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LIMIT THEOREMS FOR WEIGHTED SUMS OF INDEPENDENT IDENTICALLY DISTRIBUTED RANDOM VECTORS

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Abstract. Criteria of weak convergence to the normal law of weighted sums of independent identically distributed random vectors are presented. The conditions of density convergence are given. Various methods of normalization of weighted growing sums are considered.

რეზიუმე. მოცემულია ერთნაირად განაწილებული დამოუკიდებელი შემთხვევითი ვექტორების აწონილი ჯამების ნორმალური განაწილებისაკენ სუსტი კრებადობისა და სიმკვრივეთა კრებადობის პირობები. განიხილება ზრდადი ჯამების ნორმირების სხვადასხვა სახე.

Weighted sums of independent, identically distributed random vectors were studied by many authors; see, e.g., [5], [18], [7], [4], [8], [6], [19], [14], [15]. Refereeing only to these papers devoted to weak convergence of distributions of sums and density convergence, we intend to present some convergence criteria for triangular arrays and growing sums in terms of characteristics of weight matrices and try to compare different normalizations of growing sums.

1. Arrays of random vectors and weight matrices. All random vectors are assumed to be given on the probability space $(\Omega, \mathcal{F}, P)$.

Denote by P_X the distribution of a random vector $X \in \mathbb{R}^k$, and let $p_X(x)$, $f_X(t)$ and $\operatorname{cov}(X)$ be respectively its density (if it exists), characteristic function (c.f.) and covariance matrix. If $|f_X(t)| \in L_r$ for an integer $r \geq 1$, we denote

$$A_X^{(r)} = (2\pi)^{-k} \int\limits_{\mathbb{R}^k} |f_X(t)|^r \, dt.$$

Next, let Φ_G be a normal distribution in $\mathbb{R}^k$ with zero mean and covariance matrix G, $\varphi_G(x)$ its density, $\Phi = \Phi_I$, $\varphi(x) = \varphi_I(x)$, where I is the unit matrix.

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A triangular array of random vectors consisting of an infinite sequence of finite collections of independent random vectors from $\mathbb{R}^k$

$$X_{nj}, \quad j=\overline{1,m_n}, \quad n=1,2,\dots$$

($m_n \to \infty$ as $n \to \infty$), for which

$$\mathbf{E}X_{nj}=0, \quad \mathbf{E}|X_{nj}|^2=\sigma_{nj}^2<\infty, \quad j=\overline{1,m_n}, \quad n=1,2,\dots,$$

is said to be an $A^{(r)}$-array ($r \geq 0$), if:

(a) for $r=0$, the set

$$N_n^{(0)}=\big\{j: 1 \leq j \leq m_n,\ p_{X_{nj}}(x) \leq A_{nj}^{(0)}<\infty,\ (A_{nj}^{(0)})^{\frac{2}{k}}\sigma_{nj}^2 \leq M^{(0)}<\infty\big\}$$

is non-empty starting from some n;

(b) for $r \geq 1$, the set

$$N_n^{(r)}=\big\{j: 1 \leq j \leq m_n,\ (A_{nj}^{(r)})^{\frac{2}{k}}\sigma_{nj}^2 \leq M^{(r)}<\infty\big\}$$

is non-empty starting from some n, $A_{nj}^{(r)}=A_{X_{nj}}^{(r)}$.

According to the Plancherel identity $\int p_X^2(x)dx=A_X^{(2)}$, we have an inclusion $N_n^{(0)} \subset N_n^{(2)}$, where $A_X^{(2)} \leq A_X^{(0)}=\sup_{x\in\mathbb{R}^k} p_X(x)$.

Denote

$$\gamma_n^2=\max_{j\in N_n^{(r)}} \sigma_{nj}^2, \quad \Sigma_n^2=\sum_{j\in N_n^{(r)}} \sigma_{nj}^2$$

and introduce the following conditions:

(q) $\Sigma_n^2 \geq q>0$ starting from some n;

(γ) $\gamma_n \to 0 \quad (n\to\infty)$.

We denote by S_n the sum of all random vectors contained in the n-th row of the given triangular array,

$$S_n=X_{n1}+\dots+X_{nm_n},$$

and by $\xrightarrow{w}$ weak convergence as $n\to\infty$.

First, we formulate the theorem on the density convergence for the introduced triangular array of random vectors [15] which we will need in the sequel for consideration of weighted sums.

Theorem 1. *If for an $A^{(r)}$-array $P_{S_n} \xrightarrow{w} \Phi_G$, $|G|>0$, and the conditions* (q) *and* (γ) *are fulfilled, then*

$$\sup_{x\in\mathbb{R}^k} \big|p_{S_n}(x)-\varphi_G(x)\big| \to 0 \quad (n\to\infty).$$

Remark 1. Obviously, if the sum of a part of the n-th row of the triangular array has in the limit normal density and the distribution of the sum of the remaining part in the limit is normal, then the density of the total sum converges to that of the composition. This circumstance is taken into consideration in the conditions of Theorem 1.

The proof of the above theorem can be carried out by the method of c.f. by using the estimates of c.f. moduli inside and outside of some balls in terms of $A_{nj}^{(r)}$ and σ_{nj}^2 and following the scheme described in [17]. The latter paper was written under the influence of Yu.V. Prokhorov's article [13], where he proved the equivalence of asymptotic normality of normalized sums of random variables uniformly distributed in different intervals and the convergence of the corresponding densities; both types of convergence are equivalent to condition (γ) of uniform limiting negligibility in the form due to Feller. (Earlier, the equivalence between the condition (γ) and asymptotic normality of the same sums was observed by Olds [11].)

Let us consider a class of triangular arrays of random vectors which is connected with the sequence

$$(X) \qquad X_1, X_2, \dots, \ \ \mathbf{E}X_1 = 0, \ \ \mathbf{E}X_1X_1^T = I,$$

of independent, identically distributed k-dimensional random vectors:

$$(B) \qquad X_{nj} = B_{nj}X_j, \ \ j = \overline{1,n}, \ \ \sum_{j=1}^{n} B_{nj}B_{nj}^T = I, \ \ n = 1, 2, \dots .$$

Here, B_{nj} are non-degenerate $k \times k$-matrices. For some supplementary condition of "uniform" non-degeneration we can, analogously to [13] and [14], prove that the following statement is valid.

Theorem 2. *If a triangular array (B) of independent random vectors is such that $P_{X_1} \neq \Phi$ and*

$$(Q) \qquad \forall n, j : \mathrm{sp}(B_{nj}B_{nj}^T) \leq Q|B_{nj}B_{nj}^T|^{\frac{1}{k}}, \ \ 0 < Q < \infty,$$

then for the distribution of the sum S_n of random vectors contained in the n-th row to converge weakly $P_{S_n} \xrightarrow{w} \Phi$, it is necessary and sufficient that

$$(\gamma) \qquad \gamma_n^2 = \max_{1 \leq j \leq n} \mathrm{sp}(B_{nj}B_{nj}^T) \to 0 \quad (n \to \infty).$$

Proof. According to Kandelaki and Sazonov's theorem [9] for $P_{S_n} \xrightarrow{w} \Phi$ it is sufficient to show that

$$L_n(\varepsilon) = \sum_{j=1}^{n} \mathbf{E}\{|B_{nj}X_j|^2 \mathbb{1}_{(|B_{nj}X_j) \geq \varepsilon)}\} \to 0 \quad (n \to \infty)$$

for each $\varepsilon > 0$, where $\mathbb{1}_A$ is the indicator of an event A. But

$$|B_{nj}X_j|^2 = X_j^T B_{nj}^T B_{nj} X_j \leq \mathrm{sp}(B_{nj}B_{nj}^T)|X_j|^2$$

and since

$$\mathbb{1}_{(|B_{nj}X_j| \geq \varepsilon)} \leq \mathbb{1}_{(\mathrm{sp}(B_{nj}B_{nj}^T)|X_j|^2 \geq \varepsilon^2)} \leq \mathbb{1}_{(|X_j| \geq \varepsilon/\gamma_n)}$$

and $\sum_{j=1}^{n} \operatorname{sp}(B_{nj}B_{nj}^T) = \operatorname{sp}(I) = k$, we have

$$L_n(\varepsilon) \le \sum_{j=1}^{n} \mathbf{E}\{\operatorname{sp}(B_{nj}^T B_{nj})|X_j|^2 \mathbb{1}_{(|X_j| \ge \varepsilon/\gamma_n)}\} =$$
$$= k\mathbf{E}\{|X_j|^2 \mathbb{1}_{(|X_1| \ge \varepsilon/\gamma_n)}\} \to 0 \quad (n \to \infty)$$

and the weak convergence $P_{S_n} \xrightarrow{w} \Phi$ is proved.

Let now $P_{X_1} \ne \Phi$, and the convergence $P_{S_n} \xrightarrow{w} \Phi$ hold true. Assume the contrary, i.e. that γ_n does not tend to zero. Then there exist sequences n_m, j_m, $m = 1, 2, \ldots$, such that $1 \le j_m \le n_m$ and

$$\gamma_{n_m} = \operatorname{sp}(B_{n_m j_m} B_{n_m j_m}^T) \to \Lambda > 0 \quad (m \to \infty).$$

When considering the sequence $B_{n_m j_m} B_{n_m j_m}^T$ we find that the matrices $B_{n_m j_m}$ have a non-degenerate limit B, $|B| > 0$, because by the condition (Q),

$$|B_{n_m j_m}|^{2/k} = |B_{n_m j_m} B_{n_m j_m}^T|^{1/k} \ge \frac{1}{Q} \operatorname{sp}(B_{n_m j_m} B_{n_m j_m}^T) \to$$
$$\to \frac{1}{Q} \operatorname{sp}(BB^T) = \Lambda/Q \quad (m \to \infty).$$

This leads to the relation

$$f_{X_1}(Bt)g(t) = \psi(t)$$

between three characteristic functions, where $\psi(t) = e^{-|t|^2/2}$, and $f_{X_1}(Bt)$ is not normal. But this is impossible by Cramér's theorem on the decomposition of c.f. of the normal law [3, Ch.X]. □

Remark 2. As is seen from the proof of the theorem, the condition (Q) is important only for proving that the condition (γ) is necessary.

Applying Theorem 1, from Theorem 2 we find that the following theorem is valid.

Theorem 3. *If for (B) the condition (Q) is fulfilled, and $p_{X_1}(x) \le A^{(0)} < \infty$ or $|f_{X_1}(t)| \in L_r$ for some integer $r \ge 1$, then*

$$P_{S_n} \xrightarrow{w} \Phi$$

(or (γ) for $P_{X_1} \ne \Phi$) and the convergence

$$\sup_{x \in \mathbb{R}^k} |p_{S_n}(x) - \varphi(x)| \to 0 \quad (n \to \infty)$$

are equivalent.

Proof. In the case of bounded density $|f_{X_1}(t)| \in L_2$. Thus we consider the case $r \geq 1$,

$$A^{(r)}_{X_{nj}} = (2\pi)^{-k} \int |f_{x_{nj}}(t)|^r dt = |B_{nj}B^T_{nj}|^{-1/2} A^{(r)}_{X_1},$$

$$(A^{(r)}_{X_{nj}})^{2/k} \sigma^r_{nj} = (A^{(r)}_{X_{nj}})^{2/k} \operatorname{sp}(B_{nj}B^T_{nj})|B_{nj}B^T_{nj}|^{-1/k} \leq$$

$$\leq (A^{(r)}_{X_{nj}})^{2/k} Q = M^{(r)} < \infty. \qquad \square$$

Remark 3. Some B_{nj} may not obey the condition (Q); they can even degenerate. By Remark 1, we can easily formulate slightly modified conditions for the convergence of P_{S_n} and $p_{S_n}(x)$ in Theorems 2 and 3 and in Corollaries 1 and 2.

2. Growing sums. In this section we formulate the corollaries of Theorems 2 and 3 for growing sums of members of a sequence

$$(Y) \qquad Y_j = C_j X_j, \quad j = 1, 2, \dots,$$

of independent random vectors generated by the sequence (X) and nondegenerate matrices C_j, $j = 1, 2, \dots$ (cf. [14]).

Corollary 1. *If the matrices C_j, $j = 1, 2, \dots,$ are such that*

$$(Q_g) \qquad \lambda_1(C_j C^T_j)/\lambda_k(C_j C^T_j) \leq Q_g < \infty, \quad j = 1, 2, \dots,$$

where $\lambda_1(\cdot)$ and $\lambda_k(\cdot)$ are respectively maximal and minimal eigenvalues of the matrix, and $P_{X_1} \neq \Phi$, then for the relation $P_{S_n} \xrightarrow{w} \Phi$ for the distribution of the normalized sum

$$S_n = D_n^{-1/2}\big(Y_1 + \dots + Y_n\big), \quad \textit{where} \quad D_n = \sum_{j=1}^n C_j C^T_j,$$

to be fulfilled, it is necessary and sufficient that

$$(\gamma_g) \qquad \lim_{n\to\infty} \frac{\max\limits_{1\leq j\leq n} \lambda_1(C_j C^T_j)}{\lambda_k D_n} = 0.$$

Corollary 2. *If in the conditions of Corollary* 1 $P_{S_n} \xrightarrow{w} \Phi$ *and* $|f_{X_1}(t)| \in L_r$ *for an integer* $r \geq 1$, *then*

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}^k} |p_{S_n}(x) - \varphi(x)| = 0.$$

To prove the corollaries we apply the variational properties of eigenvalues of matrices (see, e.g., [1, Ch. X]). Note that if $\operatorname{cov}(X_1) = T \neq I$, $|T| > 0$, then the conditions (Q) and (γ) as well as (Q_g) and (γ_g) remain unchanged.

Consider now the version of the central limit theorem in $\mathbb{R}^k$ from Cramér's book [3, Ch. X] for the sequence (Y): If for $n \to \infty$

$$(D_0) \qquad \frac{1}{n}\sum_{j=1}^{n} C_j C_j^T = \frac{1}{n}\, D_n \to D_0, \quad \mathrm{sp}(D_0) > 0,$$

and

$$(L) \qquad \forall \varepsilon > 0 \;\; \frac{1}{n}\sum_{j=1}^{n} \mathbf{E}\{|C_j X_j|^2 \mathbb{1}_{|C_j X_j| \geq \varepsilon\sqrt{n}}\} \to 0,$$

then for the distribution of the normalized sum $U_n = \frac{1}{\sqrt{n}}\, D_n$ we have

$$P_{U_n} \xrightarrow{w} \Phi_{D_0}.$$

It is easy to see that under the condition (D_0), where $|D_0| > 0$, the implications

$$P_{S_n} \xrightarrow{w} \Phi \Rightarrow \sup_{x\in\mathbb{R}^k} |p_{S_n}(x) - \varphi(x)| \to 0$$

and

$$P_{U_n} \xrightarrow{w} \Phi_{D_0} \Rightarrow \sup_{x\in\mathbb{R}^k} |p_{U_n}(x) - \varphi_{D_0}(x)| \to 0$$

are equivalent.

This means that under the conditions (Q_g) and (γ_g) the sums can be normalized by using both methods.

Example. Let among the matrices C_j be only a finite number s of different ones; then the condition (L) from the Cramér's theorem is fulfilled, and when the condition (D_0) is fulfilled, too, we are, in the situation of the so-called s-sequences of independent random variables (with s different distributions of the members of (Y) [12, Ch. 7, §2]). The statement

$$|f_{X_1}(t)| \in L_r \quad \text{for an integer} \quad r \geq 1 \Rightarrow$$
$$\Rightarrow \sup_{x\in\mathbb{R}^k} |p_{U_n}(x) - \varphi_{D_0}(x)| \to 0 \quad (n \to \infty)$$

is valid due to Corollaries 1 and 2 and the above equivalence of implications.

A different way to prove the latter statement if to use, just as in [2], [10] and [16], the decomposition

$$P_{\sqrt{n}U_n} = P_{M_1 X_1}^{\nu_n(1)*} * \cdots * P_{M_s X_1}^{\nu_n(s)*},$$

where $M_1, \dots, M_s$ are different matrices among C_j, $j = 1, 2, \dots,$ and $\nu_{n(i)}$ is the frequency of M_i among $C_1, \dots, C_n$, $i = 1, \dots, s$, $\nu_{n(1)} + \cdots + \nu_{n(s)} = n$ (for the sake of simplicity, one can assume that all $\nu_n(i)$ tend to infinity as $n \to \infty$).

Corollaries 1 and 2, the version of the central limit theorem from Cramér's book and the above-mentioned decomposition provide us with natural means allowing one to study weak convergence and density convergence in considering an ergodic random choice from a finite number of weight matrices.

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