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Mathematics

Continuity Estimates with Respect to Volatility for the American Foreign Exchange Option

Malkhaz Shashiashvili*, Nasir Rehman**

* *A. Razmadze Mathematical Institute, Tbilisi, Abdus Salam School of Mathematical Sciences, GC University, Lahore, Pakistan*
** *Abdus Salam School of Mathematical Sciences, GC University, Lahore, Pakistan*

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ABSTRACT. We consider the American foreign exchange put option problem in one-dimensional diffusion model for exchange rate. The volatility is assumed to be an arbitrary strictly positive bounded function of time.

We establish several continuity estimates for the American option value process, the optimal hedging portfolio and the corresponding consumption process with respect to volatility function. *© 2008 Bull. Georg. Natl. Acad. Sci.*

Key words: *foreign exchange option, continuity estimates, volatility function, optimal hedging portfolio, estimates for the Snell envelope.*

Introduction

Let (Ω, F, P) be a probability space and (W_t) $0 \le t \le T$ a one-dimensional standard Brownian motion on it. We denote by (F_t) $0 \le t \le T$ the P-completion of natural filtration of the Brownian motion. Throughout the paper we shall assume that the time horizon T is finite.

On the filtered probability space (Ω, F, F_t, P) $0 \le t \le T$ we consider a financial market with two currencies: domestic and foreign with their corresponding interest rates $r^d(t)$ and $r^f(t)$, being two bounded positive measurable functions of time, i.e.

$$0 \le r^d(t) \le \bar{r}, \quad 0 \le r^f(t) \le \bar{r}, \quad 0 \le t \le T. \tag{1.1.}$$

The exchange rate processes (Q_t, F_t) $0 \le t \le T$ and $(\widetilde{Q}_t, F_t)$ $0 \le t \le T$ (with different volatility functions $\sigma(t)$, $0 \le t \le T$ and $\widetilde{\sigma}(t)$, $0 \le t \le T$ are strong solutions of the following linear stochastic differential equations

$$dQ_t = Q_t \cdot (r^d(t) - r^f(t)) \cdot dt + Q_t \cdot \sigma(t) \cdot dW_t, \; Q_0 > 0, \; 0 \le t \le T. \tag{1.2}$$

$$d\widetilde{Q}_t = \widetilde{Q}_t (r^d(t) - r^f(t)) \cdot dt + \widetilde{Q}_t \cdot \widetilde{\sigma}(t) \cdot dW_t, \; \widetilde{Q}_0 = Q_0, \; 0 \le t \le T$$

where the volatility functions $\sigma(t)$ and $\widetilde{\sigma}(t)$, $0 \le t \le T$ are two arbitrary measurable functions of time, such that

$$0 < \underline{\sigma} \le \sigma(t) \le \overline{\sigma}, \; 0 < \underline{\sigma} \le \widetilde{\sigma}(t) \le \overline{\sigma}, \; 0 \le t \le T. \tag{1.3}$$

We note here that the probability measure P denotes the so-called domestic risk-neutral probability measure and Q_t (respectively $\widetilde{Q}_t$) gives the units of domestic currency per unit of foreign currency at time t.

In this article we study the American foreign exchange put option problem with payoff function

$$g(x) = (K - x)^+,\ x \geq 0 \tag{1.4}$$

and its dependence on volatility function $\sigma(t),\ 0 \leq t \leq T$.

It is well-known (see, for example, Karatzas, Shreve [1], chapter 2) that the value process of the American foreign exchange put written on exchange rate process $Q_t\ \ 0 \leq t \leq T$ (with the volatility function $\sigma(t),\ 0 \leq t \leq T$) is defined as follows

$$V_t = \operatorname*{esssup}_{t \leq \tau \leq T} E(e^{-\int_t^\tau r^d(u)du} \cdot (K - Q_t)^+ / F_t),\quad 0 \leq t \leq T, \tag{1.5}$$

where the essential supremum is taken over all $(F_u)\ 0 \leq u \leq T$ -stopping times τ such that $t \leq \tau \leq T$.

We remind here the fundamental connection between the value process $(V_t, F_t)\ 0 \leq t \leq T$ and the replicating portfolio for the American put option.

An $(F_t)\ 0 \leq t \leq T$ - progressively measurable process $\Pi = (\Pi_t)\ 0 \leq t \leq T$ with $\int_0^T \Pi_t^2 dt < \infty$ (a.s.), is called portfolio process and $(F_t)\ 0 \leq t \leq T$ -adapted process $C=C_t\ 0 \leq t \leq T$ with nondecreasing continuous paths and with $C_0 = 0,\ C_T < 8$ (a. s.) is called consumption process.

It was established by Bensoussan [2] and Karatzas [3] that the value process of the American option can be precisely replicated by a certain portfolio process $\Pi = (\Pi_t)\ 0 \leq t \leq T$ and the related consumption process $C=(C_t)$ $0 \leq t \leq T$, that is

$$dV_t = r^d(t) \cdot V_t \cdot dt + \Pi_t \cdot \sigma(t) \cdot dW_t - dC_t,\ 0 \leq t \leq T, \tag{1.6}$$

with some initial condition $V_0,\ V_0 \geq 0$.

Such a pair of processes $(\Pi, C) = (\Pi_t, C_t)\ 0 \leq t \leq T$, is called the optimal portfolio and consumption processes pair. The quantity Π_t shows the amount of investment in foreign currency at time t. Thus the process $(V_t)\ 0 \leq t \leq T$ is also the value process of the optimal hedging portfolio with consumption process $(C_t)\ 0 \leq t \leq T$, until the time of exercise by the buyer of the option.

The monotone dependence properties with respect to volatility function for the European as well as American option values have been intensively investigated by several authors during the past decade, see for example, El Karoui, Jeanblanc-Picque and Shreve [4], Hobson [5] and Ekstrom [6].

In this paper we derive continuity estimates with respect to the volatility function for the American foreign exchange put value process, optimal hedging portfolio and the related consumption process. The derivation of the latter estimates is essentially based on a new *a priori* inequalities for the difference of two Snell envelopes and its components established in Danelia, Dochviri and Shashiashvili [7].

2. The Lipschitz property of the American put value process with respect to volatility

The value process of the American put option written on exchange rate with volatility function $\widetilde{\sigma}(t)$ is defined as follows (see formula 1.5)

$$\widetilde{V}_t = \operatorname*{esssup}_{t \leq \tau \leq T} E(e^{-\int_t^\tau r^d(u)du \cdot} (K - \widetilde{Q}\tau)^+ / F_t), \tag{2.1}$$

where the essential supremum is taken over all $(F_u)\ 0 \leq u \leq T$ - stopping times τ, such that $t \leq \tau \leq T$.

Introduce the American put's discounted payoff processes

$$X_t = e^{-\int_0^t r^d(u)du \cdot} (K - Q_t)^+,\ \widetilde{X}_t = e^{-\int_0^t r^d(u)du} \cdot (K - \widetilde{Q}_t)^+,\ 0 \leq t \leq T, \tag{2.2}$$

and their corresponding Snell envelopes

$$Y_t = \operatorname*{esssup}_{t \leq \tau \leq T} E(X_\tau / F_t),\ \ \widetilde{Y}_t = \operatorname*{esssup}_{t \leq \tau \leq T} E(\widetilde{X}_\tau / F_t),\ \ 0 \leq t \leq T \tag{2.3}$$

The following relationships between Value processes and the corresponding Snell envelopes are now obvious

$$V_t = e^{\int_0^t r^d(u)du} \cdot Y_t, \; \widetilde{V}_t = e^{\int_0^t r^d(u)du} \cdot \widetilde{Y}_t, \; 0 \le t \le T. \tag{2.4}$$

Consider also the Doob-Meyer decomposition of the Snell envelopes

$$Y_t = M_t - B_t, \quad B_0 = 0, \; \widetilde{Y}_t = \widetilde{M}_t - \widetilde{B}_t, \quad \widetilde{B}_0 = 0, \; 0 \le t \le T \tag{2.5}$$

where (M_t, F_t) $0 \le t \le T$ and $(\widetilde{M}_t, F_t)$ $0 \le t \le T$ are two uniformly integrable continuous martingales and (B_t, F_t) and $(\widetilde{B}_t, F_t)$ $0 \le t \le T$ - two nonnegative nondecreasing continuous intergrable stochastic processes.

We shall need for further use one technical result.

Lemma 1. *The following estimate is valid for the uniform distance between the exchange rate processes*

$$E(\sup_{0\le\tau\le T} \left|Q_t - \widetilde{Q}_t\right|)^2 \le c \cdot Q_0^2 \cdot \left\|\sigma - \widetilde{\sigma}\right\|^2_{L_2[0,T]} \tag{2.6}$$

where $\left\|\sigma - \widetilde{\sigma}\right\|^2_{L_2[O,T]} = \int_0^T (\sigma(s) - \widetilde{\sigma}(s))^2 ds$, *and c is some constant dependent on* $\bar{r}$, $\bar{\sigma}$ *and T.*

Proof. We shall use the standard techniques of stochastic differential equations together with Gronwall inequality to obtain the latter bound (see, for example Karatzas, Shreve [8], chapter 5).

Denote by $\hat{Q}$ the difference $Q_t - \widetilde{Q}_t$

$$\hat{Q}_t = Q_t - \widetilde{Q}_t, \quad 0 \le t \le T \tag{2.7}$$

where $\hat{Q}_0 = 0$

Then we have

$$\hat{Q}_t = \int_0^t \hat{Q}_u \cdot (r^d(u) - r^f(u)) \cdot du + \int_0^t [Q_u \cdot (\sigma(u) - \widetilde{\sigma}(u)) + \hat{Q}_u \cdot \widetilde{\sigma}(u)] dW_u, \tag{2.8}$$

From here we can write

$$\sup_{0\le s\le t} \hat{Q}_s^2 \le 2\int_0^t \hat{Q}_u^2 \cdot du \cdot \int_0^T (r^d(u) - r^f(u))^2 \cdot du + 2 \sup_{0\le s\le t} (\int_0^t [Q_u \cdot (\sigma(u) - \widetilde{\sigma}(u)) + \hat{Q}_u \cdot \widetilde{\sigma}(u)] \cdot dW_u)^2$$

Taking the mathematical expectation on both sides of the latter inequality together with the use of Doob's classical maximal inequality we get

$$E \sup_{0\le s\le t} \hat{Q}_s^2 \le 2 \cdot \int_0^t E\hat{Q}_u^2 du \cdot \int_0^T (r^d(u) - r^f(u))^2 \cdot du + 8 \cdot \int_0^t E[Q_u \cdot (\sigma(u) - \widetilde{\sigma}(u) + \hat{Q}_u \cdot \widetilde{\sigma}(u)]^2 du \cdot \tag{2.9}$$

Denote $\varphi(t) = E \sup_{0\le s\le t} \hat{Q}_s^2$, $0 \le t \le T$

then from the latter inequality and assumptions (1.1) – (1.3) we obtain

$$\phi(t) \le 2 \cdot \bar{r}^2 \cdot T \cdot \int_0^t \phi(u) du + 16 \cdot \bar{\sigma}^2 \cdot \int_0^t \phi(u) \cdot du + 16 \cdot \int_0^t EQ_u^2 \cdot (\sigma(u) - \widetilde{\sigma}(u))^2 du, \; 0 \le t \le T.$$

Now we use the standard bound (see, for example, Karatzas, Shreve [8], chapter 5, theorem 2.9)

$$EQ_u^2 \le b \cdot Q_0^2, \; 0 \le u \le T,$$

where constant b depends on $\bar{r}$, $\bar{\sigma}$ and T.

Therefore the previous inequality becomes

$$\phi(t) \le (2 \cdot \bar{r}^2 \cdot T + 16 \cdot \bar{\sigma}^2) \cdot \int_0^t \phi(u) du + 16 \cdot b \cdot Q_0^2 \cdot \int_0^t (\sigma(u) - \widetilde{\sigma}(u))^2 \cdot du, \; 0 \le t \le T. \tag{2.10}$$

Finally applying the classical Gronwall inequality we get

$$E(\sup_{0\le t\le T}(Q_t-\widetilde{Q}_t)^2 \le c\cdot Q_0^2\cdot\int_0^T (\sigma(u)-\widetilde{\sigma}(u))^2\cdot du \tag{2.11}$$

where the constant c depends on $\bar{r}$, $\bar{\sigma}$ and T.

Now we formulate and prove the Lipschitz continuity property of the American put value process with respect to volatility function.

Theorem 1. *The following estimate holds for the American foreign exchange put value process*

$$E\sup_{0\le t\le T}\left|V_t-\widetilde{V}_t\right|\le c_1\cdot Q_0\cdot\|\sigma-\widetilde{\sigma}\|_{L_2[0,T]} \tag{2.12}$$

where the constant c_1 depends on $\bar{r}$, $\bar{\sigma}$ and T.

Proof. One can write from equalities (1.5) and (2.1)

$$\left|V_t-\widetilde{V}_t\right|\le \operatorname*{esssup}_{t\le\tau\le T}E(\left|(K-Q_\tau)^+-(K-\widetilde{Q}_\tau)^+\right|/F_t)\le E\ (\sup_{t\le u\le T}\left|Q_u-\widetilde{Q}_u\right|/F_t).$$

After taking the supremum on both sides of the latter bound we get

$$\sup_{0\le t\le T}\left|V_t-\widetilde{V}_t\right|\le \sup_{0\le t\le T}E\ (\sup_{0\le u\le T}\left|Q_u-\widetilde{Q}_u\right|/F_t). \tag{2.13}$$

Introduce the notation

$$m_t=E(\eta(w)/F_t),\ \text{where}\ \eta(w)=\sup_{0\le u\le T}\left|Q_u-\widetilde{Q}_u\right|.$$

Then we may apply the classical Doob's maximal inequality and obtain

$$\mathrm{E}(\sup_{0\le t\le T} m_t)^2\le 4E\,\eta^2(w).$$

Hence from the inequality (2.13) we come to the estimate

$$E\ (\sup_{0\le t\le T}\left|V_t-\widetilde{V}_t\right|)^2\le 4E(\sup_{0\le t\le T}\left|Q_t-\widetilde{Q}_t\right|)^2 \tag{2.14}$$

From the latter inequality using Lemma 1 we get the desired estimate (2.12).

3. Continuity estimates for the optimal hedging portfolio process and the related consumption with respect to volatility

We are in a complete financial market model defined by conditions (1.1)-(1.3) and we consider the optimal hedging portfolio processes $\Pi=(\Pi_t)$ $0\le t\le T$ and $\widetilde{\Pi}=(\widetilde{\Pi}_t)$ $0\le t\le T$ for the American put option problem corresponding to volatility functions $\sigma(t)$, $0\le t\le T$ and $\widetilde{\sigma}(t)$, $0\le t\le T$.

Theorem 2. *For the optimal hedging portfolio processes Π and $\widetilde{\Pi}$ the following continuity estimate does hold*

$$E\int_0^T(\Pi_s\cdot\sigma(s)-\widetilde{\Pi}_s\cdot\widetilde{\sigma}(s))^2ds\le c_2\cdot K\cdot Q_0\cdot\|\sigma-\widetilde{\sigma}\|_{L_2[0,T]}, \tag{3.1}$$

where c_2 is some constant dependent on $\bar{r}$, $\bar{\sigma}$ and T.

Proof. We apply the Ito product differentiation rule to the equality

$$V_t=e^{\int_0^t r^d(u)du}\cdot Y_t\ ,\ 0\le t\le T,$$

then we get

$$dV_t=r^d(t)\cdot V_tdt+e^{\int_0^t r^d(u)du}\cdot dM_t-e^{\int_0^t r^d(u)du}\cdot dB_t,\ 0\le t\le T\ . \tag{3.2}$$

On the other, hand by the relation (1.6) we have

$$dV_t=r^d(t)\cdot V_tdt+\Pi_t\cdot\sigma(t)\cdot dW_t-dC_t,\ 0\le t\le T\ . \tag{3.3}$$

By the uniqueness of the canonical decomposition of continuous semimartingale (V_t,F_t) $0 \le t \le T$ we obtain

$$\int_0^t \Pi_s \cdot \sigma(s) dWs = \int_0^t \gamma_s \cdot dM_s, \ C_t = \int_0^t \gamma_s \cdot dB_s, \ 0 \le t \le T \tag{3.4}$$

where

$$\gamma_t = e^{\int_0^t r^d(u)du}, \ 0 \le t \le T \tag{3.5}$$

Similarly for the optimal hedging portfolio process $\widetilde{\Pi}$ with the consumption component $\widetilde{C}$ we have

$$\int_0^t \widetilde{\Pi}_s \cdot \widetilde{\sigma}(s) dW_s = \int_0^t \gamma_s \cdot d\widetilde{M}_s, \ \widetilde{C}_t = \int_0^t \gamma_s \cdot d\widetilde{B}_s, \ 0 \le t \le T \tag{3.6}$$

Therefore we can write

$$\int_0^t (\Pi_s \cdot \sigma(s) - \widetilde{\Pi}_s \cdot \widetilde{\sigma}(s)) \cdot dW_s = \int_0^t \gamma_s \cdot d(M_s - \widetilde{M}_s), \tag{3.7}$$

$$C_t - \widetilde{C}_t = \int_0^t \gamma_s \cdot d(B_s - \widetilde{B}_s), 0 \le t \le T \tag{3.8}$$

We obtain from the equality (3.7)

$$E\int_0^T (\Pi_s \cdot \sigma(s) - \widetilde{\Pi}_s \cdot \widetilde{\sigma}(s))^2 ds = E\int_0^T \gamma_s^2 \cdot d < M - \widetilde{M} > s \le e^{2 \cdot \bar{r} T} \cdot E < M - \widetilde{M} >_T .$$

Now we apply the crucial estimates from the paper by Danelia, Dochviri and Shashiashvili [7], in particular theorem 2.4 therein, from which we obtain

$$E < M - \widetilde{M} >_T \le 9 \cdot \left\| (X - \widetilde{X})_T^* \right\|_2 \cdot (\left\| X_T^* \right\|_2 + \left\| \widetilde{X}_T^* \right\|_2), \tag{3.9}$$

where $\left\| U_T^* \right\|_2 = \left[E(\sup\limits_{0 \le t \le T} |Ut|)^2 \right]^{1/2}$.

From equalities (2.2) we write

$$|X_t| \le K, \ |\widetilde{X}_t| \le K,$$

while from Lemma 1 we have

$$\left\| (X - \widetilde{X})_T^* \right\|_2 \le \sqrt{c} \cdot Q_0 \cdot \left\| \sigma - \widetilde{\sigma} \right\|_{L_2[0,T]} \tag{3.10}$$

Therefore ultimately we get

$$E\int_0^T (\Pi_s \cdot \sigma(s) - \widetilde{\Pi}_s \cdot \widetilde{\sigma}(s))^2 ds \le c_2 \cdot K \cdot Q_0 \cdot \left\| \sigma - \widetilde{\sigma} \right\|_{L_2[o,T]} .$$

Theorem 3. *For the consumption processes* C_t, $0 \le t \le T$ *and* $\widetilde{C}_t$, $0 \le t \le T$ *corresponding to the volatility functions* $\sigma(t)$, $0 \le t \le T$ *and* $\widetilde{\sigma}(t)$, $0 \le t \le T$, *the following continuity estimate is valid*

$$E \sup_{0 \le t \le T} \left| C_t - \widetilde{C}_t \right| \le c_3 \cdot K^{1/2} \cdot Q_0^{1/2} \cdot \left\| \sigma - \widetilde{\sigma} \right\|_{L_2[0,T]}^{1/2}, \tag{3.11}$$

where the constant c_3 *depends only on* $\bar{r}$, $\bar{\sigma}$ *and T.*

Proof. we get from the equality (3.8)

$$C_t - \widetilde{C}_t = \int_0^t \gamma_s \cdot d(B_s - \widetilde{B}_s) = \gamma_t \cdot (B_t - \widetilde{B}_t) - \int_0^t (B_s - \widetilde{B}_s) \cdot \gamma_s \cdot r^d(s)ds, \; 0 \le t \le T .$$

Therefore the following inequality holds

$$\left| C_t - \widetilde{C}_t \right| \le \gamma_t \cdot \left| B_t - \widetilde{B}_t \right| + \sup_{0 \le s \le t} \left| B_s - \widetilde{B}_s \right| \cdot \bar{r} \cdot \gamma_t \cdot t \;, \; 0 \le t \le T \; .$$

Let us take the supremum with respect to time argument on both sides of the latter inequality, we shall get

$$\sup_{0 \le t \le T} \left| C_t - \widetilde{C}_t \right| \le \gamma_T \cdot \sup_{0 \le t \le T} \left| B_t - \widetilde{B}_t \right| + \bar{r} \cdot T \cdot \gamma_T \cdot \sup_{0 \le t \le T} \left| B_t - \widetilde{B}_t \right| \; ,$$

hence

$$E \sup_{0 \le t \le T} \left| C_t - \widetilde{C}_t \right| \le \gamma_T \cdot (1 + \bar{r} \cdot T) \cdot E \sup_{0 \le t \le T} \left| B_t - \widetilde{B}_t \right| \; . \tag{3.12}$$

From the estimate 2.22 in Danelia, Dochviri and Shashiashvili [7] we have

$$E \sup_{0 \le t \le T} \left| B_t - \widetilde{B}_t \right| \le 9 \cdot \left\| (X - \widetilde{X})_T^* \right\|_2^{1/2} \cdot \sqrt{2K}$$

and from the inequality (3.10) we get

$$\left\| (X - \widetilde{X})_T^* \right\|_2^{1/2} \le c^{1/4} \cdot Q_0^{1/2} \cdot \left\| \sigma - \widetilde{\sigma} \right\|_{L_2[0,T]}^{1/2} .$$

Finally we come to the estimate

$$E \sup_{0 \le t \le T} \left| C_t - \widetilde{C}_t \right| \le c_3 \cdot K^{1/2} \cdot Q_0^{1/2} \cdot \left\| \sigma - \widetilde{\sigma} \right\|_{L_2[0,T]}^{1/2} \quad 0 \le t \le T \; .$$

მათემატიკა

უწყვეტობის შეფასებები ვოლატილობის მიმართ ამერიკული სავალუტო ოფციონისათვის

მ. შაშიაშვილი*, ნ. რეჰმანი**

** ა. რაზმაძის მათემატიკის ინსტიტუტი, თბილისი*
*** აბდუ სალამის მათემატიკურ მეცნიერებათა სკოლა, GC უნივერსიტეტი, ლაჰორი, პაკისტანი*

(წარმოდგენილია აკადემიის წევრის ე. ნადარაიას მიერ)

ავტორები ამერიკული სახის გაყიდვის სავალუტო ოფციონის ამოცანას განიხილავენ სავალუტო კურსის ერთგანზომილებიან დიფუზიურ მოდელში. იგულისხმება, რომ ვოლატილობა წარმოადგენს დროის რაიმე მკაცრად დადებით შემოსაზღვრულ ფუნქციას.

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