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ON THE METHOD OF MONOTONICITY IN PROBLEMS WITH AN IMPLICIT OBSTACLE

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Abstract. In the present work we consider the variational and quasi-variational inequalities for the second order elliptic, coercive partial differential operator. These problems are, in fact, the quasi-variational inequalities with a unilateral implicit obstacle considered for the Dirichlet and Neumann boundary conditions. Such kind of problems are encountered in the control theory. In investigating these problems we have used the method of monotonicity when the obstacle operator is not monotone in an ordinary sense. The existence of minimal and maximal solutions is proved. It is stated that these solutions minimize and maximize the "control energy".

რეზიუმე. ნაშრომში განხილულია ვარიაციული და კვაზივარიაციული უტოლობები მეორე რიგის ელიფსური კოერციტული კარძო წარმოებულიანი ოპერატორისათვის. ეს ამოცანები წარმოადგენენ ვარიაციულ უტოლობებს ცალმხრივი არაცხადი წინაღობით, რომლებიც განხილული არიან დირიხლესა და ნეიმანის სასაზღვრო პირობების მიმართ. ასეთი ამოცანების გამოკვლევაში გამოყენებულია მონოტონურობის მეთოდი იმ შემთხვევაში, როცა წინაღობის ოპერატორი არ არის მონოტონური ჩვეულებრივი აზრით. დამტკიცებულია მინიმალური და მაქსიმალური ამონახსნების არსებობა. ეს ამონახსნები მინიმალურს და მაქსიმალურს ხდის აგრეთვე მართვის ენერგიასაც .

1. Introduction

The problems we consider here trace back to the control theory. They are formulated in the form of the elliptic variational and quasi-variational inequalities. We are concerned only with the question of the existence of solutions and present a somewhat different version of the method of monotonicity in the problems of an implicit obstacle. Let us make some definitions.

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Let Ω be a bounded domain in $\mathbb{R}^n$, Γ be the boundary of $\Omega : \Gamma = \partial\Omega$, $\Gamma \in C^1$, ν be the outward unit normal vector to Ω; moreover, let $H^s(\Omega)$ and $H^s(\Gamma)$ be the real Sobolev spaces. Define $\widetilde{H}^1(\Omega) := \{v \in H^1(\Omega) : \ v|_\Gamma = 0\}$. Further, suppose $V = \widetilde{H}^1(\Omega)$, or $V = H^1(\Omega)$, and V' denotes the dual space of the space V. The norm in these spaces will be denoted by $\|\cdot\|_V$ and $\|\cdot\|_{V'}$, respectively:

$$\|\varphi\|_{V'} = \sup_{\psi \in V} \frac{\langle \varphi\,,\,\psi \rangle}{\|\psi\|_V}, \qquad \forall \varphi \in V',$$

where $\langle \cdot\,,\cdot \rangle$ is the duality brackets between V and V'.

Define the bilinear form on the space $H^1(\Omega) \times H^1(\Omega)$ as follows:

$$\begin{gathered} a(u,v) = \sum_{i,j=1}^{n} \int\limits_{\Omega} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dx + \sum_{i=1}^{n} \int\limits_{\Omega} a_i \frac{\partial u}{\partial x_i} v dx + \int\limits_{\Omega} a_0 uv dx, \\ a_{ij}, a_i, a_0 \in L^\infty(\mathbb{R}^n), \qquad \sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \geq \alpha_0 |\xi|^2, \\ a_0(x) \geq a^0, \quad a^0 = \text{const} > 0. \end{gathered} \tag{1.1}$$

Suppose that the form $a(u,v)$ is coercive:

$$a(u,u) \geq \alpha ||u||^2_{1,\Omega}, \quad \forall u \in H^1(\Omega), \tag{1.2}$$

where $\|\cdot\|_{1,\Omega}$ is the norm in $H^1(\Omega)$.

Define the following operators:

$$\begin{aligned} A(x,\partial) &= -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial}{\partial x_i}\right) + \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} + a_0, \\ \frac{\partial}{\partial \nu_A} &= \sum_{i,j=1}^{n} a_{ij}\nu_j \frac{\partial}{\partial x_i}. \end{aligned} \tag{1.3}$$

As is known, if $u \in H^1(\Omega)$ and $Au \in L_2(\Omega)$, then $\dfrac{\partial u}{\partial \nu_A} \in H^{-\frac{1}{2}}(\Gamma)$, and the following Green's formula

$$a(u,v) = \Big\langle \frac{\partial u}{\partial \nu_A}, v \Big\rangle_\Gamma + \int\limits_{\Omega} Auv dx, \quad \forall v \in H^1(\Omega) \tag{1.4}$$

is valid. Here $\langle \cdot\,,\cdot \rangle_\Gamma$ is the relation of duality between $H^{\frac{1}{2}}(\Gamma)$ and $H^{-\frac{1}{2}}(\Gamma)$.

Introduce the order in the spaces V and V'.

$$\begin{gathered} If\;\; u, v \in V, \quad then \\ u \geq v \quad \Leftrightarrow \quad u \geq v \quad a.e.\ in\ \Omega; \\ If\;\; u, v \in V', \quad then \\ u \geq v \quad \Leftrightarrow \quad \langle u\,, \varphi \rangle \geq \langle v\,, \varphi \rangle, \quad \forall \varphi \in V,\ \varphi \geq 0. \end{gathered} \tag{1.5}$$

Here we set the variational inequality with the unilateral restrictions: Find $u \in K$, such that

$$\begin{gathered} a(u, v-u) \geq \langle f\,,\, v-u \rangle, \qquad \forall v \in K, \\ K = \{ v \in V : v \geq h, \;\; in\, \Omega \}, \end{gathered} \tag{1.6}$$

where $f \in V'$ and $h \in V$.

Problem (1.6) is called the variational inequality with Dirichlet or Neumann boundary conditions correspondingly to $V = \widetilde{H}^1(\Omega)$ or $V = H^1(\Omega)$. Due to the classical theory of the variational inequalities (see [1], [2], [3], [6]), the coercivity property of the form (1.1), the closure and convexity of the set K imply a unique solvability of problem (1.6) when $f \in V'$ and $h \in V$. In this problem, h denotes an obstacle. When the set K depends on the solution u, i.e., when $K = K(u)$, then problem (1.6) is called a quasi-variational inequality. A significant role in this kind of problems play variational inequalities with an implicit obstacle such as, for example, the case $h = M(u)$ in problem (1.6). Quasi-variational inequalities were introduced by Alain Bensoussan and Jacus Luis Lions in 1973. The main tool in the investigation of problems with an implicit obstacle is the monotonicity method which is applied when the obstacle operator M is monotone. In \$3 we will consider the quasi-variational inequality and apply the monotonicity method in the case when the operator M is not monotone with respect to the order in V.

Define the linear operator

$$\begin{gathered} \mathcal{A} :\, V \longrightarrow V'; \\ \langle \mathcal{A}u\,,\, v \rangle = a(u, v), \qquad \forall v \in V. \end{gathered} \tag{1.7}$$

From formulas (1.4) and (1.3) it is clear that if $V = \widetilde{H}^1(\Omega)$, then $\mathcal{A} = A$. To demonstrate that the functional $\mathcal{A}u$ defined by means of (1.7) belongs to the space V', we have to need to notice that the form (1.1) is bounded:

$$a(u, v) \leq c\|u\|_V \|v\|_V, \quad \forall u, v \in V. \tag{1.8}$$

In this inequality we can take $c = \max\limits_{1 \leq i,j \leq n} (\|a_{ij}\|_{L^\infty}, \|a_i\|_{L^\infty}, \|a_0\|_{L^\infty})$. Thus $\mathcal{A}u \in V'$ and by the definition of the norm in V' we have

$$\|\mathcal{A}u\|_{V'} \leq c\|u\|_V, \quad \forall u \in V. \tag{1.9}$$

By definition (1.7), inequality (1.6) can be rewritten as follows:

$$u \in K, \ \ \langle \mathcal{A}u\,,\, v-u\rangle \geq \langle f\,,\, v-u\rangle, \qquad \forall v \in K. \tag{1.10}$$

Here $\langle \cdot\,,\cdot \rangle$ is the duality brackets between V and V'.

The operator $\mathcal{A}$ is a bijective mapping from V to V'. To prove this, it is equivalent to prove that the equation

$$\langle \mathcal{A}u\,,\, v\rangle = \langle f\,,\, v\rangle, \quad \forall v \in V \tag{1.11}$$

has a unique solution $u \in V$ for every $f \in V'$. Problem (1.11) is the Dirichlet, or the Newmann problem with the homogeneous boundary conditions according to whether $V = \widetilde{H}^1(\Omega)$ or $V = H^1(\Omega)$. The unique solvability of problem (1.11) follows from the fact that it can be considered in the capacity of the variational inequaity (1.11) for $K = V$. In this case problem (1.11) has a unique solution, since V is a closed, convex set. Therefore problem (1.11) is uniquely solvable, and $u = \mathcal{A}^{-1}f$. Thus $\mathcal{A}$ possesses the inverse operator $\mathcal{A}^{-1} : V' \longrightarrow V$.

Let us get back to problem (1.6). This problem can be interpreted in various ways. The probabilistic interpretations of variational and quasi-variational inequalities as optimal stopping-time problems can be found in [1] and [2]. For different physical interpretations see [3]. Following [3], we present here only thermostatic interpretation of problem (1.6). Under the solution $u(x)$ we mean the temperature of the body Ω at the point x. Further, f denotes an external thermal energy received from the body Ω. Our goal is to keep the temperature of the body not lower than the preassigned one $h(x)$. This can be achieved by the so-called regulators which are distributed at the points of the body Ω. The regulators start their work, that is energy emission, only in the case when the temperature $u(x)$ will equate to $h(x)$. The whole energy received by the body we denote by $\mathcal{A}u$. Thus as it follows from the above-said, $\mathcal{A}u - f$ is the energy emitted by the regulators, i.e., the control energy. It can be shown that if $\mathcal{A}h - f$ remains invariable, then $\mathcal{A}u - f$ is likewise invariable. Indeed, let

$$\overline{h} = h - \mathcal{A}^{-1}f, \quad \overline{K} = \{\overline{v} \in V : \overline{v} \geq \overline{h}, \ \ in\ \Omega\}.$$

Consider the variational inequality

$$\overline{u} \in \overline{K}, \quad \langle \mathcal{A}\overline{u}\,,\, \overline{v} - \overline{u}\rangle \geq 0, \qquad \forall \overline{v} \in \overline{K} \tag{1.12}$$

and check that $\overline{u} = u - \mathcal{A}^{-1}f$, where u is the solution of problem (1.10) with the given f and h. Towards this end, we have only to notice that $(v - \mathcal{A}^{-1}f) \in \overline{K}$ for every $v \in K$, where K is defined in (1.6).

Thus if $\mathcal{A}h - f$ is invariable, then $\overline{h} = \mathcal{A}^{-1}(\mathcal{A}h - f)$ is likewise invariable, which defines $\overline{u} = \mathcal{A}^{-1}(\mathcal{A}u - f)$ uniquely by formula (1.12). Hence $\mathcal{A}u - f$ remains invariable too.

Relying on this fact, we can consider the mapping

$$\begin{aligned} \mathcal{T} :\ & V' \longrightarrow V'. \\ \mathcal{T} :\ & \mathcal{A}h - f \longrightarrow \mathcal{A}u - f. \end{aligned} \tag{1.13}$$

It is clear that this mapping is equivalent to the dependence

$$\mathcal{A}\overline{h} \longrightarrow \mathcal{A}\overline{u}$$

which is realized by means of inequality (1.12).

In Section 2 we will study the properties of the operator defined in (1.13) and use them in quasi-variational inequalities.

2. The properties of the operator $\mathcal{T}$

To study the properties of the operator $\mathcal{T}$, we will first present the lemma proven in [4] and then use the lemma from [5] (Lemma 1.2). The similar result can be found in [6].

Lemma 2.1. *Let u be a solution of problem* (1.6)*, then*

$$\begin{gathered} \mathcal{A}^{-1} f \le u \le \mathcal{A}^{-1} v, \\ \forall v \in V',\ v \ge f,\ \mathcal{A}^{-1} v \ge h, \end{gathered} \tag{2.1}$$

where f and h are from problem (1.6)*.*

Now let us prove the lemma, which is of importance in this section.

Lemma 2.2. *Let u_i be solutions of the variational inequality* (1.6) *with $h_i \in V$ and $f_i \in V'$, $i = 1, 2$, and*

$$\mathcal{A}(h_1 - h_2) \ge f_1 - f_2, \tag{2.2}$$

in the sense of (1.5)*. Then*

$$\mathcal{A}(u_1 - u_2) \ge f_1 - f_2 \quad \textit{and} \quad u_1 - u_2 \le h_1 - h_2 \tag{2.3}$$

likewise in the sense of (1.5)*.*

Proof. Consider the problem: Find $w \in V$, such that

$$\begin{gathered} a(w, v - w) \ge \langle f_1 - f_2\,,\, v - w\rangle, \quad \forall v \in V,\ v \ge h_1 - u_2, \\ w \ge h_1 - u_2. \end{gathered} \tag{2.4}$$

Since (2.2) holds, and $h_1 - h_2 \ge h_1 - u_2$, then on the basis of the right-hand side estimate (2.1) of Lemma 2.1, for problem (2.4) we can write

$$w \le h_1 - h_2.$$

Denote $z := w + u_2$ and prove that z is a solution of problem (1.6) considered for h_1 and f_1. Indeed, first we observe that $z \ge h_1$. Further, for every $v \in V$, $v \geq h_1$ we have

$$a(z, v-z) = a(w+u_2, v-w-u_2) \geq \langle f_1 - f_2\,,\, v-w-u_2\rangle +$$
$$a(u_2, v-w-u_2) \geq \langle f_1 - f_2\,,\, v-w-u_2\rangle + \langle f_2\,,\, v-w-u_2\rangle =$$
$$\langle f_1\,,\, v-z\rangle.$$

We have respectively used $v - u_2 \geq h_1 - u_2$ and $v - w \geq v - h_1 + h_2 \geq h_2$. Thus z solves problem (1.6) formulated for h_1 and f_1, and due to the unique solvability of this problem, $z = u_1$.

Consequently, $w = u_1 - u_2$, and since $\mathcal{A}u \geq f$ in problem (1.6), for problem (2.4) we obtain

$$\mathcal{A}w = \mathcal{A}(u_1 - u_2) \geq f_1 - f_2. \quad \square$$

Thus, the "control energy" $\mathcal{A}u - f$ depends monotonicaly on $\mathcal{A}h - f$.

To present a general theorem on the properties of the operator $\mathcal{T}$, it is necessary to prove one more lemma.

Lemma 2.3. *The operator* $\mathcal{A}^{-1} : V' \longrightarrow V$ *is monotone increasing with respect the order* (1.5) *in* V'*, i.e.,*

$$\mathcal{A}^{-1}v - \mathcal{A}^{-1}f \geq 0, \quad \forall v, f \in V',\ v \geq f.$$

Proof. The assertion of this lemma is the easy consequence of Lemma 2.1 with $h = \mathcal{A}^{-1}v$. $\square$

Now we can prove the above-mentioned theorem.

Theorem 2.4. *The operator* $\mathcal{T}$ *defined from* (1.13)*, is monotone increasing with respect to the order* (1.5) *which possesses the following properties:*

$$\text{(i)} \qquad 0 \leq \mathcal{T}v \leq g, \qquad \forall g, v \in V',\ g \geq 0,\ g \geq v. \tag{2.5}$$

$$\text{(ii)} \qquad \sup_{g \in V', g \leq z} \|\mathcal{T}g\|_{V'} = d(z) < +\infty, \qquad \forall z \in V'. \tag{2.6}$$

Proof. The monotonicity of $\mathcal{T}$ follows from definition (1.13) and Lemma 2.2.

The left-hand side of the estimate of claim (i) is clear since $\mathcal{A}u \geq f$ in problem (1.6). The right-hand side of the estimate is the well-known Lewy-Stampacchia inequality. In the particular case inequality (2.5) can be rewritten as

$$0 \leq \mathcal{T}v \leq v^+, \qquad \forall v \in L_2(\Omega),$$

where $v^+ = \max(v, 0)$ a. e. in Ω. This inequality has been proved in [1] and [2] for variational inequalities with the Dirichlet and Neumann boundary conditions, respectively. These proofs differ from each other. We derive (2.5) from Lemma 2.2. Indeed, let g and v satisfy the conditions of claim (i) of Theorem 2.4. Take

$$f_1 = f_2 = 0, \ \ \textit{and} \ \ h_1 = \mathcal{A}^{-1}g, \ h_2 = \mathcal{A}^{-1}v$$

in Lemma 2.2. Then it is easy to see that $u_1 = h_1$ in the above-mentioned lemma. Hence by Lemma 2.2, since $\mathcal{A}h_1 \geq \mathcal{A}h_2$, we have $g = \mathcal{A}u_1 \geq \mathcal{A}u_2 = \mathcal{T}v$.

Let us prove the second claim. Without loss of generality, we can take $f = 0$ in (1.13). Further, for g and z from (2.6) consider the problem

$$\begin{gathered} a(u, v-u) \geq 0, \quad \forall v \in V,\ v \geq \mathcal{A}^{-1}g, \\ u \in V,\ u \geq \mathcal{A}^{-1}g. \end{gathered} \tag{2.7}$$

Observe that $\mathcal{T}g = \mathcal{A}u$. By Lemma 2.3, $\mathcal{A}^{-1}z \geq \mathcal{A}^{-1}g$, and we can take $v = \mathcal{A}^{-1}z$ in (2.7) to obtain

$$a(u, u) \leq a(u, \mathcal{A}^{-1}z).$$

Since the form (1.1) is coercive and bounded,

$$\begin{gathered} \alpha \|u\|_V^2 \leq c\, \|u\|_V\, \|\mathcal{A}^{-1}z\|_V, \\ \|u\|_V \leq \frac{c}{\alpha}\, \|\mathcal{A}^{-1}z\|_V, \end{gathered}$$

where α and c are from (1.2) and (1.8), respectively. Then according to (1.10), we have

$$\|\mathcal{T}g\|_{V'} = \|\mathcal{A}u\|_{V'} \leq c\, \|u\|_V \leq \frac{c^2}{\alpha}\, \|\mathcal{A}^{-1}z\|_V. \tag{2.8}$$

Now we can assume that $d(z) = \dfrac{c^2}{\alpha}\, \|\mathcal{A}^{-1}z\|_V$.

Thus the proof is complete. □

3. The Problem with an Implicit Obstacle

In this section we consider the variational inequalities with implicit obstacles which are stated as follows:

Find $u \in K(u)$ such that

$$\begin{gathered} a(u, v-u) \geq \langle f\,,\, v-u \rangle, \qquad \forall v \in K(u), \\ K(u) = \{\, v \in V : v \geq M(u),\, \}, \end{gathered} \tag{3.1}$$

for $f \in V'$ and $M : V \to V$.

Consider the mapping

$$P := T_f M\ :\ V \longrightarrow V, \tag{3.2}$$

where $f \in V'$, and $T_f\ :\ V \to V$ realizes the dependence of the solution $u = u(h,\, f)$ on the data h with fixed f in problem (1.6), i. e.,

$$T_f\ :\ h \ \longrightarrow\ u = u(h,\, f), \quad \forall h \in V. \tag{3.3}$$

It is easy to see that every fixed point of the mapping (3.2) is a solution of problem (3.1), and vice versa. Hence a number of solutions of problem (3.1) coincides with that of fixed points of the mapping P.

As is known, the operator T_f is monotone increasing with respect to the order (1.5) in V. The results of this type can be found in [1], [2] and [5]. Therefore if M is monotone increasing in V, then P is likewise monotone increasing, and due to the theorems about fixed points of monotone increasing mappings in Banach spaces, the operator (3.2) under certain conditions has a minimal and a maximal fixed point which is a minimal and a maximal solution of problem (3.1) (see [7] and [8]).

Define

$$\mathcal{P} := \mathcal{A}P\mathcal{A}^{-1} : V' \longrightarrow V'. \tag{3.4}$$

It is easy to see that

$$\mathcal{P} = \mathcal{T}_f \mathcal{M}, \tag{3.5}$$

where

$$\begin{gathered} \mathcal{M} : V' \longrightarrow V', \quad \mathcal{T}_f : V' \longrightarrow V', \\ \mathcal{M} = \mathcal{A}M\mathcal{A}^{-1} \quad \textit{and} \quad \mathcal{T}_f = \mathcal{A}T_f\mathcal{A}^{-1}. \end{gathered} \tag{3.6}$$

Hence the fixed points of the mapping (3.2) correspond uniquely to these of the mapping (3.4). Thus the question on the solvability and on a number of solutions of problem (3.1) can be reduced to finding of a number of fixed points of the operator $\mathcal{P}$.

By definitions (1.13) and (3.3),

$$\mathcal{T}_f g = \mathcal{T}(g+f) + f, \quad \forall g \in V'. \tag{3.7}$$

holds. To describe the properties of the operator $\mathcal{T}_f$, we present the theorem which is similar to Theorem 2.4.

Theorem 3.1. *The operator* $\mathcal{T}_f : V' \to V'$ *defined from* (3.5) *is monotone increasing with respect to the order* (1.5) *and possesses the following properties:*

(i) $f \le \mathcal{T}_f v \le g, \qquad \forall g, v \in V', \quad g \ge f, \quad g \ge v. \qquad (3.8)$

(ii) *For each* $z \in V'$ *there exists such constant* $d_f(z)$, *that*

$$\|\mathcal{T}_f v\|_{V'} \le d_f(z), \qquad \forall v \in V',\ v \le z \quad \textit{and} \tag{3.9}$$

$$d_f(z_1) \ge d_f(z_2), \quad \forall z_1, \quad z_2 \in V',\ z_1 \ge z_2. \tag{3.10}$$

Due to Theorem 2.4, the assertions of this theorem become clear if we take into account (3.7). In item (ii) of Theorem 3.1 we can take $d_f(z) = d(z+f) + \|f\|_{V'}$.

Now we are able to prove the main theorem of this paper. Towards this end, we will use the technique of proving the theorems on finding fixed points of monotone increasing mappings which are introduced, for example, in [7] and [8]. But in these works the spaces, in which the monotone operators act, differ from the space V' to which we apply our proof.

Theorem 3.2. *Let in problem* (3.1) *the operator* $\mathcal{M}$ *defined from* (3.6) *be a monotone increasing mapping from* S_f *to* V'*; i.e.,*

$$\begin{gathered} u, v \in S_f,\ u \geq v \quad \Rightarrow \quad \mathcal{M}u \geq \mathcal{M}v, \\ S_f = \{v \in V' : \quad v \geq f\}. \end{gathered} \tag{3.11}$$

Then if there exists $z \in S_f$ *such that* $z \geq \mathcal{M}z$*, then the quasi-variational inequality* (3.1) *has minimal and maximal solutions* $\underline{u}$ *and* $\overline{u}$ *which satisfy the following properties:*

$$\mathcal{A}\underline{u} \leq \mathcal{A}u \leq \mathcal{A}\overline{u} \leq z. \tag{3.12}$$

Here u *is an arbitrary solution of problem* (3.1) *with* $\mathcal{A}u \leq z$*.*

Proof. From (3.12) it is clear that

$$\underline{u} \leq u \leq \overline{u} \quad \text{in} V',$$

were u is an arbitrary solution of problem (3.1) with $\mathcal{A}u \leq z$. This justifies the notation $\underline{u}$ and $\overline{u}$.

From the above reasoning it follows that the existence of a minimal and a maximal solution of the implicit obstacle problem (3.1) (without additional properties (3.12)) is equivalent to the proof that the set of fixed points of the operator P defined by means of (3.2) is not empty and possesses a minimal and a maximal element. These elements will be a minimal and a maximal solution of problem (3.1). Further, the fixed points of the operators P and $\mathcal{P}$ are linked by the law:

If

$$P\psi = \psi, \quad \text{then} \quad \mathcal{P}\varphi = \varphi \ \text{ if and only if } \ \psi = \mathcal{A}^{-1}\varphi.$$

If the operator $\mathcal{P}$ possesses minimal and maximal fixed points, then by the monotonicity of the operator $\mathcal{A}^{-1}$ they must be $\mathcal{A}\underline{u}$ and $\mathcal{A}\overline{u}$, where $\underline{u}$ and $\overline{u}$ are the minimal and maximal fixed points of P.

Thus the equivalent for the assertion of Theorem 3.2 is the proof that the operator $\mathcal{P}$, defined by (3.4) under the requirements of our theorem, possesses a minimal and a maximal element on a set of fixed points which are less than, or equal to z.

Let us prove this assertion.

First we observe that due to (3.5), (3.11) and Theorem 3.1, the operator $\mathcal{P} : S_f \to S_f$ is monotone increasing.

Define

$$\begin{aligned} S^+ &= \{v \in V' : \quad f \leq v \leq z,\ \|v\|_{V'} \leq d_f(z),\ v \leq \mathcal{P}v\}, \\ S^- &= \{v \in V' : \quad f \leq v \leq z,\ \|v\|_{V'} \leq \max(d_f(z),\ \|z\|_{V'}),\ \mathcal{P}v \leq v\}. \end{aligned}$$

We notice that $S^+ \subset S_f$ and $S^- \subset S_f$. These sets are not empty because $f \in S^+$ and $z \in S^-$. Indeed, from Theorem 3.1 it follows that since

$$\mathcal{P}f = \mathcal{T}_f \mathcal{M} f \geq f,$$
$$\mathcal{P}z = \mathcal{T}_f \mathcal{M} z \leq z, \quad \text{because of } \ z \geq f \ \text{ and } \ z \geq \mathcal{M}z.$$

Let us prove the existence of a maximal fixed point of $\mathcal{P}$.

First we prove that $\mathcal{P} : S^+ \to S^+$. Indeed, let $v \in S^+$. Then by the monotonicity of $\mathcal{P}$, $\mathcal{M}$ and d_f, with regard for (3.8) we obtain

$$f \leq \mathcal{P}f \leq \mathcal{P}v \leq \mathcal{P}z \leq z,$$
$$\mathcal{P}v \leq \mathcal{P}(\mathcal{P}v), \quad \text{since} \quad v \leq \mathcal{P}v,$$
$$\|\mathcal{P}v\|_{V'} \leq \|\mathcal{T}_f \mathcal{M} v\|_{V'} \leq d_f(\mathcal{M}v) \leq d_f(\mathcal{M}z) \leq d_f(z).$$

Thus $\mathcal{P}v \in S^+$.

Now we notice that if S^+ possesses a maximal element v^*, then v^* will be a maximal element among fixed points of $\mathcal{P}$, which are $\leq z$. Indeed, if $v^* \in S^+$, then $\mathcal{P}v^* \in S^+$ and $\mathcal{P}v^* \leq v^*$ which implies $\mathcal{P}v^* = v^*$. It is clear that each $\varphi = \mathcal{P}\varphi$ with $\varphi \leq z$ belongs to S^+ and $\varphi \leq v^*$. Therefore we have only to prove that S^+ has a maximal element. To this end, we need to prove that every ascending from S^+ chain

$$w_1 \leq w_2 \leq \cdots \leq w_n \leq w_{n+1} \leq \cdots \quad w_n \in S^+, \tag{3.13}$$

has the upper bound w^* in S^+. Then the existence of a maximal element in S^+ will be the consequence of Zorn's Lemma.

First we prove that for every chain from (3.13) the sequence of real numbers converges for each $\psi \in V$:

$$\langle w_1, \psi \rangle, \ \langle w_2, \psi \rangle, \cdots \langle w_n, \psi \rangle \cdots$$

Indeed, if $\psi \geq 0$, then

$$\langle f, \psi \rangle \leq \langle w_1, \psi \rangle \leq \langle w_2, \psi \rangle \leq \cdots \leq \langle w_n, \psi \rangle \leq \cdots \leq \langle z, \psi \rangle.$$

Hence

$$\langle f, \psi \rangle \leq \lim_{n \to \infty} \langle w_n, \psi \rangle \leq \langle z, \psi \rangle, \quad \forall \psi \geq 0. \tag{3.14}$$

In the case in which ψ is an arbitrary element of V, we can represent it as a difference of two nonnegative elements from V as follws:

$$\psi = \psi^+ - \psi^-,$$

where $\psi^+ = \max(\psi, 0)$ and $\psi^- = \max(-\psi, 0)$. The results proving that $\psi^+ \in V$ can be found in [6]. Thus

$$\lim_{n \to \infty} \langle w_n, \psi \rangle = \lim_{n \to \infty} \langle w_n, \psi^+ \rangle - \lim_{n \to \infty} \langle w_n, \psi^- \rangle.$$

Therefore we can consider on V the functional w^*:

$$\langle w^*, \psi \rangle = \lim_{n\to\infty} \langle w_n, \psi \rangle, \quad \forall \psi \in V.$$

Prove that $w^* \in S^+$. Since $|\langle w_n, \psi \rangle| \le d_f(z)\|\psi\|_V$ for every w_n from S^+, we have $|\langle w^*, \psi \rangle| \le d_f(z)\|\psi\|_V$, which means that

$$w^* \in V' \text{ and } \|w^*\|_{V'} \le d_f(z).$$

Further, (3.14) implies that $f \le w^* \le z$. Let us show that $w^* \le \mathcal{P}w^*$. It is easy to see that w^* is the least upper bound in V' of the chain $\{w_n\}$ from (3.13). Hence by the monotonicity of $\mathcal{P}$ we can write

$$\mathcal{P}w^* \ge \mathcal{P}w_n \ge w_n \text{ and } \mathcal{P}w^* \ge w^*.$$

Thus $w^* \in S^+$, and w^* is the upper bound of the chain (3.13) in S^+.

Now Zorn's Lemma allows us to conclude that S^+ possesses a maximal element v^* which is a maximal fixed point of the mapping $\mathcal{P} : S^+ \to S^+$.

Analogously we can prove the existence of a minimal fixed point v_* of the mapping $\mathcal{P}$ by using the set S^-. Now a minimal and a maximal solution of problem (3.1) can be given by the following formulas

$$\underline{u} = \mathcal{A}^{-1}v_*, \quad \overline{u} = \mathcal{A}^{-1}v^*.$$

They satisfy conditions (3.12) as well. □

As it is obvious from (3.5), a minimum and a maximum solution of problem (3.1) minimizes and maximizes the "control energy".

Finally, as an example we present here the obstacle operator $M : V \to V$ for which the operator $\mathcal{M}$ possesses the property (3.11).

Let

$$Mu = a(u, \varphi)\,\psi, \quad \text{for} \quad \varphi,\, \psi \in V,\ \varphi \ge 0,\ \mathcal{A}\psi \ge 0.$$

Since $Mu = \langle \mathcal{A}u, \varphi \rangle\,\psi$, from (3.6) it follows that $\mathcal{M}v = \langle v, \varphi \rangle \mathcal{A}\psi$ for every $v \in V'$, and in view of $\varphi \ge 0$ and $\mathcal{A}\psi \ge 0$, we conclude that $\mathcal{M} : V' \to V'$ is a monotone increasing operator.

Consider the obstacle operator

$$Mu = a(u, u)\,\psi, \quad \text{for} \quad \psi \in V,\ \ \mathcal{A}\psi \ge 0.$$

Then

$$\mathcal{M}v = \langle v, \mathcal{A}^{-1}v \rangle \mathcal{A}\psi, \quad \forall v \in V'.$$

If $f \ge 0$, then $\mathcal{M} : S_f \to S_f$ is a monotone increasing operator, where S_f is from (3.11). Indeed, if $v_1,\, v_2 \in S_f$, $v_1 \ge v_2 \ge f \ge 0$,then $\mathcal{A}^{-1}v_1 \ge \mathcal{A}^{-1}v_2 \ge \mathcal{A}^{-1}f \ge 0$ and

$$\mathcal{M}v_1 - \mathcal{M}v_2 = (\langle v_1, \mathcal{A}^{-1}v_1 \rangle - \langle v_2, \mathcal{A}^{-1}v_2 \rangle)\mathcal{A}\psi =$$
$$= \big(\langle v_1 - v_2, \mathcal{A}^{-1}v_1 \rangle + \langle v_2, \mathcal{A}^{-1}(v_1 - v_2) \rangle\big)\mathcal{A}\psi \ge 0.$$

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