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PERIODICALLY MIXED SERIES AND APPROXIMATIONS OF MULTIVARIATE FUNCTIONS

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Abstract. In the present paper the notion of a periodically mixed power function series is introduced. A theorem asserting the existence of a universal periodically mixed power function series such that any continuous multivariate function can be uniformly approximated by the corresponding subsequence of partial sums of this series is formulated.

Introduction

Fekkete was the first (see [4]) who proved that there exists a real power series $\sum\limits_{n=1}^{\infty} a_n t^n$ on $[-1,1]$, such that to every continuous function g on $[-1,1]$ with $g(0)=0$ there exists an increasing sequence $(m_k)_{k=1}^{\infty}$ of positive integers such that $\sum\limits_{n=1}^{m_k} a_n t^n \to g(t)$ uniformly as $k\to\infty$. Later, Mazurkiewicz [3] and Sierpinski [5], proved, that there exists a real power series (see, also, [1, pp. 74–75])

$$\sum_{n=1}^{\infty} a_n t^n, \quad t\in[0,1]$$

such that to every continuous function f on $[0,1]$ there exists an increasing sequence $(m_k)_{k=1}^{\infty}$ of positive integers such that

$$\sum_{n=1}^{m_k} a_n t^n \to f(t) - f(0)$$

uniformly as $k\to\infty$.

Mentioned phenomenon is called a universality in the sense of uniform approximation (see [2]). In the present paper the notion of d-periodically mixed function series, where d is a natural number such that $d\geq 2$, is introduced and some properties of such a series are established.

Note, that a d-periodically mixed function series is a single function series and every term of this series is a function of one variable (see Definition 1, below). Notions of d-periodically mixed power type series and d-periodically mixed power series are also introduced (see, below Definition 2 and Definition 3, respectively).

The existence of a d-periodically mixed power type series such that for every continuous on $[0,1]^d$ function there exists a sequence of partial sums of this series which uniformly approximates this function on $[0,1]^d$ is established (see, Theorem 1, below). It holds the analogous proposition for d-periodically mixed power series (see, Theorem 2, below). So, there exists a universal single function series such that every term of this series is a function of one variable and every continuous multivariate function can be uniformly approximated by subsequences of this series. The latter is a generalization of the above mentioned known results.

1. Notation, Definitions, Theorems

Let N be the set of all positive integer numbers, d be a natural number such that $d\geq 2$, R^d be the d-dimensional Euclidean space, $[0,1]^d$ be a d-dimensional unit cube, $x=(x_1,\ldots,x_d)$ be a point of $[0,1]^d$, $\theta=(0,\ldots,0)\in[0,1]^d$. As usual $C[0,1]$ stands for the set of all continuous on $[0,1]$ functions

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and $C[0,1]^d$ stands for the set of all continuous on $[0,1]^d$ functions. $\Phi = \{\varphi_n(t)\}_{n=1}^{\infty}$ be a system of functions defined on $[0,1]$.

Consider a series with respect to Φ, i. e.,

$$\sum_{n=1}^{\infty} a_n \varphi_n(t), \quad t \in [0,1]. \tag{1}$$

Let $S_m(t)$ be the m-th partial sum of this series. i. e.,

$$S_m(t) = \sum_{n=1}^{m} a_n \varphi_n(t).$$

Let $d \geq 2$ be a fixed natural number and

$$d(n) = n - \left[\frac{n-1}{d}\right] d$$

for every positive integer n.

Note, that $(d(n))_{n=1}^{\infty}$ is the following periodic sequence of natural numbers:

$$1, 2, \dots, d, 1, 2, \dots, d, 1, 2, \dots, d, \dots.$$

Every positive integer number n may uniquely be presented in the following form:

$$n = j + (i-1)d$$

where i and j are positive integer numbers and $1 \leq j \leq d$.

For every j, $1 \leq j \leq d$ consider the following set

$$N_j = \{n \in N : \ n = j + (i-1)d, \quad \text{where} \quad i \in N\}$$

then

$$N = N_1 \bigcup \cdots \bigcup N_d, \quad \text{where} \quad N_i \bigcap N_j = \emptyset, \quad \text{if} \quad i \neq j.$$

Therefore for every $n \in N$ there exists j, such that $n \in N_j$ and

$$d(n) = n - \left[\frac{n-1}{d}\right] d = j.$$

So, for every positive integer n and $x = (x_1, \dots, x_d) \in [0,1]^d$ we have

$$x_{d(n)} = x_j \in [0,1].$$

Definition 1. We say that a single series

$$\sum_{n=1}^{\infty} a_n \varphi_n\left(x_{d(n)}\right), \quad \text{where} \quad x = (x_1, \dots, x_d) \in [0,1]^d \tag{2}$$

is a d-periodically mixed series with respect to variables.

Let $S_m^{(d)}(x)$ be the m-th partial sum of a d-periodical mixed function series at the point $x \in [0,1]^d$, i. e.,

$$S_m^{(d)}(x) = S_m^{(d)}(x_1, \dots, x_d) = \sum_{n=1}^{m} a_n \varphi_n\left(x_{d(n)}\right).$$

It is obvious that d-periodical mixed function series (2) is a generalization of the series (1) in the sense that series (1) and (2) coincides with each other at points $t \in [0,1]$ and $(t, \dots, t) \in [0,1]^d$ respectively. So, it holds the following equality for the m-th partial sums of (1) and (2):

$$S_m(t) = S_m^{(d)}(t, \dots, t).$$

In the present paper we consider a system of functions $\Phi = (\varphi_n(t))_{n=1}^{\infty}$ with $\varphi_n(t) = t^{p_n}$, where $t \in [0,1]$, $n = 1, 2, \dots$ and $(p_n)_{n=1}^{\infty}$ is a strictly increasing sequence of positive real numbers.

Definition 2. We say that a series

$$\sum_{n=1}^{\infty} a_n t^{p_n}, \quad t \in [0,1] \tag{3}$$

where $(p_n)_{n=1}^{\infty}$ is an increasing sequence of positive real numbers is a power type series and a series

$$\sum_{n=1}^{\infty} a_n x_{d(n)}^{p_n}, \quad x \in [0,1]^d \tag{4}$$

is a d-periodically mixed power type series.

We denote by $\sigma_m(t)$ and $\sigma_m^{(d)}(x)$ the m-th partial sums of series (3) and (4) respectively. i. e.

$$\sigma_m(t) = \sum_{n=1}^{m} a_n t^{p_n}, \quad t \in [0,1],$$

and

$$\sigma_m^{(d)}(x) = \sum_{n=1}^{m} a_n x_{d(n)}^{p_n}, \quad x \in [0,1]^d.$$

If $p_n = n$ for any positive integer n, then the series (3) is the power series

$$\sum_{n=1}^{\infty} a_n t^{n}, \quad t \in [0,1] \tag{5}$$

and the series (4) is the series

$$\sum_{n=1}^{\infty} a_n x_{d(n)}^{n}, \quad x \in [0,1]^d. \tag{6}$$

Definition 3. We say that the series (6) is a d-periodically mixed power series.

We denote by $\tau_m(t)$ and $\tau_m^{(d)}(x)$ the m-th partial sums of series (5) and (6) recpectively, that is

$$\tau_m(t) = \sum_{n=1}^{m} a_n t^{n}, \quad t \in [0,1]$$

and

$$\tau_m^{(d)}(x) = \sum_{n=1}^{m} a_n x_{d(n)}^{n}, \quad x \in [0,1]^d.$$

It is obvious that if $t \in [0,1]$ and $(t,\dots,t) \in [0,1]^d$ then for every positive integer m we have:

$$\sigma_m(t) = \sigma_m^{(d)}(t,\dots,t) \quad \text{and} \quad \tau_m(t) = \tau_m^{(d)}(t,\dots,t).$$

If $(f_k(x))_{k=1}^{\infty}$ is a sequence of functions defined on $[0,1]^d$, then it is meant that there exists a limit $\lim_{k\to\infty} f_k(x) = t \in [0,1]$ at the point $x \in [0,1]^d$ if the symbol $S_m\left(\lim_{k\to\infty} f_k(x)\right)$ is applied.

For d-periodically mixed power type series it holds the following:

Theorem 1. *Let d be a natural number, such that $d \geq 2$ and $(p_n)_{n=1}^{\infty}$ be an increasing sequence of positive real numbers such that*

$$\sum_{n=1}^{\infty} \frac{1}{p_n} = \infty$$

then there exist a sequence of real numbers $(a_n)_{n=1}^{\infty}$ and a strictly increasing sequence of positive integers $(M_{k,q})_{k=1}^{\infty}$, where $q = 1,2,\dots,2d+1$, such that for the d-periodically mixed power type series (4) with a_n coefficients we have:

$$\left\{\lim_{k\to\infty} \sigma_{M_{k,q}}^{(d)}(x), \quad \textit{where} \quad x \in [0,1]^d\right\} = [0,1], \quad q = 1,2,\dots,2d+1$$

and also for any function $F \in C[0,1]^d$, there exists an increasing sequence of positive integers $(M_k)_{k=1}^{\infty}$ such that:

$$F(x) - F(\theta) = \sum_{q=1}^{2d+1} \lim_{k\to\infty} \sigma_{M_k} \left(\lim_{k\to\infty} \sigma^{(d)}_{M_{k,q}}(x) \right)$$

uniformly on $[0,1]^d$ and $[0,1]$ for indicated limits respectively.

Note, that one direct consequence of Theorem 1 is the following theorem related to d-periodically mixed power series.

Theorem 2. *Let d be a natural number, such that $d \geq 2$, then there exist a sequence of real numbers $(a_n)_{n=1}^{\infty}$ and a strictly increasing sequence of positive integers $(M_{k,q})_{k=1}^{\infty}$, where $q = 1, 2, \dots, 2d+1$, such that for the d-periodically mixed power series (6) with a_n coefficients we have:*

$$\left\{ \lim_{k\to\infty} \tau^{(d)}_{M_{k,q}}(x), \quad \text{where} \quad x \in [0,1]^d \right\} = [0,1], \quad q = 1, 2, \dots, 2d+1$$

and also for any function $F \in C[0,1]^d$, there exists an increasing sequence of positive integers $(M_k)_{k=1}^{\infty}$ such that:

$$F(x) - F(\theta) = \sum_{q=1}^{2d+1} \lim_{k\to\infty} \tau_{M_k} \left(\lim_{k\to\infty} \tau^{(d)}_{M_{k,q}}(x) \right)$$

uniformly on $[0,1]^d$ and $[0,1]$ for indicated limits respectively.

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