + g_2[\{\varrho_n\}^+]^+\right) dS \le (\mathcal{G},\xi) + \langle\varphi, a\rangle_S \le$$

$$\le \int\limits_S \left(-g_1[\{\varrho_n\}^+]^+ + g_2[\{\varrho_n\}^+]^-\right) dS. \tag{4.4}$$

This inequality is the necessary condition for the variational inequality (4.1) to be solvable.

Assume that we have a strict inequality in (4.4). Then by taking into consideration that the space $\mathcal{R}$ has finite dimension, $\dim \mathcal{R} = 6$, it is easy to see that (4.4) is equivalent to the relation

$$\int\limits_S \left(-g_1[\{\varrho_n\}^+]^+ + g_2[\{\varrho_n\}^+]^-\right) dS - (\mathcal{G},\xi) - \langle\varphi, a\rangle_S \ge C\|\xi\|_{[L_2(\Omega)]^6} \tag{4.5}$$

with some positive constant C and $\forall\xi \in \mathcal{R}$.

Let $\mathcal{P}_\mathcal{R}$ be an orthogonal projection operator of the space $[H^1(\Omega)]^6$ on $\mathcal{R}$ in the sense of the space $[L_2(\Omega)]^6$, i.e., $\forall V \in [H^1(\Omega)]^6 : V = W + \xi$, where $\xi = (\varrho, a)^\top = \mathcal{P}_\mathcal{R} V \in \mathcal{R}$ and

$$W = (\eta,\varsigma)^\top \in \mathcal{R}^\perp := \left\{U \in [H^1(\Omega)]^6 : (U,\xi) = 0 \quad \forall\, \xi \in \mathcal{R}\right\}.$$

Due to inequality (2.9) and Lemma 5.1 in [11] the bilinear form B then will be semicoercive , i.e., there is a positive constant α_0 such that

$$B(V,V) \ge \alpha_0 \,\|V - \mathcal{P}_\mathcal{R} V\|^2_{[H^1(\Omega)]^6} = \|W\|^2_{[H^1(\Omega)]^6} \quad \forall\, V \in [H^1(\Omega)]^6. \tag{4.6}$$

It is easy to see that the norm $\|V\|_{[H^1(\Omega)]^6}$ is equivalent the norm $\|W\|_{[H^1(\Omega)]^6} + \|\xi\|_{[L_2(\Omega)]^6}$. Therefore for all $V = W + \xi \in [H^1(\Omega)]^6$ with $W = (\eta,\varsigma)^\top$ and $\xi = (\varrho, a)^\top$, due to (4.6) and (4.5) we have

$$\begin{aligned}\mathcal{J}(V) = \mathcal{J}(W+\xi) = \frac{1}{2}B(W,W) + j(\eta+\varrho) - j(\varrho) - (\mathcal{G}, W) -\\ - \left\langle\varphi, \{\varsigma\}^+\right\rangle_S + j(\varrho) - (\mathcal{G},\xi) - \langle\varphi, a\rangle_S \ge C\|W\|^2_{[H^1(\Omega)]^6} +\\ +C_1\|\xi\|_{[L_2(\Omega)]^6} - C_2\|W\|_{[H^1(\Omega)]^6} + j(\eta+\varrho) - j(\varrho),\end{aligned}$$

for some positive constants C, C_1 and C_2.

Let us now estimate $j(\eta+\varrho) - j(\varrho)$. We have

$$j(\eta+\varrho) - j(\varrho) = \int\limits_S \Big\{ -g_1\left([\{\eta_n+\varrho_n\}^+]^+ - [\{\varrho_n\}^+]^+\right) +$$

$$+g_2\left([\{\eta_n+\varrho_n\}^+]^- - [\{\varrho_n\}^+]^-\right)\Big\} dS \le \int\limits_S \left(-g_1[\{\eta_n\}^+]^+ + g_2[\{\eta_n\}^+]^-\right) dS \le$$

$$\leq C\int\limits_S \left([\{\eta_n\}^+]^+ + [\{\eta_n\}^+]^-\right) dS = C\int\limits_S \left|\{\eta_n\}^+\right| dS \leq \|W\|_{[H^1(\Omega)]^6}. \quad (4.7)$$

Analogously, we obtain

$$j(\varrho)-j(\eta+\varrho)\leq \int\limits_S \left(-g_1[\{\eta_n\}^+]^- + g_2[\{\eta_n\}^+]^+\right) dS \leq C\|W\|_{[H^1(\Omega)]^6}. \quad (4.8)$$

From estimates (4.7) and (4.8) we have

$$j(\eta+\varrho)-j(\varrho)\geq -C\|W\|_{[H^1(\Omega)]^6},$$

where the positive constant C is independent of η and ϱ.

Taking into account this inequality, we finally obtain

$$\mathcal{J}(V)\geq C\|W\|^2_{[H^1(\Omega)]^6} + C_1\|\xi\|_{[L_2(\Omega)]^6} - C_2\|W\|_{[H^1(\Omega)]^6},$$

whence it follows that

$$\mathcal{J}(V)\longrightarrow\infty, \quad \text{as} \quad \|V\|_{[H^1(\Omega)]^6}\longrightarrow\infty, \quad V\in[H^1(\Omega)]^6.$$

Thus the functional $\mathcal{J}$ is coercive and the minimization problem for this functional is solvable. Consequently, the corresponding variational inequality (4.1) is solvable as well (see [5], [17]). Further, for two possible solutions U and $\widetilde{U}$ of the class $[H^1(\Omega)]^6$ to the variational inequality (4.1), we easily derive $B(U-\widetilde{U},U-\widetilde{U})=0$, which implies $U-\widetilde{U}=([a\times x]+b,a)\,, a,b\in\mathbb{R}^3$. So we have the following existence and uniqueness theorem.

Theorem 4.1. *Let $S_1=\emptyset$, $\mathcal{G}\in[L_2(\Omega)]^6$, $\varphi\in[H^{-1/2}(S)]^3$, $g_i\in L_\infty(S)$ $(i=1,2)$, $g_1\leq 0\leq g_2$ and the condition* (4.5) *be fulfilled. Then the variational inequality* (4.1) *is solvable in the space* $[H^1(\Omega)]^6$. *Moreover, solutions are defined modulo generalized rigid displacement vectors.*

Remark 4.2. Let the boundary $S:=\partial\Omega$ fall into three mutually disjoint parts S_1, S_T and S_2, such that $\overline{S}_1\cup\overline{S}_T\cup\overline{S}_2=S, \overline{S}_1\cap\overline{S}_2=\emptyset$. Analogously to the coercive case, we can study the problem, when on the part of the boundary S_T the traction boundary condition

$$r_{S_T}\{T(\partial,n)U\}^+=Q$$

is given, where $Q\in[H^{-1/2}(S_T)]^6$. The conditions of the parts S_1 and S_2 in this case remain the same as in Problem (A).

In that case, we have the following variational inequality:

Find $U=(u,\omega)^\top\in\mathcal{K}$ such that $\forall V=(v,w)^\top\in\mathcal{K}$

$$\begin{gathered}B(U,V-U)+j(v)-j(u)\geq(\mathcal{G},V-U)+\\+\big\langle Q,r_{S_T}\{V-U\}^+\big\rangle_{S_T}+\big\langle\varphi,r_{S_2}\{w-\omega\}^+\big\rangle_{S_2},\end{gathered}$$

where the functional j is defined by formula (3.8).

The proof of the existence and uniqueness theorem for this case can be carried out by the word for word arguments.

Remark 4.3. Analogously to the non-coercive case, we can study the problem, when on the part of the boundary S_1 instead of the Dirichlet condition (3.3) the traction boundary condition

$$r_{S_1}\{T(\partial,n)U\}^+ = Q$$

is given, where $Q \in [\widetilde{H}^{-1/2}(S_1)]^6$.

In that case, instead of (4.1) we have the following variational inequality: *Find* $U = (u,\omega)^\top \in [H^1(\Omega)]^6$ *such that* $\forall V = (v,w)^\top \in [H^1(\Omega)]^6$

$$B(U,V-U) + j(v) - j(u) \geq (\mathcal{G}, V-U) + \\ + \langle r_{S_1}Q, r_{S_1}\{V-U\}^+\rangle_{S_1} + \langle r_{S_2}\varphi, r_{S_2}\{w-\omega\}^+\rangle_{S_2}, \tag{4.9}$$

where $\varphi \in [\widetilde{H}^{-1/2}(S_2)]^3$.

The necessary condition for the variational inequality (4.9) to be solvable now reads as

$$-\int\limits_{S_2} \big(-g_1[\{\varrho_n\}^+]^- + g_2[\{\varrho_n\}^+]^+\big)\,dS \leq (\mathcal{G},\xi) + \langle r_{S_2}\varphi, a\rangle_{S_2} + \\ + \langle r_{S_1}Q, r_{S_1}\{\xi\}^+\rangle_{S_1} \leq \int\limits_{S_2} \big(-g_1[\{\varrho_n\}^+]^+ + g_2[\{\varrho_n\}^+]^-\big)\,dS,$$

where $\xi = (\varrho, a)^\top \in \mathcal{R}$ is an arbitrary generalized rigid displacement vector.

Let us assume that we have the strict inequality in this necessary condition. Since $\mathcal{R}$ is finite-dimensional we can show that the strict inequality is equivalent to the condition: there is a positive constant C such that for all $\xi \in \mathcal{R}$ the following inequality holds

$$\int\limits_{S_2} \big(-g_1[\{\varrho_n\}^+]^+ + g_2[\{\varrho_n\}^+]^-\big)\,dS - (\mathcal{G},\xi) - \\ - \langle r_{S_2}\varphi, a\rangle_{S_2} - \langle r_{S_1}Q, r_{S_1}\{\xi\}^+\rangle_{S_1} \geq C\|\xi\|_{[L_2(\Omega)]^6}.$$

This is a sufficient condition for the solvability of the variational inequality (4.9).

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