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Mathematics

On some Properties of Sets of Uniqueness of Functional Series

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ABSTRACT. The structure and some properties of uniqueness sets of functional series are presented.

Key words: *functional series, set of uniqueness, U set, M set.*

It is well known that many deep researches are dedicated to the uniqueness problem of functional series with respect to different systems of functions. These researches started from Cantor's fundamental result [1], according to which the empty set is a set of uniqueness (a *U* set) for trigonometric series. This theorem was generalized by Young [2], who proved that any countable set is a *U* set for trigonometric series.

It should be noted [3: 193] that any subset of [0,1] with positive Lebesgue measure is not a *U* set. So, any such set is an *M* set for trigonometric series. Especially important is an example of trigonometric null-series constructed by Menshov [4]. The existence of such series directly implies the existence of an M set with zero measure for trigonometric series.

Uniqueness problem for Walsh, Haar, Rademacher and trigonometric systems was investigated by Bari [5], Rajchman [6], Marcinkiewich and Zygmund [7], Salem and Zygmund [8], Vilenkin [9], Schneider [10], Fine [11], Stechkin and Ulyanov [12], Skvortsov [13], Arutunyan and Talalyan [14] and by other known mathematicians.

In the present paper we formulate some of our theorems connected with some properties of uniqueness sets of functional series.

Let $\Phi = \{\varphi_n(x)\}_{n=1}^{\infty}$ be a system of finite functions defined on [0,1]. By *m* we denote the set of all number sequences and by *a* and *b* we denote some elements of the set *m*. So,

$$a = (a_1, a_2, \ldots, a_n, \ldots) \text{ and } b = (b_1, b_2, \ldots, b_n, \ldots).$$

We say that *a*=*b* if and only if $a_n = b_n$ for every whole number $n \geq 1$. The sequence $(0,0,0\ldots)$ we denote by θ. Let $m_0 = \{a \in m: \ a \neq \theta\}$.

Consider a series with respect to Φ system

$$\sum_{n=1}^{\infty} a_n \varphi_n(x). \qquad (a)$$

Definition 1. *A set $A \subset [0,1]$ is called a set of uniqueness, or a U set, if the equality*

$$\sum_{n=1}^{\infty} a_n \varphi_n(x) = 0 \quad \textit{for all} \quad x \in [0,1] \setminus A$$

i*mplies that $a_n = 0$ for every whole $n \geq 1$. Otherwise the set A is called an M set.*

Definition 2. *We say that a set $E \subset [0,1]$ belongs to a class $V(\Phi)$ if the equality*

$$\sum_{n=1}^{\infty} a_n \varphi_n(x) = 0 \quad \textit{for all} \quad x \in E$$

implies that $a_n = 0$ for every whole number $n \geq 1$.

It is obvious that if $E \in V(\Phi)$, then the set $[0,1] \setminus E$ is a *U* set and if $E \notin V(\Phi)$, then the set $[0,1] \setminus E$ is an *M* set. Also, if $V(\Phi) \neq \varnothing$, then $[0,1] \in V(\Phi)$. Below everywhere $V(\Phi) \neq \varnothing$.

Let us formulate some of our theorems and some properties of uniqueness sets.

Theorem 1. *A set $E \subset [0,1]$ belongs to $V(\Phi)$ if and only if for any series (a), where $a \in m_0$, there exists a point $x^*(a) \in E$, such that*

$$\sum_{n=1}^{\infty} a_n \varphi_n\left(x^*(a)\right) \neq 0.$$

The set of all $x^*(a)$ points for any series (*a*), where $a \in m_0$, we denote by $I^*(a)$. So,

$$I^*(a) = \left\{ x^*(a) \in [0,1] : \quad \sum_{n=1}^{\infty} a_n \varphi_n\left(x^*(a)\right) \neq 0 \right\}.$$

Property 1. $E \in V(\Phi)$ *if and only if*

$$\bigcup_{a \in m_0} \left\{ x^*(a) \right\} \subset E.$$

Property 2. *A set A is a U set if and only if*

$$A \subset [0,1] \setminus \bigcup_{a \in m_0} \left\{ x^*(a) \right\}.$$

Property 3. *A set A is an M set if and only if there exists $a \in m_0$, such that $I^*(a) \subset A$.*

Property 4. *If a system of functions Φ is such that for any whole number $n \geq 1$ and any $x \in [0,1]$ the equality $\varphi_n(1-x) = \varphi_n(x)$ holds, then $\left[0, \frac{1}{2}\right] \in V(\Phi)$ and $\left[\frac{1}{2}, 1\right] \in V(\Phi)$.*

Property 5. *If a system of functions* Φ *is such that for any whole number* $n \ge 1$ *and* $x \in [0,1]$ *the equality* $\varphi_n(1-x) = -\varphi_n(x)$ *holds, then* $\left[0, \frac{1}{2}\right] \in V(\Phi)$ *and* $\left[\frac{1}{2}, 1\right] \in V(\Phi)$.

Theorem 2. *Let* $E \subset [0,1] \setminus A$, *where* $A = \bigcup_{n=1}^{\infty} \{x \in [0,1]: \ \varphi_n(x) = 0\}$. *If for any series* (a) *there exist a point* $x_0 \in E$ *and a strictly increasing to infinity subsequence* $m_k \uparrow \infty$ *of whole numbers, such that for every whole number* $k \ge 1$ *the following inequality holds:*

$$\sum_{j=2}^{m_k} \left(\sum_{i=1}^{j-1} a_i \varphi_i(x_0) \right) a_j \varphi_j(x_0) \ge 0,$$

then $E \in V(\Phi)$.

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