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ON COEFFICIENTS OF SERIES WITH RESPECT TO THE RADEMACHER SYSTEM

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1. Notation, Definitions and Known Results

The Rademacher function $r_k(t)$, where $k = 0, 1, 2, \dots$, can be defined as follows (see [1]):

$$r_0(t) = \begin{cases} 1, & \text{for} \quad t \in \Big[0, \dfrac{1}{2}\Big), \\ -1, & \text{for} \quad t \in \Big[\dfrac{1}{2}, 1\Big). \end{cases}$$

We continue the function $r_0(t)$ over the entire number axis with the condition

$$r_0(t) = r_0(t + 1),$$

where t is an arbitrary real number.

By the definition, for every natural number k and for any real number t,

$$r_k(t) = r_0(2^k t).$$

A system of functions $\{r_k(t)\}_{k=0}^{\infty}$, where $t \in [0, 1)$, is called the Rademacher system.

Here we present some known results both for single and for d-multiple ($d \geq 2$) series with respect to the Rademacher system.

Single Series

Below by μE will be denoted the Lebesgue linear measure of the set $E \subset [0, 1)$. The symbol $\overset{\bullet}{+}$ denotes summation which is defined in [1] (see [1], Chs. I and II).

Let σ be any rearrangement of nonnegative integers.

Consider the series

$$\sum_{k=0}^{\infty} a_k \, r_{\sigma(k)}(t) \tag{1}$$

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with respect to the rearranged Rademacher system.

For the series (1), S. Stechkin and P. Ul'yanov have proved the uniqueness theorem (see [2]) which we present here in a form, convinient for us.

Theorem A. *Let the series* (1) *converges to some constant* C *on the set* $E \subset [0, 1)$*. Then:*

1) *if* $\mu E > \frac{1}{2}$*, then* $a_k = 0$ *for every* $k \geq 0$ *and, as a consequence,* $C = 0$*;*

2) *if* $\mu E > 0$*, then there exists the natural number* k_0 *such that for every* $k \geq k_0$ *the equality* $a_k = 0$ *holds, i.e., the series* (1) *is a polynomial.*

The following strengthening of the Theorem A is known (see [3]).

Theorem B. *Let the set* $E \subset [0, 1)$*. Then:*

1) *if* $\mu E > \frac{1}{2}$*, then there exists a countable set* $P \subset E$ *such that the convergence of the series* (1) *on the set* P *to an arbitrary constant* C *implies that* $a_k = 0$ *for any* $k \geq 0$ *and* $C = 0$*;*

2) *if* $\mu E > 0$*, then there exists a countable set* $P_1 \subset E$ *such that the convergence of the series* (1) *to zero on the set* P_1 *implies that* $a_k = 0$ *for all* $k \geq k_0$*, for some* k_0*, i.e., the series* (1) *is a polynomial.*

Multiple Series

Let $d \geq 2$ be a natural number, R^d be the d-dimensional Euclidean space, $[0, 1)^d$ be a unit cube in R^d, Z_0^d be the set of all points from R^d with integer nonnegative coordinates, $x = (x_1, \ldots, x_d)$ be a point of a unit cube $[0, 1)^d$ and let $n = (n_1, \ldots, n_d)$ and $m = (m_1, \ldots, m_d)$ be the points of the set Z_0^d, and μ_d be the Lebesgue measure in R^d. Consider a d-multiple series with respect to the Rademacher system

$$\sum_{n=0}^{\infty} a_n \, r_n(x) = \sum_{n_1=0}^{\infty} \cdots \sum_{n_d=0}^{\infty} a_{n_1,\ldots,n_d} \prod_{j=1}^{d} r_{r_j}(x_j). \tag{2}$$

For multiple series with respect to the Rademacher system, the following existence theorem, well known (see [4] and [5]).

Theorem C. *Let the series* (2) *converges to some constant* C *on the set* $E \subset [0, 1)^d$*. Then:*

1) *if* $\mu_d E > 1 - \frac{1}{2^d}$*, then* $a_n = 0$ *for every* $n \in Z_0^d$ *and* $C = 0$*;*

2) *if* $\mu_d E > 1 - \frac{1}{2^{d-1}}$*, then there exists a natural number* k_0 *such that* $a_n = 0$ *for any* $n \in Z_0^d$ *with* $\max\limits_{1 \leq j \leq d} \{n_j\} \geq k_0$*, i.e., the series* (2) *is a polynomial.*

Note, that in Theorem C under the convergence of the series (2) is meant the convergence of this series in Pringsheim's sense.

2. Statements of the Theorems

We have proved the theorems which generalize Theorems A and B for single series and Theorem C for d-multiple series.

For single series (1) the following statements hold.

Theorem 1. *Let the series* (1) *converges to the function* $f(t)$ *on the set* $E \subset [0,1)$*, i.e.,*

$$\sum_{k=0}^{\infty} a_k\, r_{\sigma(k)}(t) = f(t), \quad t \in E. \tag{3}$$

Then, if the set E *is such that for some* k *there exists the point* $t_k \in E$ *such that* $t_k \dot{+} \dfrac{1}{2^{\sigma(k)+1}} \in E$*, then*

$$a_k = \frac{1}{2}\, r_{\sigma(k)}(t_k) \left[f(t_k) - f\left(t_k \dot{+} \frac{1}{2^{\sigma(k)+1}} \right) \right]. \tag{4}$$

Theorem 2. *Let the series* (1) *converges to the function* $f(t)$ *on the set* $E \subset [0,1)$*. Then:*

1) *if* $\mu E > \dfrac{1}{2}$*, then for any* $k \geq 0$ *there exists the point* $t_k \in E$ *such that* $t_k \dot{+} \dfrac{1}{2^{\sigma(k)+1}} \in E$ *and for any* $k \geq 0$ *the equalities* (4) *are fulfilled;*

2) *if* $\mu E > 0$*, then there exists the number* k_0 *such that for any* $k \geq k_0$ *there exists the point* $t_k \in E$ *such that* $t_k \dot{+} \dfrac{1}{2^{\sigma(k)+1}} \in E$*, and for any* $k \geq k_0$ *the equalities* (4) *are fulfilled.*

Theorem 3. *Let the series* (1) *converges to the function* $f(t)$ *on the set* $E \subset [0,1)$*. Then:*

1) *if* $\mu E > \dfrac{1}{2}$*, then for any* $k \geq 0$ *there exist the points* $t_k \in E$ *and* $t_k \dot{+} \dfrac{1}{2^{\sigma(k)+1}} \in E$*. If the function* $f(t)$ *is such that for some* $p \geq 0$

$$f(t_p) = f\left(t_p \dot{+} \frac{1}{2^{\sigma(p)+1}} \right), \tag{5}$$

then $a_p = 0$*. In particular, if* (5) *is fulfilled for any* $p \geq 0$*, then* $a_p = 0$ *for any* $p \geq 0$ *and* $f(t) \equiv 0$ *on the set* E*;*

2) *if* $\mu E > 0$*, then there exists the number* k_0 *such that for any* $k \geq k_0$ *there exist the points* $t_k \in E$ *and* $t_k \dot{+} \dfrac{1}{2^{\sigma(k)+1}} \in E$*.*

If $f(t)$ *is such that for some* $p \geq k_0$ *the equality* (5) *is fulfilled, then* $a_p = 0$*. In particular, if* (5) *is fulfilled for any* $p \geq k_0$*, then* $a_p = 0$ *for any* $p \geq k_0$*, i.e., the series* (1) *is a polynomial.*

Obviously, Theorem 3 is the strengthening of Theorems A and B.

Multiple Series

We cite here the results for d-multiple series with respect to the Rademacher system which strengthen Theorem C. In these results, under the convergence of the series (2) one may mean different types of convergences. In particular, one may mean the convergence of the series (2) with respect to cubes or with respect to spheres.

Theorem 4. *Let the series* (2) *converges to the function* $f(x)$ *on the set* $E \subset [0,1)^d$, *i.e.,*

$$\sum_{n=0}^{\infty} a_n\, r_n(x) = f(x), \quad x \in E.$$

Then:

1) *if* $\mu_d\, E > 1 - \dfrac{1}{2^d}$, *then for any* $n \in Z_0^d$, *the set* E *contains* 2^d *points* $x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(2^d)}$ *which are the vertices of the* d*-dimensional parallelepiped and are such that if the function* $f(x)$ *is such that for some* $m \in Z_0^d$,

$$f(x_m^{(1)}) = f(x_m^{(2)}) = \cdots = f(x_m^{(2^d)}), \tag{6}$$

then $a_m = 0$. *In particular, if* (6) *is fulfilled for any* $m \in Z_0^d$, *then* $a_m = 0$ *for any* $m \in Z_0^d$ *and* $f(x) \equiv 0$ *on the set* E*;*

2) *if* $\mu_d\, E > 1 - \dfrac{1}{2^{d-1}}$, *then there exists the nonnegative number* k_0 *such that for any* $n \in Z_0^d$ *with* $\max\limits_{1 \le j \le d}\{n_j\} \ge k_0$ *the set* E *contains* 2^d *points* $x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(2^d)}$ *which are the vertices of the* d*-dimensional parallelepiped and are such that if for some* $m \in Z_0^d$ *with* $\max\limits_{1 \le j \le d}\{m_j\} \ge k_0$ *the equality* (6) *is fulfilled, then* $a_m = 0$. *In particular, if* (6) *is fulfilled for any* $m \in Z_0^d$ *with* $\max\limits_{1 \le j \le d}\{m_j\} \ge k_0$, *then* $a_m = 0$ *for any* $m \in Z_0^d$ *with* $\max\limits_{1 \le j \le d}\{m_j\} \ge k_0$, *i.e., the series* (2) *is a polynomial.*

Theorem 4 implies the following

Corollary. *Let the set* $E \subset [0,1)^d$. *Then:*

1) *if* $\mu_d E > 1 - \dfrac{1}{2^d}$, *then there exists a countable set* $P \subset E$ *such that the convergence of the series* (2) *on the set* P *to some constant* C *implies that* $a_n = 0$ *for any* $n \in Z_0^d$ *and* $C = 0$*;*

2) *if* $\mu_d E > 1 - \dfrac{1}{2^{d-1}}$, *then there exists a countable set* $P_1 \subset E$ *such that the convergence of the series* (2) *to some constant* C *on the set* P_1 *implies that* $a_n = 0$ *for any* $n \in Z_0^d$ *with* $\max\limits_{1 \le j \le d}\{n_j\} \ge k_0$ *for some* k_0, *i.e., the series* (2) *is a polynomial.*

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