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Effective codescent morphisms, amalgamations and factorization systems

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Abstract

The paper deals with the question whether it is sufficient, when investigating the problem of the effectiveness of a descent morphism, to restrict the consideration only to the descent data (C, γ, ξ), where γ lies in a certain morphism class. The notion of a factorization system and the dual to the amalgamation property in the sense of Kiss, Marki, Pröhle and Tholen play the key role in our discussion.

It is shown that a category $\mathbf{C}$ inherits from a category $\mathbf{X}$ the property that all descent morphisms are effective if either $\mathbf{X}$ is regular and $\mathbf{C}$ is a full coreflective, closed under pullbacks of certain epimorphisms, subcategory of $\mathbf{X}$ or $\mathbf{C}$ is regular, $\mathbf{X}$ has coequalizers and there exists a topological functor $\mathbf{C} \to \mathbf{X}$. This implies that in the category of topological spaces, all regular monomorphisms are effective codescent morphisms (the result of Mantovani). The same is shown to be valid also for the categories of compact Hausdorff topological spaces, normal topological spaces, Banach spaces, (quasi-)uniform spaces, and (quasi-)proximity spaces. Moreover, the effectiveness of all codescent morphisms is established for the categories of Hausdorff topological spaces and (complete) metric spaces. The internal characterization of such morphisms $p : B \to E$ is given for the category of Hausdorff topological spaces, in the case of compact B and regular E.

MSC: 18C20; 18A32; 18B30; 46M99

1. Introduction

The problem of describing effective descent morphisms in the categories dual to particular ones rests on the problem of constructing pushouts in the initial categories, which, generally speaking, is difficult in itself. Frequently, the available constructions of pushouts of any morphisms are cumbersome and inconvenient for work. However, these constructions sometimes become much simpler if we deal with the pushouts of morphisms from some special classes of monomorphisms. Hence there naturally arises the question: when we investigate the problem of the effectiveness of a descent morphism p, is it sufficient to consider only those descent data[1] (C, γ, ξ), where $p\gamma$ (resp. γ) belong to a certain morphism class $\mathcal{E}$? One of the aims of this paper is to give a positive, in a certain sense, answer to this question. It is shown that, for any category $\mathbf{C}$ with pullbacks, the following statement is valid if $\mathcal{E}$ is the part of the factorization system $(\mathcal{E}, \mathcal{M} = \mathcal{E}^{\downarrow})$ with $\mathcal{M} \subset \mathcal{M}ono\ \mathbf{C}$:

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[1] For the terminology and notation see Section 2.

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Theorem. *A descent morphism p is effective if and only if for any p' lying in $\mathcal{E}$ and being the pullback of p along some $\mathcal{M}$-morphism, and for any descent data (C', γ', ξ') with respect to p' such that $p'\gamma' \in \mathcal{E}$, there exists f' from the corresponding slice category, whose image under the comparison functor $\Phi^{p'}$ is isomorphic to (C', γ', ξ'). If $\mathcal{E}$ is stable under pullback along $\mathcal{E}$-morphisms, then the statement remains valid provided that "$p'\gamma' \in \mathcal{E}$" is replaced by "$\gamma' \in \mathcal{E}$".*

In particular, it follows that when investigating the question whether all descent morphisms of **C** are effective, we can confine the consideration only to descent morphisms p from $\mathcal{E}$ and to descent data (C, γ, ξ) with respect to p, such that $p\gamma \in \mathcal{E}$ (resp. $\gamma \in \mathcal{E}$ if the corresponding condition is imposed on $\mathcal{E}$).

Another aim of the paper consists in finding restrictions on a functor $F : \mathbf{C} \to \mathbf{X}$ with **C** and **X** being categories with pullbacks, under which the following implication holds:
(∗) *If every descent morphism of* **X** *is effective, then the same is true for* **C**.

The problem closely related to this issue is to establish when F reflects effective descent morphisms. These problems have been studied in the papers of Janelidze and Tholen [9], Mantovani [12], Roque [16] and Mesablishvili [13]. In [9], along with many other results, the effective descent morphisms of **C** are described, assuming that **C** is a full, closed under pullbacks, subcategory of **X**, whose effective descent morphisms are known. If, in addition to this, the inclusion functor $F : \mathbf{C} \to \mathbf{X}$ has a left adjoint, while the corresponding reflection is semi-left-exact in the sense of Cassidy, Hébert and Kelly [3], or, equivalently, admissible in the sense of Janelidze [7], then, as shown in [12], F reflects effective descent morphisms. According to [16], if **C** has and F preserves the coequalizers of equivalence relations and, moreover, preserves pullbacks and reflects isomorphisms, then (*) holds whenever every regular epimorphism of **X** is a descent morphism. In [13] it is shown that even without imposing the latter restriction on **X** when it has coequalizers, F reflects effective descent morphisms (resp. certain effective descent morphisms) under the same conditions on **C** and F, where the coequalizers of equivalence relations are replaced by all coequalizers (resp. by another kind of coequalizers).

In this paper, first, with the help of the theorem, the general conditions are found for F, under which (*) is valid. As is different from the above-cited works, here it is required of F to preserve not all pullbacks, but only the pullbacks of morphisms from $\mathcal{E}$, where $\mathcal{E}$ is again the part of the factorization system $(\mathcal{E}, \mathcal{M})$, but this time it is assumed that $(\mathcal{E}, \mathcal{M})$ is proper (i.e. $\mathcal{E} \subset \mathcal{E}pi\,\mathbf{C}$, $\mathcal{M} \subset \mathcal{M}ono\,\mathbf{C}$) and that $\mathcal{E}$ is stable under pullback along $\mathcal{E}$-morphisms. As to the other conditions imposed on F and mentioned as the conditions (C1)–(C3) in Section 3, we only observe that (C1) is some modification of the condition that every $\mathcal{E}$-morphism is weakly coCartesian; the condition (C2) is also formulated in terms of $\mathcal{E}$-morphisms and can be considered as a dual version of the well-known solution set condition, since, under some natural restriction, the dual to (C2) can replace the latter condition in the Freyd adjoint functor theorem; the condition (C3) requires that descent morphisms belonging to $\mathcal{E}$ be mapped to descent morphisms. There are two important particular cases where all these conditions are satisfied and therefore (*) holds:

(i) **X** is regular and **C** is a full coreflective, closed under pullbacks of $\mathcal{E}$-morphisms, subcategory of **X** (F is the inclusion functor);

(ii) **C** is regular ($\mathcal{E} = \mathcal{R}eg\mathcal{E}pi\,\mathbf{C}$), **X** has coequalizers and F is a topological functor. It follows that *every regular epimorphism of a regular topological category* **C** *is an effective descent morphism*. This generalizes the result of Mantovani [12] which deals with the case, where **C** is the category dual to the category of topological spaces.

In case (i), assuming additionally that **C** is closed under all pullbacks and applying the above-mentioned result of Mesablishvili, we easily conclude that F reflects effective descent morphisms. In this paper we show that this is true for descent morphisms (i.e. that a descent morphism p is effective whenever $F(p)$ is an effective descent morphism) in the general case of an arbitrary functor F which preserves all pullbacks and satisfies the conditions (C1), (C2). An interesting particular case is again the case (ii).

The results obtained in the paper are illustrated by examples in concrete categories. Namely, it is shown that all regular monomorphisms of the categories of compact Hausdorff topological spaces, normal topological spaces, uniform spaces, quasi-uniform spaces, proximity spaces, quasi-proximity spaces, extended pseudo-normed spaces, extended pseudo-metric spaces and Banach spaces are effective codescent morphisms, while every regular monomorphism of the category of compact metric spaces, for which there exist all pushouts along it, is also an effective codescent morphism. It is shown that every codescent morphism of the categories of Hausdorff topological spaces, metric spaces and complete metric spaces is an effective codescent morphism. Furthermore, for the category of Hausdorff topological spaces, we give the internal characterization of such morphisms $p : B \to E$ in the case of compact B and regular E. By this characterization, a closed embedding p is an effective codescent morphism if and only if for any completely separable open subsets U_1 and U_2 of B, there exist disjoint open subsets V_1 and V_2 of the space E such that $U_1 = B \cap V_1$ and $U_2 = B \cap V_2$. It is shown that for arbitrary B and regular E if we omit from the latter statement "and only if" and "completely separable", then we obtain the valid one. In particular, p is an effective codescent morphism if B is clopen (and E is regular).

As to the categories of metric spaces and complete metric spaces, we also find for them the corresponding sufficient condition: p is an effective codescent morphism if the inequality

$$d(x, y) \leq \inf_{b\in B} d(x, b) + \inf_{b\in B} d(b, y)$$

is fulfilled for any $x, y \in E \backslash B$.

Finally, note that the dual to the above-mentioned condition that $\mathcal{E}$ is stable under pullback along $\mathcal{E}$-morphisms, playing the important role in our discussion is precisely the amalgamation property in the sense of Kiss, Marki, Pröhle and Tholen [11]. The present paper gives already the second example of the usefulness of the amalgamation properties in descent theory. We encountered these properties for the first time in [21], where, in particular, codescent morphisms are described in regular categories satisfying the strong amalgamation property.

2. Effective descent morphisms and factorization systems

We begin with the needed definitions from descent theory [9].

Let $\mathbf{C}$ be a category with pullbacks, and $p : E \to B$ be its morphism. Let $\mathbb{E}$ be a morphism class which is closed under composition with isomorphisms and stable under pullback. By definition, $\mathbb{E}$-descent data with respect to p is a triple (C, γ, ξ) with $C \in \mathrm{Ob}\,\mathbf{C}$ and γ, ξ being morphisms $C \longrightarrow E$ and $E \times_B C \longrightarrow C$, respectively, such that $\gamma \in \mathbb{E}$ and the following equalities are valid (see Figs. 1 and 2):

$$\gamma\xi = \pi_1, \tag{2.1}$$

$$\xi(\gamma, 1_C) = 1_C, \tag{2.2}$$

$$\xi(1_E \times_B \pi_2) = \xi(1_E \times_B \xi). \tag{2.3}$$

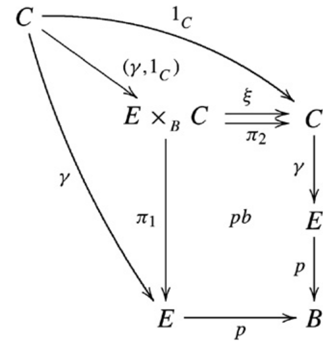


Fig. 1.

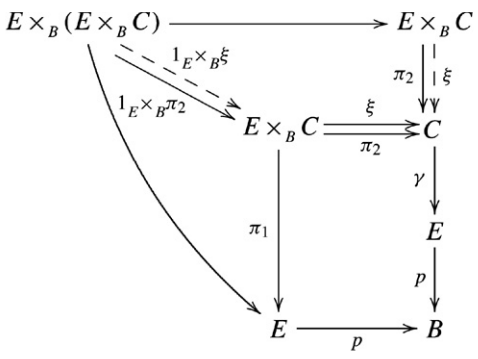
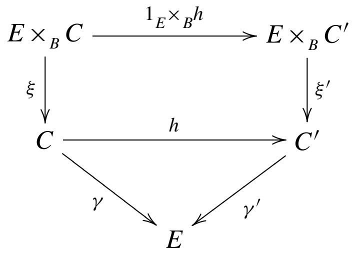


Fig. 2.

$\mathbb{E}$-descent data (with respect to p) form the category $\mathbf{Des}_{\mathbb{E}}(p)$ if a morphism $(C, \gamma, \xi) \longrightarrow (C', \gamma', \xi')$ is defined as a $\mathbf{C}$-morphism $h : C \longrightarrow C'$ such that the following diagram commutes



Denote the full subcategory of the slice category $\mathbf{C}/B$ with objects being morphisms from $\mathbb{E}$ by $\mathbb{E}/B$. We have the comparison functor

$$\Phi^p : \mathbb{E}/B \longrightarrow \mathbf{Des}_{\mathbb{E}}(p)$$

which sends $f : D \longrightarrow B$ to

$$(E \times_B D, \pi_1', 1_E \times_B \pi_2'),$$

where π_1' and π_2' are the pullbacks of f and p, respectively, along each other.

p is called an $\mathbb{E}$-descent (resp. effective $\mathbb{E}$-descent) morphism if Φ^p is full and faithful (resp. the equivalence of categories). If $\mathbb{E}$ is the class of all $\mathbf{C}$-morphisms, then we omit the prefix $\mathbb{E}$.

If $\mathbb{E}$ is closed under composition with p from the left ($e \in \mathbb{E} \Rightarrow pe \in \mathbb{E}$), then the change-of-base functor

$$p^* : \mathbb{E}/B \longrightarrow \mathbb{E}/E$$

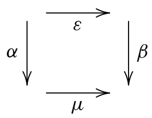
(which pulls back along p) has the evident left adjoint $p_!$ and the category $\mathbf{Des}_{\mathbb{E}}(p)$ is isomorphic to the Eilenberg–Moore category of the monad induced by the adjunction

$$p_! \dashv p^*, \tag{2.4}$$

while Φ^p is isomorphic to the comparison functor obtained from (2.4).[2] Therefore, in that case, p is an $\mathbb{E}$-descent (resp. an effective $\mathbb{E}$-descent) morphism if and only if p^* is premonadic (resp. monadic). Applying this fact, Janelidze and Tholen proved [8,9] that when $\mathbf{C}$ has coequalizers (and $\mathbb{E}$ is closed under composition with p from the left), p is an $\mathbb{E}$-descent morphism if and only if it is an $\mathbb{E}$-universal regular epimorphism, i.e. a morphism such that its pullback along any $\mathbb{E}$-morphism is a regular epimorphism.

Let $(\mathcal{E}, \mathcal{M})$ be a factorization system on $\mathbf{C}$ in the usual sense of Freyd and Kelly [5]. Recall that this means that $\mathcal{E}$ and $\mathcal{M}$ are morphism classes, both closed under composition with isomorphisms and such that

(i) every morphism f admits an $(\mathcal{E}, \mathcal{M})$-factorization, i.e. there are morphisms $\varepsilon \in \mathcal{E}$ and $\mu \in \mathcal{M}$ with $f = \mu\varepsilon$;
(ii) for any $\varepsilon \in \mathcal{E}$ and $\mu \in \mathcal{M}$ we have $\varepsilon \downarrow \mu$, which means that for any commutative square



there exists a unique δ with $\alpha = \delta\varepsilon$ and $\beta = \mu\delta$.

It is well-known that $\mathcal{E}^{\downarrow} = \mathcal{M}$ and $\mathcal{M}^{\uparrow} = \mathcal{E}$, where the symbol $\mathcal{E}^{\downarrow}$ (resp. $\mathcal{M}^{\uparrow}$) denotes the class of all morphisms φ with $\varepsilon \downarrow \varphi$ (resp. $\varphi \downarrow \mu$) for all $\varepsilon \in \mathcal{E}$ (resp. $\mu \in \mathcal{M}$). Moreover, the class $\mathcal{E}$ (resp. $\mathcal{M}$) contains isomorphisms, is closed under composition and stable under pushout (resp. pullback). In this section it is additionally assumed that $\mathcal{M} \subset \mathcal{M}ono\,\mathbf{C}$ and hence, as is well known, $\mathcal{E}$ is right cancellable ($\beta\alpha \in \mathcal{E} \Rightarrow \beta \in \mathcal{E}$). In particular, it contains split epimorphisms.[3]

[2] Note that, for arbitrary $\mathbb{E}$, even if p^* has a left adjoint, $\mathbf{Des}_{\mathbb{E}}(p)$ may not be equivalent to the corresponding category of algebras [9].

[3] In fact, if $\mathbf{C}$ has binary products, then, for any factorization system, the inclusions $\mathcal{M} \subset \mathcal{M}ono\,\mathbf{C}$, $\mathcal{S}plit\mathcal{E}pi\,\mathbf{C} \subset \mathbb{E}$ and the right cancellability of $\mathcal{E}$ are equivalent [1].

Suppose further that $\mathbb{E}$ is closed under composition with $\mathcal{M}$-morphisms both from the left and from the right, and that $\mathbb{E}$ is left cancellable under $\mathcal{M}$-morphisms ($\mu\gamma \in \mathbb{E}$, $\mu \in \mathcal{M} \Rightarrow \gamma \in \mathbb{E}$). Note that if $\mathcal{M} \subset \mathbb{E}$, $\mathbb{E}$ is closed under composition and left cancellable under its morphisms, then these conditions are obviously satisfied. This is, for instance, the case if either $\mathbb{E} = \mathcal{M}$ or $\mathbb{E} = \mathcal{M}or\,\mathbf{C}$.

Lemma 2.1. *The class of (effective) $\mathbb{E}$-descent morphisms is stable under pullback along $\mathcal{M}$-morphisms.*

Proof. The assertion follows from arguments similar to those of Sobral [17]. □

Consider $\mathbb{E}$-descent data (C, γ, ξ) with respect to p. Take the $(\mathcal{E}, \mathcal{M})$-factorization $\mu\varepsilon$ of $p\gamma$ and then represent the pullback shown in Fig. 1 as the concatenation of pullbacks (I) and (II) (see Fig. 3). Since $\gamma\xi = \mu'\alpha$, $\mu' \in \mathcal{M}$ and ξ is a split epimorphism, there exists a morphism γ' with $\gamma'\xi = \alpha$ and $\mu'\gamma' = \gamma$.

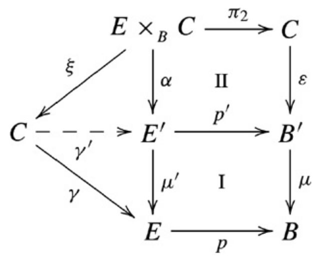


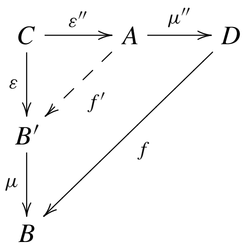
Fig. 3.

Lemma 2.2. *p' lies in $\mathcal{E}$, $\varepsilon = p'\gamma'$ and (C, γ', ξ) is $\mathbb{E}$-descent data with respect to p'.*

Proof. Since π_2 is a split epimorphism, $\varepsilon\pi_2$ lies in $\mathcal{E}$ and therefore so does p'. Moreover, $\mu\varepsilon\xi = p\gamma\xi = \mu p'\gamma'\xi$. It follows that $\varepsilon = p'\gamma'$. The rest of the proof is trivial. □

Lemma 2.3. *The following conditions are equivalent:*
(i) $(C, \gamma, \xi) \approx \Phi^p(f)$ *for some* $f \in \mathrm{Ob}\,\mathbb{E}/B$;
(ii) $(C, \gamma', \xi) \approx \Phi^{p'}(f')$ *for some* $f' \in \mathrm{Ob}\,\mathbb{E}/B'$.

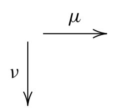
Proof. (i)⇒(ii): Let f' be the diagonal morphism in the diagram



where $\mu''\varepsilon''$ is the $(\mathcal{E}, \mathcal{M})$-factorization of the projection $C = E \times_B D \to D$. The conditions imposed on $\mathbb{E}$ imply that $f' \in \mathbb{E}$.

(ii)⇒(i): Take $f = \mu f'$. □

Before we continue our discussion, let us recall a definition [11]. $\mathbf{C}$ is said to satisfy the amalgamation (resp. transferability) property (with respect to $(\mathcal{E}, \mathcal{M})$ or, for simplicity, with respect to $\mathcal{M}$) if, for any span



(2.5)

with $\mu, \nu \in \mathcal{M}$ (resp. $\mu \in \mathcal{M}$), there exists a commutative square

$$\begin{array}{ccc} \cdot & \xrightarrow{\mu} & \cdot \\ {\scriptstyle \nu}\downarrow & & \downarrow{\scriptstyle \nu_1} \\ \cdot & \xrightarrow{\mu_1} & \cdot \end{array} \tag{2.6}$$

with $\mu_1, \nu_1 \in \mathcal{M}$ (resp. $\mu_1 \in \mathcal{M}$.). When $\mathcal{E} \subset \mathcal{E}pi\,\mathbf{C}$ and (2.5) admits a pushout

$$\begin{array}{ccc} \cdot & \xrightarrow{\mu} & \cdot \\ {\scriptstyle \nu}\downarrow & & \downarrow{\scriptstyle \nu_2} \\ \cdot & \xrightarrow{\mu_2} & \cdot \end{array} \tag{2.7}$$

then the latter condition is obviously equivalent to the requirement that $\mu_2, \nu_2 \in \mathcal{M}$ (resp. $\mu_2 \in \mathcal{M}$). Hence, for $\mathbf{C}$ with pushouts, the transferability property implies the amalgamation one. This assertion remains valid if we replace "pushouts" by "binary products" [11]. It is easy to see that the transferability property is satisfied if $\mathbf{C}$ has enough injectives (with respect to $\mathcal{M}$).

Remark 2.4. By adding the prefix "co" to a notion, it is obviously meant that we have its dual. At that, we assume throughout the paper that the category dual to $\mathbf{C}$ is equipped with the factorization system $(\mathcal{M}, \mathcal{E})$ so that $\mathbf{C}$ satisfies the coamalgamation property if and only if $\mathcal{E}$ is stable under pullback along $\mathcal{E}$-morphisms. In that case the morphism α shown in Fig. 3 lies in $\mathcal{E}$ and hence, so does γ'.

Lemmas 2.1–2.3 and Remark 2.4 imply

Theorem 2.5. *An $\mathbb{E}$-descent morphism p is effective if and only if for any p' lying in $\mathcal{E}$ and being the pullback of p along an $\mathcal{M}$-morphism, and any $\mathbb{E}$-descent data (C', γ', ξ') with respect to p', such that $p'\gamma' \in \mathcal{E}$, there exists $f' \in \mathrm{Ob}\,\mathbb{E}/B'$ with $(C', \gamma', \xi') \approx \Phi^{p'}(f')$. In particular, all $\mathbb{E}$-descent morphisms are effective if and only if for any $\mathbb{E}$-descent morphism p lying in $\mathcal{E}$ and any $\mathbb{E}$-descent data (C, γ, ξ) with respect to p, such that $p\gamma \in \mathcal{E}$, there exists $f \in \mathrm{Ob}\,\mathbb{E}/B$ with $(C, \gamma, \xi) \approx \Phi^{p}(f)$.*

If $\mathbf{C}$ satisfies the coamalgamation property, then the statement remains valid provided that "$p'\gamma' \in \mathcal{E}$" (resp. "$p\gamma \in \mathcal{E}$") is replaced by "$\gamma' \in \mathcal{E}$" (resp. "$\gamma \in \mathcal{E}$").

In view of Theorem 2.5, let us make

Remark 2.6. Let $\mathbf{C}$ admit coequalizers. Then it is obvious that every regular epimorphism is a descent morphism if and only if $\mathbf{C}$ is regular. When $\mathcal{F} = (\mathcal{R}eg\mathcal{E}pi\,\mathbf{C}, \mathcal{M}ono\,\mathbf{C})$ is a factorization system, these conditions are equivalent also to the cotransferability property with respect to $\mathcal{F}$. Moreover, if the pair $\mathcal{F}' = (\mathcal{E}pi\,\mathbf{C}, \mathcal{R}eg\,\mathcal{M}ono\,\mathbf{C})$ is a factorization system, then every epimorphism of $\mathbf{C}$ is regular if and only if $\mathbf{C}$ satisfies the dual to the intersection property of amalgamations [10,11,14,20]. Recall that the latter property means that for any commutative square (2.6) with monomorphic μ, ν, μ_1 and ν_1, there exists a pullback

$$\begin{array}{ccc} \cdot & \xrightarrow{\mu} & \cdot \\ {\scriptstyle \nu}\downarrow & & \downarrow{\scriptstyle \nu_3} \\ \cdot & \xrightarrow{\mu_3} & \cdot \end{array} \tag{2.8}$$

where μ_3 and ν_3 are also monomorphic. When pushout (2.7) exists, it can be assumed without loss of generality that (2.8) coincides with (2.7) [11].

Example 2.7. Consider a codescent morphism p of the category **Top** of topological spaces (and continuous mappings) and codescent data (C, γ, ξ) with respect to p. Suppose that both p and γ are embeddings. Then $C \coprod_B E$ is merely the disjoint union of sets C and $E \backslash p(B)$ equipped with the evident topology. From $(2.2)^{op}$ it follows that the restrictions of ξ and π_2 on $C \backslash \gamma(E)$ are equal. This fact together with $(2.1)^{op}$ implies that $(C, \gamma, \xi) \approx \Phi_p(f)$ for $f : B \longrightarrow (C \backslash \gamma(E)) \cup \gamma p(B)$ mapping each b from B to $\gamma p(b)$. Since **Top** is a coregular category, Theorem 2.5 implies the following:

Every regular monomorphism in the category of topological spaces (i.e. an embedding) is an effective codescent morphism.

Note that the same result was obtained by Mantovani [12] in a different way.

3. Effective descent morphisms and functors preserving certain pullbacks

Let $\mathbf{C}$ and $\mathbf{X}$ be categories with pullbacks, and let $F : \mathbf{C} \to \mathbf{X}$ be a functor. Like in Section 2, it is assumed that $\mathbf{C}$ is equipped with a factorization system $(\mathcal{E}, \mathcal{M})$.[4] In what follows we will deal with the following conditions:

(C1) for any span

$$\begin{array}{ccc} C & \xrightarrow{\varepsilon} & D \\ {\scriptstyle \varepsilon'}\downarrow & & \\ D' & & \end{array}$$

in $\mathbf{C}$ with $\varepsilon, \varepsilon' \in \mathcal{E}$ and any $g : F(D) \to F(D')$ such that $gF(\varepsilon) = F(\varepsilon')$, there exists a morphism $f : D \to D'$ with $F(f) = g$ and $f\varepsilon = \varepsilon'$;

(C2) for any object C in $\mathbf{C}$ and any morphism $x : F(C) \to X$ in $\mathbf{X}$, there exist $\varepsilon : C \to D$ and $\alpha : F(D) \to X$ such that $\varepsilon \in \mathcal{E}$, $\alpha F(\varepsilon) = x$ and α is an F-monomorphism, i.e. a morphism such that $g = h$, for any $g, h : D' \to D$ with $\alpha F(g) = \alpha F(h)$;

(C3) F maps descent morphisms lying in $\mathcal{E}$ to descent morphisms.

Remark 3.1. (i) The condition (C1) is fulfilled obviously if F is full and faithful. (C1) is also satisfied when $\mathcal{E}$ is the class of regular epimorphisms, F is faithful and maps regular epimorphisms to epimorphisms.

(ii) Let $\mathcal{M} \subset \mathcal{M}ono\,\mathbf{C}$. The condition (C2) is satisfied if F has a right adjoint R. Indeed, take the adjunct $\overline{x} : C \longrightarrow R(X)$ of x and then its $(\mathcal{E}, \mathcal{M})$-factorization $\overline{x} = \mu\varepsilon$. The morphism ε and the coadjunct $\overline{\mu}$ of μ are the desired ones.

Moreover, (C2) trivially implies the dual to the well-known solution set condition when $\mathbf{X}$ is F-well-powered, i.e. for any object X of $\mathbf{X}$ there is only a set of non-isomorphic pairs (C, α) with $C \in \mathrm{Ob}\,\mathbf{C}$ and α being an F-monomorphism $F(C) \to X$. Hence, in that case, for cocomplete $\mathbf{C}$, F has a right adjoint if and only if it preserves all (small) colimits and the condition (C2) is satisfied.

If the pair $(\mathcal{R}eg\mathcal{E}pi\,\mathbf{X}, \mathcal{M}ono\,\mathbf{X})$ is a factorization system on $\mathbf{X}$, then the condition (C2) is obviously equivalent to its weak version (C2′), when the first "morphism" in (C2) is replaced by "regular epimorphism".

(iii) When $\mathbf{C}$ has coequalizers, F preserves regular epimorphisms, and every regular epimorphism of $\mathbf{X}$ is descent, then the condition (C3) is fulfilled obviously. It also holds if both $\mathbf{C}$ and $\mathbf{X}$ have coequalizers, while F has a fully faithful left adjoint and also preserves regular epimorphisms. Indeed, for a morphism $p : E \to B$, the pullback of $F(p)$ along an arbitrary φ is isomorphic to the image of the pullback p' of p along the adjunct of φ.

Remark 3.2. Let $\mathcal{E}'$ be any morphism class in $\mathbf{X}$ such that every F-monomorphism lies in $\mathcal{E}'^{\downarrow}$. For any epimorphism class $\mathcal{E}$ of $\mathbf{C}$ closed under composition with isomorphisms and such that $F(\mathcal{E}) \subset \mathcal{E}'$, the condition (C2) together with the condition (C1′) obtained by replacing in (C1) "$\varepsilon, \varepsilon' \in \mathcal{E}$" by "$\varepsilon \in \mathcal{E}$", implies that $(\mathcal{E}, \mathcal{E}^{\downarrow})$ is a factorization system on $\mathbf{C}$. To prove that this is so, let us consider any morphism $f : C \to C'$ in $\mathbf{C}$ and, according to (C2), represent $F(f)$ as $\alpha F(\varepsilon)$. From (C1′) we obtain a morphism $\mu : D \longrightarrow C'$ with $F(\mu) = \alpha$ and $f = \mu\varepsilon$. Let us verify that $\mu \in \mathcal{E}^{\downarrow}$. To this end, consider a commutative square $\psi\varepsilon_1 = \mu\varphi$ with $\varepsilon_1 \in \mathcal{E}$. Passing to $\mathbf{X}$, we obtain a diagonal morphism δ in the commutative square $F(\psi)F(\varepsilon_1) = \alpha F(\varphi)$. Again applying (C1′), we get a morphism ρ with $\rho\varepsilon_1 = \varphi$.

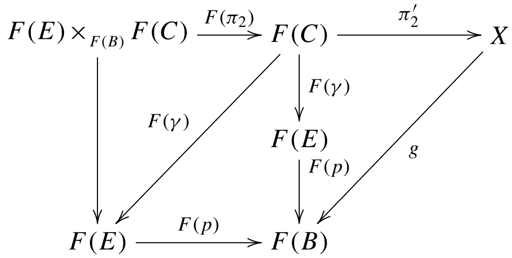
Lemma 3.3. *Let $(\mathcal{E}, \mathcal{M})$ be a proper factorization system (i.e. $\mathcal{E} \subset \mathcal{E}pi\,\mathbf{C}$ and $\mathcal{M} \subset \mathcal{M}ono\,\mathbf{C}$) and $\mathbf{C}$ satisfy the coamalgamation property with respect to it. Let F preserve the pullbacks of $\mathcal{E}$-morphisms and the conditions* (C1), (C2) *be satisfied. For an $\mathcal{E}$-morphism $p : E \to B$, if (C, γ, ξ) is descent data with respect to p and γ lies in $\mathcal{E}$, then $(F(C), F(\gamma), F(\xi))$ is descent data with respect to $F(p)$. Moreover, for the statements*

[4] At this stage of our discussion we do not impose any restriction on $(\mathcal{E}, \mathcal{M})$.

(i) $(C, \gamma, \xi) \approx \Phi^p(f)$ *for some* $f \in \mathrm{Ob}\,\mathbf{C}/B$,
(ii) $(F(C), F(\gamma), F(\xi)) \approx \Phi^{F(p)}(g)$ *for some* $g \in \mathrm{Ob}\,\mathbf{X}/F(B)$,
we have (ii) $\Rightarrow$ (i). *If* F *preserves all pullbacks along* $\mathcal{E}$*-morphisms, then* (i) *and* (ii) *are equivalent.*

If $\mathbf{X}$ *has coequalizers, then one can replace the condition* (C2) *in the statement by* (C2$'$) *and "*$p : E \to B$*" by "*$p : E \to B$ *with* $F(p)$ *being a descent morphism".*

Proof. (ii)$\Rightarrow$(i): Consider the diagram



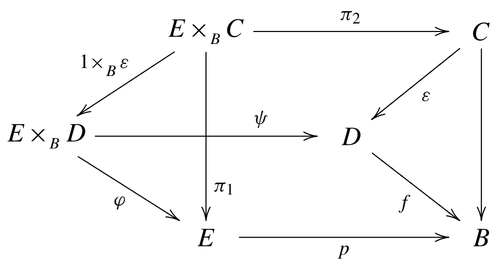
where both quadrangles are pullbacks, and the morphisms ε and α indicated in the condition (C2) for $x = \pi_2'$. (C1) implies that there exists $f : D \to B$ such that $F(f) = g\alpha$ and $f\varepsilon = p\gamma$. It will be shown that $(C, \gamma, \xi) \approx \Phi^p(f)$. We have

$$F(\xi) = 1_{F(E)} \times_{F(B)} \pi_2' = 1_{F(E)} \times_{F(B)} (\alpha F(\varepsilon))$$
$$= (1_{F(E)} \times_{F(B)} \alpha)(1_{F(E)} \times_{F(B)} F(\varepsilon)).$$

Since $f \in \mathcal{E}$, we obtain that

$$F(\xi) = (1_{F(E)} \times_{F(B)} \alpha) F(1_E \times_B \varepsilon). \tag{3.1}$$

Since the upper square in the diagram



is a pullback, by the coamalgamation property of $\mathbf{C}$ we conclude that $1 \times_B \varepsilon$ also lies in $\mathcal{E}$. Now, by the condition (C1), from (3.1) we obtain a morphism $\beta : E \times_B D \to C$ with $F(\beta) = 1_{F(E)} \times_{F(B)} \alpha$ and such that

$$\xi = \beta(1_E \times_B \varepsilon). \tag{3.2}$$

Let us show that β is the inverse for the canonical morphism $\theta : C \longrightarrow E \times_B D$ induced by γ and ε. Since

$$\pi_2' F(\pi_2) = \pi_2'(1_{F(E)} \times_{F(B)} \pi_2') = \pi_2' F(\xi),$$

we have

$$\varepsilon\pi_2 = \varepsilon\xi.$$

This, together with (2.1) implies that

$$\psi\theta\xi = \varepsilon\xi = \psi(1_E \times_B \varepsilon)$$

and

$$\varphi\theta\xi = \gamma\xi = \varphi(1_E \times_B \varepsilon).$$

Thus we obtain

$$\theta\xi = 1_E \times_B \varepsilon. \tag{3.3}$$

Since both ξ and $1_E \times_B \varepsilon$ are epimorphisms, from (3.2) and (3.3) we conclude that β and θ are the mutually inverse isomorphisms. □

Before continuing our consideration, let us recall that the class of effective descent morphisms is stable under pullback as was proved by Sobral and Tholen for a category with coequalizers [19] and later by Sobral in the general case of an arbitrary category (with pullbacks) [17]. Applying this result, from Lemma 3.3 and Theorem 2.5 we get

Theorem 3.4. *Let* $(\mathcal{E}, \mathcal{M})$ *be a proper factorization system on* **C** *and* **C** *satisfy the coamalgamation property with respect to it. Let* F *preserve the pullbacks of* $\mathcal{E}$*-morphisms and the conditions* (C1)–(C3) *be satisfied. If every descent morphism of* **X** *is effective, then the same is valid for* **C**.

If the condition (C3) *is replaced by the requirement that* F *preserve pullbacks, then a descent morphism* p *of* **C** *is effective whenever* $F(p)$ *is an effective descent morphism.*

The statement remains valid if we replace (C2) *by* (C2′) *but require of* **X** *to have coequalizers.*

Proof. The first part of the theorem is obvious. As to the second one, we observe that for any pullback p' of p, the morphism $F(p')$, being a pullback of $F(p)$, is an effective descent morphism. □

Example 3.5. Let **Ban** be the category of Banach spaces and linear contractions, and **Vect** be the category of vector spaces. The forgetful functor F: **Ban** $\longrightarrow$ **Vect** does not preserve all pushouts (for instance, it does not preserve the pushout of any non-surjective dense linear contraction along the zero mapping). Nevertheless, as can be easily shown, F preserves the pushouts of isometric embeddings. Since **Ban** has enough injectives with respect to the class $\mathcal{M}$ of such injections (Cohen [4], Banaschewski [2]), **Ban** satisfies the transferability property with respect to $\mathcal{M}$. Moreover, $\mathcal{M}$ is the part of the factorization system ($\mathcal{E} =$ *dense linear contractions*, $\mathcal{M}$) and the duals to the conditions (C1)–(C3) are obviously satisfied. Since each monomorphism of **Vect** is an effective codescent morphism, Theorem 3.4 implies that

Every regular monomorphism of **Ban** *(i.e. an isometric embedding) is an effective codescent morphism.*

Theorem 3.6. *Let* F *be a topological functor. Let, in addition,* **C** *admit the factorization system* $(\mathcal{R}eg\mathcal{E}pi\,\mathbf{C}, \mathcal{M}ono\,\mathbf{C})$ *and satisfy the coamalgamation property with respect to it (these conditions are obviously satisfied if* **C** *is regular). A descent morphism* p *of* **C** *is effective whenever* $F(p)$ *is an effective descent morphism of* **X**. *In particular, if* **X** *(and therefore* **C***) has coequalizers and every descent morphism of* **X** *is effective, then the same holds for* **C**. *Hence every regular epimorphism of a regular topological category is an effective descent morphism.*

Proof. As is known, F is faithful, has both right and left adjoints, and these adjoints are fully faithful. Thus, the statement follows from Remark 3.1 and Theorem 3.4. □

Remark 3.7. (i) For a topological functor F, the pair $(\mathcal{R}eg\mathcal{E}pi\,\mathbf{C}, \mathcal{M}ono\,\mathbf{C})$ is a factorization system on **C** if $(\mathcal{R}eg\mathcal{E}pi\,\mathbf{X}, \mathcal{M}ono\,\mathbf{X})$ is a factorization system on **X**. Indeed, in that case, the pair $(\mathcal{E} = \mathcal{M}^{\uparrow}, \mathcal{M} = F^{-1}(\mathcal{M}ono\,\mathbf{X}))$ is a factorization system on **C**, and $\mathcal{E}$ is the class of coCartesian morphisms over regular epimorphisms [22]. Moreover, $\mathcal{M} = \mathcal{M}ono\,\mathbf{C}$ and, as it is easy to verify, $\mathcal{E} = \mathcal{R}eg\mathcal{E}pi\,\mathbf{C}$.

Further, if, in addition, every epimorphism of **C** is regular, then in Theorem 3.6 the requirement that **C** possess the coamalgamation property can be replaced by imposing the same restriction on **X**. However, in the general case such a replacement in Theorem 3.6 cannot be made. As an example let us consider the forgetful functor **Top** → **Set**. As shown by Reiterman and Tholen in [15] and by Sobral in [18], not every descent morphism of **Top** is effective.

(ii) In Theorem 3.6, the requirement that p be a descent morphism cannot be omitted as is shown by the example where F is again the forgetful functor from topological spaces to sets and p is an injective continuous mapping that is not an embedding.

Example 3.8. We already know that every regular monomorphism of the category of topological spaces is an effective codescent morphism. From Theorem 3.6 we also conclude that

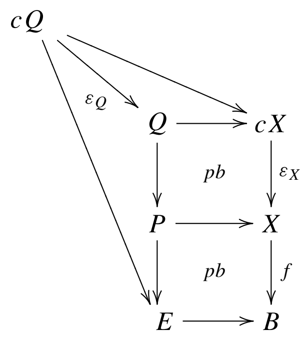
Every regular monomorphism of the categories of uniform spaces, quasi-uniform spaces, proximity spaces, quasi-proximity spaces, extended[5] *pseudo-normed spaces, and extended pseudo-metric spaces is an effective codescent morphism.*

Theorem 3.9. *Let* **C** *be a full coreflective subcategory of* **X**, *and let* **C** *have a proper factorization system* $(\mathcal{E}, \mathcal{M})$, *with respect to which* **C** *satisfies the coamalgamation property (the two last assumptions are obviously satisfied if* **C** *is regular).*

(i) *Let* **C** *have coequalizers and be closed under pullbacks of* $\mathcal{E}$*-morphisms. If every regular epimorphism of* **X** *is an effective descent morphism, then every descent morphism of* **C** *is effective.*

(ii) *Let* **C** *be closed under pullbacks. A descent morphism of* **C** *is effective in* **C** *whenever it is an effective descent morphism in* **X**. *If, moreover, the pair* $(\mathcal{R}eg\mathcal{E}pi\,\mathbf{X}, \mathcal{M}ono\,\mathbf{X})$ *is a factorization system on* **X**, *the counit* ε *of the coreflection is a componentwise regular epimorphism and every descent morphism of* **X** *is effective, then every descent morphism of* **C** *too is effective.*

Proof. (i) immediately follows from Remark 3.1 and Theorem 3.4. As to (ii), we observe that if $\mathcal{R}eg\mathcal{E}pi\,\mathbf{X}$ is right cancellable, and, for any $X \in \mathrm{Ob}\,\mathbf{X}$, ε_X is a regular epimorphism, then every descent morphism $p : E \to B$ of **C** is descent in **X** too. Indeed, for any **X**-morphism $f : X \to B$, the outer quadrangle in the commutative diagram



is a pullback in the category **C**. □

Remark 3.10. (i) If **C** is complete and well-powered, then in Theorem 3.9 one can replace the condition that **C** has a proper factorization system, with respect to which **C** satisfies the coamalgamation property, by the same requirement for **X**. Indeed, if $(\mathcal{E}, \mathcal{M})$ is a proper factorization system on **X**, then, according to [22], the pair $(\mathcal{E}' = F^{-1}(\mathcal{E}), \mathcal{M}' = \mathcal{E}'^{\downarrow})$ is a factorization system on **C**, where F is the inclusion functor, and $\mathcal{M}' \subset \mathcal{M}ono\,\mathbf{C}$. For the rest we observe that $\mathcal{E}' = \mathcal{E} \cap \mathcal{M}or\,\mathbf{C}$.

(ii) When **X** (and therefore **C**) has coequalizers, the statement (ii) of Theorem 3.9 immediately follows from the result of Mesablishvili mentioned in Section 1.

Example 3.11. The category **Haus** of Hausdorff topological spaces is not closed in **Top** under all pushouts. However, it is closed under pushouts of closed embeddings. From Theorem 3.9 we conclude:

Every codescent morphism of **Haus** *is effective*.

Observe that not every regular monomorphism of **Haus** (i.e. a closed embedding) is a codescent morphism (Kelly [10]). We have the corresponding necessary condition, which turns out to be also sufficient in a certain particular case. Before we formulate it, let us recall that subsets U_1 and U_2 of a topological space B are called completely separable if there exists a continuous mapping from B to the interval [0, 1] such that $f(U_1) = \{0\}$ and $f(U_2) = \{1\}$. When B is normal, the latter condition is equivalent to the requirement that the closures of U_1 and U_2 are disjoint.

For a closed embedding $p : B \to E$ *in* **Haus** *and for the conditions*

(i) *p is an effective codescent morphism,*

[5] Here we use “extended” to clarify the fact that we consider vector spaces V equipped with the mapping from V to the *extended* set of real numbers (satisfying the well-known conditions).

(ii) *for any completely separable open subsets* U_1 *and* U_2 *of* B, *there exist disjoint open subsets* V_1 *and* V_2 *of* E *such that* $U_1 = B \cap V_1$ *and* $U_2 = B \cap V_2$,

(iii) *for any open subsets* U_1 *and* U_2 *of* B, *there exist disjoint open subsets* V_1 *and* V_2 *of* E *such that* $U_1 = B \cap V_1$ *and* $U_2 = B \cap V_2$,

one has (i) $\Rightarrow$ (ii) $\Leftarrow$ (iii). *If* E *is regular, then the implication* (iii) $\Rightarrow$ (i) *is also valid. If, in addition,* B *is compact, then* (i) *and* (ii) *are equivalent. If, again,* E *is regular and each two disjoint open subsets of* B *are completely separable (as in the case of a discrete* B), *then all three conditions are equivalent.*

(i)$\Rightarrow$(ii): We begin by recalling that the pushout in **Haus** of

$$\begin{array}{ccc} B & \xrightarrow{p} & E \\ {\scriptstyle f}\downarrow & & \\ A & & \end{array} \tag{3.4}$$

is constructed as follows: first we take the pushout D of (3.4) in **Top** (D clearly being, as a set, the disjoint union of A and $E \backslash B$) and, after that, the quotient of D by the closure of the diagonal Δ of $D \times D$. The space $D/\overline{\Delta}$ with the evident mappings $p' : A \longrightarrow D/\overline{\Delta}$ and $f' : E \longrightarrow D/\overline{\Delta}$ is the desired pushout.

Suppose p is codescent, and $f : B \longrightarrow [0, 1]$ is a continuous mapping such that $f(U_1) = \{0\}$ and $f(U_2) = \{1\}$. Since $p'(0) \neq p'(1)$, there exist disjoint open neighborhoods W_1 and W_2 of 0 and 1, respectively, in D. One has the representation $W_i = X_i \cup Y_i$ $(i = 1, 2)$, where X_i is contained in A (=[0, 1]) and is open in it, while $Y_i \subset E \backslash B$ and $Y_i \cup f^{-1}(X_i)$ is open in E. Obviously, $U_i \subset f^{-1}(X_i)$ and the subsets $(Y_i \cup f^{-1}(X_i))$ $(i = 1, 2)$ of E are disjoint. Moreover, since U_i is open in B, there exists open Z_i in E such that $Z_i \cap B = U_i$. Hence $V_i = (Y_i \cup f^{-1}(X_i)) \cap Z_i$ $(i = 1, 2)$ are the desired open subsets of E.

(iii)$\Rightarrow$(i): Assume that E is a regular space. Let us show that p' is injective. To this end, consider any different points a_1, a_2 of A and their open disjoint neighborhoods U_1 and U_2, respectively. Then $f^{-1}(U_1)$ and $f^{-1}(U_2)$ are also open and disjoint. Therefore E has subsets V_1 and V_2 of the same kind and with $V_i \cap B = f^{-1}(U_i)$ $(i = 1, 2)$. It is obvious that $U_i \cup (V_i \backslash f^{-1}(U_i))$ $(i = 1, 2)$ are disjoint open (in D) neighborhoods of a_1 and a_2, respectively, and hence $p'(a_1) \neq p'(a_2)$.

Consider a closed subset F of A. We are to verify that the union of those $\overline{\Delta}$-cosets of D which intersect F is closed. Assume that $x \in F$ and $(x, y) \in \overline{\Delta}$. Then, since p' is injective, $y \in E \backslash B$. There exist disjoint open subsets U and V of E such that $y \in U$ and $B \subset V$. Clearly, U is a neighborhood of y in D. Let W be any open neighborhood of x in D. Then $W = X \cup Y$, where X is an open subset of A, $Y \subset E \backslash B$ and $Y \cup f^{-1}(X)$ is open. Clearly, for $V' = (Y \cup f^{-1}(X)) \cap V$, the set $X \cup (V' \backslash B)$ is a neighborhood of y and does not intersect U, a contradiction.

(ii)$\Rightarrow$(i): Let E be regular and B be compact. To show that p' is injective, let us first assume that A is normal. Then, for any different points a_1 and a_2 of A, there exist open neighborhoods U_1 and U_2, respectively, such that their closures are disjoint. Then $f^{-1}(U_1)$ and $f^{-1}(U_2)$ are disjoint open and completely separable. Let V_1 and V_2 be open subsets of E mentioned in (ii). Like in the preceding case, $U_i \cup (V_i \backslash f^{-1}(U_i))$ $(i = 1, 2)$ are open disjoint neighborhoods of a_1 and a_2, respectively. Therefore $p'(a_1)$ and $p'(a_2)$ are different.

Let now A be an arbitrary Hausdorff space. Since **Haus** satisfies the amalgamation property with respect to the class of closed embeddings, we can restrict our consideration only to surjective continuous mappings f so that A can be assumed to be compact. Consider different points a_1 and a_2 of A. Since, according to the Tietze–Urysohn theorem, the interval [0, 1] is an injective object (with respect to the class of embeddings) in the category of compact Hausdorff spaces, there exists a continuous mapping $g : A \longrightarrow [0, 1]$ with $g(a_1) \neq g(a_2)$. Then, as we have already shown, the pushout p'' of p along gf is injective and thus $p'(a_1) \neq p'(a_2)$.

We have already seen that p' maps closed subsets to closed ones. This implies that p' is a closed embedding.

From the above we in particular obtain,

An embedding $p : B \rightarrowtail E$ *with regular* E *and clopen* B *is an effective codescent morphism of* **Haus**.

Example 3.12. The category **CompHaus** of compact Hausdorff spaces is closed in **Top** under pushouts of injective continuous mappings. Moreover, **CompHaus** is coregular. Applying Theorem 3.9, we obtain that,

Every monomorphism of **CompHaus** *(i.e. an injective continuous mapping) is an effective codescent morphism.*

The requirement that **C** admit all pullbacks in the definition of an (effective) descent morphism p can be replaced by the milder condition that **C** have all pullbacks along p. Theorem 3.4 remains valid in this general case, too.

Example 3.13. Though the category $\mathbf{Top}_4$ of normal topological spaces is not reflective in **Top**, the inclusion functor $\mathbf{Top}_4 \longrightarrow \mathbf{Top}$ satisfies the conditions dual to (C1)–(C3) for ($\mathcal{E}$ = *dense cont. mappings*, $\mathcal{M}$ = *closed embeddings*). Moreover, $\mathbf{Top}_4$ is closed under pushouts along $\mathcal{M}$-morphisms and hence it has pushouts along them. This also implies that $\mathbf{Top}_4$ satisfies the transferability property. Thus, from Theorem 3.4 we obtain that

Every regular monomorphism (i.e. a closed embedding) of $\mathbf{Top}_4$ *is an effective codescent morphism.*

Example 3.14. The category **Met** of metric spaces (with contractions) does not admit all pushouts even along regular monomorphisms [23]. Nevertheless, it admits pushouts of isometric embeddings, and, moreover, as we will see below, there are closed isometric embeddings, the pushouts of which along arbitrary morphisms exist. Not all of the existing pushouts are preserved by the forgetful functor $F : \mathbf{Met} \longrightarrow \mathbf{Set}$, though it preserves the pushouts of closed isometric embeddings, as follows from the arguments of [23]. Since ($\mathcal{E}$ = *dense contractions*, $\mathcal{M}$ = *closed isometric embeddings*) is a factorization system on **Met**, with respect to which it satisfies the amalgamation property, by Theorem 3.4 we conclude that

Every codescent morphism of **Met** *is effective.*

Moreover,

For a regular monomorphism (i.e. a closed isometric embedding) $p : B \rightarrowtail E$ *of* **Met**, *the following conditions are equivalent and imply that* p *is an effective codescent morphism:*

(i) for any contraction $f : B \longrightarrow A$, there exists a pushout

$$\begin{array}{ccc} B & \xrightarrow{p} & E \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ A & \xrightarrow{p'} & C \end{array} \tag{3.5}$$

in **Met** and the restriction of f' on $E\backslash B$ is an isometric embedding;

(ii) there exists a pushout

$$\begin{array}{ccc} B & \xrightarrow{p} & E \\ \downarrow & & \downarrow{\scriptstyle f'} \\ \{\cdot\} & \xrightarrow{p'} & C \end{array} \tag{3.6}$$

in **Met** and the restriction of f' on $E\backslash B$ is an isometric embedding;

(iii) for any $x, y \in E\backslash B$, there exists a contraction $f' : E \rightarrow C$ retracting B to a point and preserving the distance between x and y[6];

(iv) for any $x, y \in E\backslash B$, one has

$$d(x, y) \leq \inf_{b\in B} d(x, b) + \inf_{b\in B} d(b, y). \tag{3.7}$$

(iii)⇒(iv): For any $b \in B$, the inequality $d(f'(x), \cdot) \leq d(x, b)$ holds and hence

$$d(f'(x), \cdot) \leq \inf_{b\in B} d(x, b). \tag{3.8}$$

Further,

$$d(x, y) = d(f'(x), f'(y)) \leq d(f'(x), \cdot) + d(\cdot, f'(y)),$$

which together with (3.8) implies (3.7).

[6] Note that since **Met** has injective hulls (with respect to the class of isometric embeddings) [6], there always exists $f'' : E \rightarrow C$ retracting B to a point and such that $f''(x) \neq f''(y)$. This implies that f' in (i) and (ii) is necessarily injective.

(iv)⇒(i): As we have already observed, the pushouts of isometric embeddings exist in **Met**. Moreover, if p and f are isometric embeddings, then so is $f'|_{E\setminus B}$. Thus, without loss of generality we can restrict the consideration only to surjective f. Take the pushout (3.5) in the category of sets so that $C = A \coprod (E\setminus B)$ with evident p' and f'. It is easy to verify that the symmetric function ρ on C defined by

$$\rho(x, y) = \begin{cases} d_E(x, y) & \text{if } x, y \in E\setminus B, \\ d_A(x, y) & \text{if } x, y \in A, \\ \inf\limits_{f(b)=y} d(x, b) & \text{if } x \in E\setminus B \text{ and } y \in A, \end{cases}$$

is a metric, and C, together with ρ, is a pushout in **Met**. Obviously, p' is a closed isometric embedding. This implies that p is a codescent morphism.

Example 3.15. Similar arguments, when applied to the forgetful functor **ComplMet** → **Set**, where **ComplMet** is the category of complete metric spaces, show that

Every codescent morphism of **ComplMet** *(i.e. an isometric embedding) is effective.*

The analogues of the conditions (i)–(iv) in Example 3.14 formulated for **ComplMet** are also equivalent.

Example 3.16. Since the category **CompMet** of compact metric spaces has enough injectives with respect to the class of isometric embeddings (Isbell [6]), **CompMet** possesses the transferability property with respect to it. Considering the forgetful functor **CompMet** → **Set** and applying arguments similar to those of Example 3.14, we obtain that

Every regular monomorphism of **CompMet** *(i.e. an isometric embedding) such that all pushouts along it exist, is an effective codescent morphism.*

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