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ON THE SUMS OF CONVERGENT TRIGONOMETRIC SERIES

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Abstract. It is proved in the present paper that the sum of everywhere convergent trigonometric series fails to have removable points of discontinuity. This statement is valid both for the multiple and for single trigonometric series.

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1. Introduction

In 1915, H. Steinhaus has proved ([1], see also [2], Ch. IX) the following Theorem A. Let a trigonometric series

$$\frac{a_0}{2}+\sum_{k=1}^{\infty} a_k \cos 2\pi k\tau + b_k \sin 2\pi k\tau$$

converge everywhere to a finite function $f(\tau)$.

If $f(\tau_0)>\alpha$, then a set of points τ, where $f(\tau)>\alpha$, has positive measure in any neighborhood of the point τ_0.

In the present work, using the results obtained in [3], we prove that Theorem A is valid for a multi-dimensional case, as well. Note that the fact dealing with the validity of Theorem A for multiple series has been announced by us in [4].

2. The Notation and Statements of the Results

Let $d\geq 2$ be some natural number, R^d the Euclidean space of dimension d, Z_0^d a set of all points from R^d with integer nonnegative coordinates.

By $n=(n_1,\dots,n_d)$ we denote the points from the set Z_0^d, and by $n'=(n_1,\dots,n_{d-1})$ those of the set Z_0^{d-1}. $x=(x_1,\dots,x_{d-1},x_d)$ denote the

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points from the unit cube $I^d = [0,1]^d$, and $x' = (x_1, \dots, x_{d-1})$ those from $I^{d-1} = [0,1]^{d-1}$. Let μ_d be the Lebesgue measute corresponding to the space R^d. If the set $E \subset I^d$, then $E_{(x_d)}$ and $E_{(x')}$ denote a cross-section of the set E at the points x_d and x', respectively, i.e., $E_{(x_d)} = \{(x_1, \dots, x_{d-1}) : (x_1, \dots, x_{d-1}, x_d) \in E\}$ and $E_{(x')} = \{(x_d : (x_1, \dots, x_{d-1}, x_d) \in E\}$.

The trigonimetric system given on $[0,1]$ we denote by $T^1 = \{t_i(\tau)\}_{i=0}^{\infty}$, where $\tau \in [0,1]$ and $t_0(\tau) \equiv 1$, $t_{2i-1}(\tau) = \sqrt{2}\sin 2\pi i\tau$, $t_{2i}(\tau) = \sqrt{2}\cos 2\pi i\tau$, $i = 1,2,3,\dots$. For every $n \in Z_0^d$, $T_n(x) = \prod_{j=1}^d t_{n_j}(x_j)$, $x \in I^d$.

Consider the d-multiple trigonometric series

$$\sum_{n=0} a_n T_n(x) = \sum_{n_1=0}^{\infty} \cdots \sum_{n_d=0}^{\infty} a_{n_1,\dots,n_d} \prod_{j=1}^{d} t_{n_j}(x_j). \tag{1}$$

Under the convergence of multiple series we mean the Pringsheim convergence.

Let the point $x^0 = (x_1, \dots, x_d^0) \in I^d$ and the number $r > 0$. By $K(x^0, r)$ we denote the cube in R^d:

$$K(x^0; r) = \{(x_1, \dots, x_d) \in R^d : |x_j - x_j^0| < \frac{r}{2}, \quad 1 \le j \le d\}.$$

The following statements are valid.

Theorem 1. *Let the series* (1) *converge everywhere to a finite function* $f(x)$. *If* $f(x^0) > \alpha$, *then the set of points* x, *where* $f(x) > \alpha$, *has the positive measure in any neighborhood of the point* x^0, *i.e., for an arbitrary* $r > 0$,

$$\mu_d\{x \in [0,1]^d : f(x) > \alpha\} \cap K(x^0; r) > 0.$$

Theorem 2. *If the series* (1) *converges everywhere to the finite function* $f(x)$, *then* $f(x)$ *fails to have removable points of discontinuity.*

3. Proofs of Theorems

In the sequel, the use will made of the following notation:

$$f(x) = f(x_1, \dots, x_{d-1}, x_d) = f(x', x_d)$$
$$a_n = a_{n_1,\dots,n_{d-1},n_d} = a_{n',n_d}.$$

Here we present two statements which we will need for the proof of the theorems. These statements are the particular cases of a more general result proven by us earlier (see [3], Corollary 7).

Statement 1. *Let the series* (1) *converge on* I^d *to a finite function* $f(x)$. *Then for every* $n' = (n_1, \dots, n_{d-1}) \in Z_0^{d-1}$ *and* $x_d \in I$ *we have*

$$\sum_{n_d=0}^{\infty} a_{n',n_d} t_{n_\alpha}(x_d) = b_{n'}(x_d), \quad |b_{n'}(x_d)| < \infty,$$

and for any $x' \in I^{d-1}$,

$$\sum_{n'=0}^{\infty} b_{n'}(x_d) \prod_{j=1}^{d-1} t_{n_j}(x_j) = f(x', x_d).$$

Statement 2. *Let the series* (1) *converge on* I^d *to a finite function* $f(x)$. *Then for every* n_d *and* $x' = (x_1, \ldots, x_{d-1}) \in I^{d-1}$ *we have*

$$\sum_{n'=0}^{\infty} a_{n_1,\ldots n_{d-1},n_d} \prod_{j=1}^{d-1} t_{n_j}(x_j) = a_{n_d}(x') < \infty,$$

and for any $x_d \in I$ *the single series*

$$\sum_{n_d=0}^{\infty} a_{n_d}(x') t_{n_d}(x_d) = f(x', x_d).$$

Proof of Theorem 1. This theorem we prove by using the method of induction with respect to d.

Since Theorem A is valid for single series, we assume that Theorem 1 is valid for $d = N-1$ and prove it for $d = N$, where $N \geq 2$. In this case, $x = (x_1, \ldots, x_N)$ and $x' = (x_1, \ldots, x_{N-1})$. Denote

$$E(\alpha) = \{x \in [0,1]^N : f(x) > \alpha\}$$

and

$$E = E(\alpha) \cap K(x^0, r).$$

Obviously, Theorem 1 will be proved if we show that for an arbitrary $r > 0$

$$\mu_N E > 0.$$

According to Statement 2, for any $x_N \in I$ we have

$$\sum_{n_N=0}^{\infty} a_{n_N}(x_1^0, \ldots, x_{N-1}^0) t_{n_N}(x_N) = f(x_1^0, \ldots, x_{N-1}^0, x_N).$$

As far as $f(x_1^0, \ldots, x_{N-1}^0, x_N^0) > \alpha$, by virtue of Theorem A we have

$$\mu_1 \{x_N : f(x_1^0, \ldots, x_{N-1}^0, x_N) > \alpha\} \cap \left\{x_N : |x_N - x_N^0| < \frac{r}{2}\right\} > 0. \tag{2}$$

Since

$$E_{(x_1^0,\ldots,x_{N-1}^0)} = \{x_N : f(x_1^0, \ldots, x_{N-1}^0, x_N) > \alpha\} \cap \left\{x_N : |x_N - x_N^0| < \frac{r}{2}\right\} \tag{3}$$

the relation (2) implies that

$$\mu_1 E_{(x_1^0,\ldots,x_{N-1}^0)} > 0. \tag{4}$$

Let $\overline{x}_N$ be the point from $E_{(x_1^0,\ldots,x_{N-1}^0)}$. Owing to (3), we have

$$\overline{x}_N \in \{x_N : f(x_1^0, \ldots, x_{N-1}^0, x_N) > \alpha\} \cap \left\{x_N : |x_N - x_N^0| < \frac{r}{2}\right\}.$$

This relation, in particular, means that

$$f(x_1^0,\dots x_{N-1}^0,\overline{x}_N)>\alpha. \tag{5}$$

By Statement 1, for every $(x_1,\dots,x_{N-1})\in I^{d-1}$ we have

$$\sum_{n'=0}^{\infty} b_{n'}(\overline{x}_N)\prod_{j=1}^{N-1} t_{n_j}(x_j)=f(x_1,\dots,x_{N-1},\overline{x}_N). \tag{6}$$

From (5) and (6) by the induction we obtain

$$\mu_{N-1}\{x':f(x',\overline{x}_N)>\alpha\}\cap\{x':|x_j-x_j^0|<\frac{r}{2};\ 1\le j\le N-1\}>0. \tag{7}$$

Since

$$E_{(\overline{x}_N)}=\{x':f(x',\overline{x}_N)>\alpha\}\cap\{x':|x_j-x_j^0|<\frac{r}{2};\ 1\le j\le N-1\},$$

for every $\overline{x}_N\in E_{(x_1^0,\dots,x_{N-1}^0)}$, by the relation (7), we find that

$$\mu_{N-1}E_{(\overline{x}_N)}>0. \tag{8}$$

Obviously, if $H=E_{(x_1^0,\dots x_{N-1}^0)}$, then

$$\mu_N E=\int_0^1 \mu_{N-1}E_{(xN)}dx_N\ge\int_H \mu_{N-1}E_{(xN)}dx_N. \tag{9}$$

It follows from the relations (4), (8) and (9) that

$$\mu_N E>0.$$

Thus Theorem 1 is proved. □

Proof of Theorem 2. For every $x\in I^d$, let

$$\sum_{n=0}^{\infty} a_n T_n(x)=f(x),\quad |f(x)|<\infty.$$

Suppose that at some point $x_0\in I^d$ there exists

$$\lim_{x\to x_0} f(x)=A, \tag{10}$$

and

$$f(x_0)=B.$$

Assume $A\ne B$. Without loss of generality, we assume that $A<B$.

Let a fixed number ε be such that

$$0<\varepsilon<\frac{B-A}{2}.$$

It follows from (10) that for an arbitrary $r>0$,

$$|f(x)-A|<\varepsilon \quad\text{for}\quad x\in K(x^0,r)\ \text{ and }\ x\ne x^0,\ \text{ that is}$$

$$f(x)<A+\varepsilon\quad\text{for}\quad x\in K(x^0,r)\ \text{ and }\ x\ne x^0.$$

Since $A+\varepsilon < B-\varepsilon$, therefore

$$f(x) < B-\varepsilon \quad \text{for} \quad x \in K(x^0; r) \quad \text{and} \quad x \neq x^0,$$

whence

$$\{x : f(x) > B-\varepsilon\} \cap K(x^0, r) = x^0. \tag{11}$$

As far as $f(x_0) = B > B-\varepsilon$, by Theorem 1 we have

$$\mu_d\{x : f(x) > B-\varepsilon\} \cap K(x^0, r) > 0,$$

but this contradicts (11).

The obtained contradiction shows that $A = B$.

Thus Theorem 2 is proved. $\square$

Remark. The above proof shows that Theorem 2 is likewise valid for single series.

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