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Mathematics

Effective Descent Morphisms and Colimits

Dali Zangurashvili

Department of Algebra, A. Razmadze Mathematical Institute, Ivane Javakhishvili Tbilisi State University, Tbilisi, Georgia

(Presented by Academy Member Hvedri Inassaridze)

ABSTRACT. One of the well known results of descent theory asserts that a descent morphism p is effective if the ground category has coequalizers and the change-of-base functor p^* is coequalizer-preserving. In this paper, the question is posed whether this statement remains valid if one replaces the coequalizers therein by colimits of some other type. Two types of colimits are considered here: pushouts and colimits of diagrams (being large in general and) formed by certain (epi)morphisms with coinciding domains. The positive answer to the posed question is obvious for the first type of colimits. For the second one an answer is given in the form of a necessary and sufficient condition (not containing the existence quantifier) for p to be effective. The approach used here to the considered question also yields a new (elementary) proof of the above-mentioned known result. © *2019 Bull. Georg. Natl. Acad. Sci.*

Key words: effective descent morphism, colimit, coequalizer, pushout, cointersection of quotient objects

It is well known that for a descent morphism p of a category $\mathbf{C}$ with pullbacks if

(*) $\mathbf{C}$ has coequalizers and the change-of-base functor p^* is coequalizer-preserving,

then p is effective [1]. A question naturally arises whether this statement remains valid if one replaces the coequalizers in (*) by the colimits of some other type. It is easy to give a positive answer to this question for the case of pushouts. In this paper a necessary and sufficient condition for p to be effective is found for the case of colimits of diagrams (being large in general and) formed by certain (epi)morphisms with coinciding domains. Note that the criterion found does not contain the existence quantifier. According to this criterion, p is effective if and only if for any descent data $d = (C, \gamma, \xi)$ we have

$$e_d \xi = e_d \pi_2,$$

for a certain morphism e_d. Here π_2 is the pullback of the morphism p along the composition $p\gamma$ (the needed notation and definitions from descent theory are given below).

Note also that the approach used here to the considered question also gives a new (elementary) proof of the well known result mentioned above.

We begin with the needed definitions from descent theory [1].

Let $\mathbf{C}$ be a category with pullbacks and $p: E \to B$ be its morphism. We will consider the adjunction

$$p_! \dashv p^* \tag{1}$$

between the slice categories, where the right adjoint is the well known change-of-base functor

$$p^* : \mathbf{C}/B \to \mathbf{C}/E$$

(pulling back a morphism along p) and the left adjoint is the functor

$$p_! : \mathbf{C}/E \to \mathbf{C}/B$$

which maps an object α of the slice category $\mathbf{C}/E$ into the composition $p\alpha$.

The morphism p is called a descent morphism (resp. an effective descent morphism) if p^* is premonadic (resp. monadic), i.e. the comparison functor

$$\Phi^p : \mathbf{C}/B \to \mathbf{Des}(p),$$

where $\mathbf{Des}(p)$ is the Eilenberg-Moore category of the monad induced by adjunction (1), is full and faithful (resp. an equivalence of categories).

Recall that objects of the category $\mathbf{Des}(p)$ are triples (C, γ, ξ) (called descent data with respect to p), where $C \in \mathrm{Ob}\mathbf{C}$, while γ and ξ are respectively morphisms $C \to E$ and $E \times_B C \to C$, and the following equalities are fulfilled (see Figs. 1 and 2):

$$\gamma\xi = \pi_1, \tag{2}$$

$$\xi(\gamma, 1_C) = 1_C, \tag{3}$$

$$\xi(1_E \times_B \xi) = \xi(1_E \times_B \pi_2). \tag{4}$$

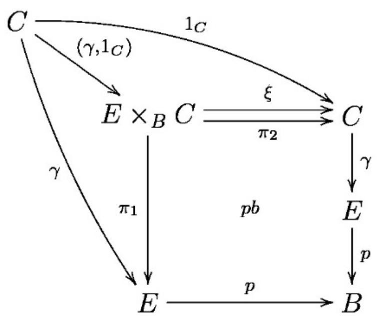


Fig. 1. The diagram related to the equalities (2), (3).

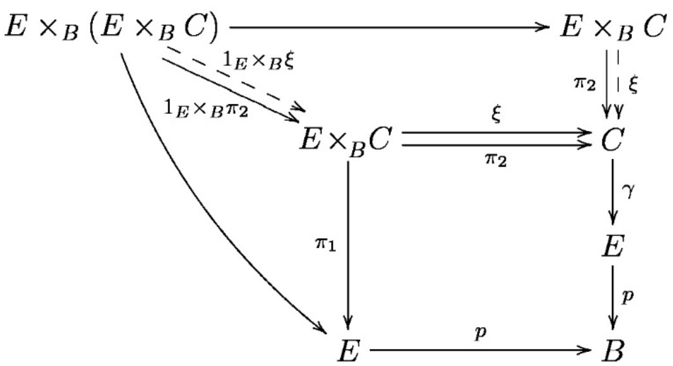
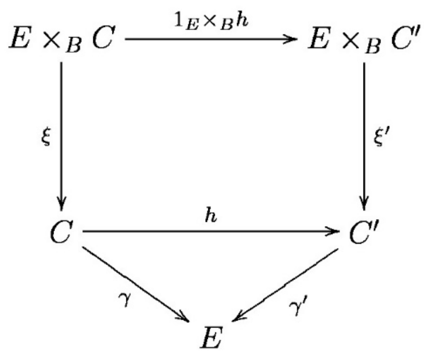


Fig. 2. The diagram related to the equality (4).

Besides, morphisms $(C,\gamma,\xi)\to(C',\gamma',\xi')$ in the category $\mathbf{Des}(p)$ are $\mathbf{C}$-morphisms $h:C\to C'$ such that the following diagram is commutative:



The functor Φ^p maps $f:D\to B$ into the descent data

$$\left(E\times_B D,\ \pi_1',1_E\times_B \pi_2'\right),$$

where π_1' and π_2' are the pullbacks of f and g, respectively, along each other.

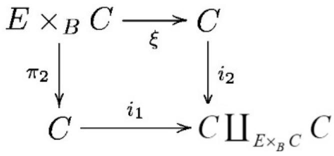
Theorem 1 [2, 1]**.** *Let* $\mathbf{C}$ *have coequalizers.* p *is a descent morphism if and only if it is a universal regular epimorphism, i.e. a regular epimorphism whose any pullback is also a regular epimorphism.*

Theorem 2 (see, e.g. [1])**.** *Let* $\mathbf{C}$ *have coequalizers, and let* p *be its descent morphism. The following conditions are equivalent:*

(i) p *is effective;*

(ii) the functor p^* *preserves the coequalizer of the pair* ξ,π_2*, for any descent data* (C,γ,ξ) *with respect to* p.

Recall that the known proof of this theorem is based on the fact that in the case of categories $\mathbf{C}$ with coequalizers there exists a left adjoint to the functor Φ^p. Moreover, the existence of all coequalizers is not essential in Theorem 2 – it is sufficient to require the existence of co equalizers for pairs mentioned in the condition (ii). This immediately implies that Theorem 2 remains valid if one replaces both "coequalizer(s)" therein by "pushout(s)". Indeed, for any descent data (C,γ,ξ) the morphisms i_1 and i_2 in the pushout



coincide by (3) and therefore i_1 is the coequalizer of the pair ξ,π_2.

Let us now consider the general case where a category $\mathbf{C}$ is not imposed by any restriction related to the existence of colimits.

Lemma 3. *Let p be a morphism of* $\mathbf{C}$ *and* (C,γ,ξ) *be descent data with respect to* p*. The following conditions are equivalent:*

(i) $(C,\gamma,\xi)\approx\Phi^p(f)$ *for some* $f\in \mathrm{Ob}\mathbf{C}/B$;

(ii) *there exist morphisms* $e: C \to D,\ f: D \to B$ *and* $\theta: E \times_B D \to C$ *with*

$$e\xi = e\pi_2, \tag{5}$$

$$p\gamma = fe, \tag{6}$$

$$\xi = \theta(1_E \times_B e). \tag{7}$$

and such that $1_E \times_B e$ *is an epimorphism* (*see Fig.* 3)

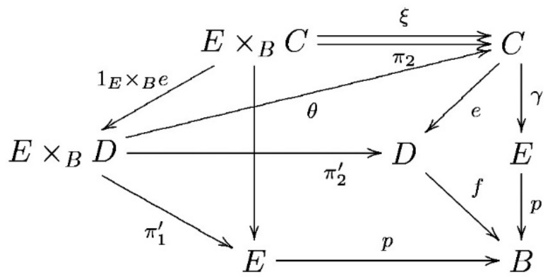


Fig. 3. The diagram related to the equalities (5)-(7).

If **C** *has coequalizers and* p *is a descent morphism, then "there exist morphisms* $e: C \to D,\ f: D \to B$ *and* $\theta: E \times_B D \to C$*" can be replaced by "there exist an epimorphism* $e: C \to D$ *and morphisms* $f: D \to B,\ \theta: E \times_B D \to C$*". Moreover, in that case, the conditions* (i), (ii) *are equivalent to the following condition:*

(iii) *for the coequalizer* $e: C \to D$ *of the pair* ξ, π_2, *there exists a morphism* $\theta: E \times_B D \to C$ *satisfying equality* (7) *and, moreover,* $1_E \times_B e$ *is an epimorphism.*

If, in addition, p^* *maps epimorphisms into morphisms* (*of the slice category*) *with epimorphic underlying morphisms, then the requirement that* $1_E \times_B e$ *be epimorphic can be omitted in* (ii) *and in* (iii).

Proof. (i)⇒(ii): From the construction of the functor Φ^p it readily follows that the morphisms $e = \pi_2'$, f and $\theta = 1_{EX_B D}$ are the desired ones.

(ii)⇒(i): Let us show that θ is the inverse for (γ, e). Indeed, we have

$$\pi_1'(\gamma, e)\theta = \gamma\theta = \pi_1', \tag{8}$$

since by (2)

$$\pi_1'(1_E \times_B e) = \pi_1 = \gamma\xi = \gamma\theta(1_E \times_B e),$$

and, moreover, $1_E \times_B e$ is an epimorphism. Further,

$$\pi_2'(1_E \times_B e) = e\pi_2 = e\xi = e\theta(1_E \times_B e),$$

whence we conclude that

$$\pi_2' = e\theta. \tag{9}$$

Therefore

$$\pi_2'(\gamma, e)\theta = e\theta = \pi_2'.$$

Taking (8) into account, we obtain $(\gamma, e)\theta = 1_{E \times_B D}$. . To prove the equality $\theta(\gamma, e) = 1_C$, it is sufficient to observe that $(\gamma, e) = (1_E \times_B e)(\gamma, 1_C)$ and use equality (3).

The equivalence of the conditions (i) and (iii) is proved in [3] (Theorem 2), but it also follows from the obvious equivalence (ii)⇔(iii). □

Note that Lemma 3 easily implies the implication (ii)⇒(i) of Theorem 2. Indeed, let us take the coequalizer e of the pair ξ, π_2 for descent data (C,γ,ξ) with respect to p. Then the morphism $1_E \times_B e$ is the coequalizer of the pair $1_E \times_B \xi, 1_E \times_B \pi_2$ and therefore by (4) there exists a morphism θ with (7).

Let $d = (C,\gamma,\xi)$ be descent data with respect to p. It is not difficult to verify that the following conditions are equivalent:

(**) the diagram formed by morphisms $e: C \to D$ such that there exist morphisms $f: D \to B$ and $\theta: E \times_B D \to C$ with (6), (7) and $1_E \times_B e$ is an epimorphism, has a colimit and the functor p^* preserves it;

(***) the full subcategory of the slice category $C/\mathbf{C}$, whose objects are morphisms $e: C \to D$ such that there exist morphisms $f: D \to B$ and $\theta: E \times_B D \to C$ with (6), (7) and $1_E \times_B e$ is an epimorphism, has a weakly terminal object e_d (i.e. an object such that for any one g of this subcategory there exists at least one morphism $g \to e_d$).

Theorem 4. *Let p be a descent morphism and the equivalent conditions* (**) *and* (***) *be fulfilled for any descent data d with respect to* p. p *is effective if and only if the equality*

$$e_d \xi = e_d \pi_2 \tag{10}$$

holds for any descent data d.

The proof easily follows from Lemma 3. □

Theorem 4 implies

Corollary 5. *Let* $\mathbf{C}$ *be co-well-powered, and let the class of epimorphisms be stable under pullback. Let* p *be a descent morphism. Assume that, for any descent data* d *with respect to* p, $\mathbf{C}$ *has cointersections of quotient objects* $e: C \to D$ *for which there exist morphisms* $f: D \to B$ *and* $\theta: E \times_B D \to C$ *with (6) and (7). Moreover, let* p^* *preserve these cointersections. Then p is an effective descent morphism if and only if the equality (10) is satisfied for any descent data* d.

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ეფექტური დაწევის მორფიზმები და კოზღვრები

დ. ზანგურაშვილი

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(წარმოდგენილია აკადემიის წევრის ხ. ინასარიძის მიერ)

დაწევის თეორიის ერთ-ერთი კარგად ცნობილი შედეგი ამტკიცებს, რომ დაწევის მორფიზმი p არის ეფექტური თუ მოცემულ კატეგორიას აქვს კოგანმატოლებლები და ბაზის შეცვლის ფუნქტორი p^* არის კოგანმატოლებლების შემნახველი. ამ სტატიაში დასმულია საკითხი იმის შესახებ, რჩება თუ არა ძალაში ეს დებულება, თუ კოგანმატოლებლებს მასში შევცვლით სხვა ტიპის კოზღვრებით. აქ ორი ტიპის კოზღვრებია განხილული: ფუშაუტები და ერთი და იმავე არის მქონე გარკვეული (ეპი)მორფიზმებით შექმნილი (საზოგადოდ არამცირე) დიაგრამების კოზღვრები. პირველი ტიპის კოზღვრებისათვის დადებითი პასუხი აღნიშნულ კითხვაზე ადვილია. მეორე ტიპის კოზღვრებისათვის პასუხი მოცემულია p-ს ეფექტურობისათვის აუცილებელი და საკმარისი პირობის სახით (რომელიც არ შეიცავს არსებობის კვანტორს). აქ გამოყენებული მიდგომა იძლევა აგრეთვე ზემოთ მოყვანილი დებულების ახალ (ელემენტარულ) დამტკიცებას.

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