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ON SETS OF UNIQUENESS OF SOME FUNCTION SERIES

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Abstract. Uniqueness theorems for function series with respect to systems of finite functions, Lebesgue measurable and finite functions, and some orthonormal systems of functions are formulated.

1. Notation and Definitions

Let $\Phi = \{\varphi_n(x)\}_{n=1}^{\infty}$ be a system of finite functions defined on $[0,1]$, $a = (a_1, a_2, \dots, a_n, \dots)$ be a sequence of real numbers, and $\theta = (0, 0, 0, \dots)$ be a constant sequence of zeros. $a \neq \theta$ means that there exists a natural number $n_0 \geq 1$ such that $a_{n_0} \neq 0$.

Let S be the set of all sequences of real numbers, $S = \{a: \ a = (a_1, a_2, \dots, a_n, \dots)\}$. Let S_0 be the set $S \backslash \{\theta\}$, i.e., $S_0 = \{a: \ (a \in S) \ \& \ (a \neq \theta)\}$.

Consider a series with respect to Φ:

$$\sum_{n=1}^{\infty} a_n \varphi_n(x). \tag{1}$$

For every fixed $x \in [0,1]$ let

$$A(x) = \left\{a: \ (a \in S_0) \ \& \left(\sum_{n=1}^{\infty} a_n \varphi_n(x) \neq 0\right)\right\}$$

and for every fixed $a \in S_0$ let

$$E(a) = \left\{x: \ (x \in [0,1]) \ \& \left(\sum_{n=1}^{\infty} a_n \varphi_n(x) \neq 0\right)\right\}.$$

Definition 1. A set $H \subset [0,1]$ is called a U-set if the convergence of a series $\sum\limits_{n=1}^{\infty} a_n \varphi_n(x)$ to zero for every $x \in [0,1] \backslash H$ implies that $a_n = 0$ for every natural number $n \geq 1$.

2. A Uniqueness Theorem for Series with Respect to Systems of Finite Functions

Let $\Phi = \{\varphi_n(x)\}_{n=1}^{\infty}$ be a system of finite functions defined on $[0,1]$, then the following assertions hold true:

Theorem 1. *A set $H \subset [0,1]$ is a U-set if and only if*

$$\bigcup_{x \in [0,1] \backslash H} A(x) = S_0.$$

Proposition 1. *A set $H \subset [0,1]$ is a U-set if and only if*

$$E(a) \bigcap ([0,1] \backslash H) \neq \emptyset \qquad \textit{for any} \qquad a \in S_0.$$

Proposition 2. *If the empty set is a U-set, then a nonempty set $H \subset [0,1]$ is a U-set if and only if*

$$\bigcup_{x \in H} A(x) \subset \bigcup_{x \in [0,1] \backslash H} A(x).$$

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3. A Uniqueness Theorem for Series with Respect to Systems of Lebesgue Measurable and Finite Functions

Let $\Phi=\{\varphi_n(x)\}_{n=1}^{\infty}$ be a system of Lebesgue measurable and finite functions defined on $[0,1]$.

In what follows, μ_* and μ^* stand for Lebesgue inner and outer linear measures of a set, respectively, and measurable is applied instead of Lebesgue measurable for the sake of brevity.

Definition 2. A series $\sum\limits_{n=1}^{\infty} a_n\varphi_n(x)$ is called a null-series with respect to Φ if $\sum\limits_{n=1}^{\infty} a_n\varphi_n(x)=0$ for almost all $x\in[0,1]$ and there exists a natural number $n_0\geq 1$ such that $a_{n_0}\neq 0$.

Definition 3. An orthonormal system of functions $\Phi=\{\varphi_n(x)\}_{n=1}^{\infty}$ defined on $[0,1]$ is called a strictly convergence system if $\sum\limits_{n=1}^{\infty} a_n^2<\infty$ implies that a series $\sum\limits_{n=1}^{\infty} a_n\varphi_n(x)$ converges almost everywhere on $[0,1]$ and $\sum\limits_{n=1}^{\infty} a_n^2=\infty$ implies that a series $\sum\limits_{n=1}^{\infty} a_n\varphi_n(x)$ diverges on a subset of $[0,1]$ of positive Lebesgue measure.

It is well known that if Φ is a strictly convergence system, then there is no null-series with respect to Φ.

Note that examples of strictly convergence systems defined on $[0,1]$, are lacunar trigonometric systems (see [3, Ch. 5, §6]), Rademacher's system (see [1, Ch. 4, §5]), Kashin's complete and orthonormal system [2].

The following assertions hold true.

Theorem 2. *If there is no null-series with respect to the system* $\Phi=\{\varphi_n(x)\}_{n=1}^{\infty}$, *then any set* $H\subset[0,1]$ *such that* $\mu_*H=0$ *is a* U*-set.*

Note that if a set $H\subset[0,1]$ is such that $\mu_*H=0$ and $\mu^*H=1$, then $\mu_*\left([0,1]\backslash H\right)=0$ and therefore, according to Theorem 2, we have

Corollary 1. *If there is no null-series with respect to the system* $\Phi=\{\varphi_n(x)\}_{n=1}^{\infty}$, *and a set* $H\subset[0,1]$ *is such that* $\mu_*H=0$ *and* $\mu^*H=1$, *then both* H *and* $[0,1]\backslash H$ *are* U*-sets.*

Corollary 1 implies:

Corollary 2. *If* $\Phi=\{\varphi_n(x)\}_{n=1}^{\infty}$ *is a strictly convergence system and a set* $H\subset[0,1]$ *is such that* $\mu_*H=0$ *and* $\mu^*H=1$, *then both* H *and* $[0,1]\backslash H$ *are* U*-sets.*

Remark. It can be proved that after appropriate modifications of the notation and definitions presented in Section 1, the assertions formulated in Section 2 remain true for multiple function series, too.

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