Georgian Math. J., Ahead of Print
DOI 10.1515/gmj-2012-0007

Stability estimates for the multidimensional elliptic obstacle problem

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Abstract. A Hölder type stability estimate in the second order Sobolev space of functions is established for the solution of the elliptic obstacle problem with respect to variations of the coefficients of the corresponding differential operator. In modern finance, this estimate provides results on the sensitivity of a perpetual American option price with respect to variations of the volatilities in the underlying assets prices. To meet the objective of this paper, the stability result with respect to external force functions, which was proved by J.-F. Rodrigues under the nondegeneracy condition, is generalized to the case of arbitrary functions belonging to the space $L^p(D)$, $p \geq 2$.

Keywords. Unilateral elliptic obstacle problem, stability estimates with respect to the coefficients of a differential operator, sensitivity of perpetual American option prices with respect to volatilities.

2010 Mathematics Subject Classification. 35J15, 35J86, 49J40, 91G80.

1 Introduction

The classical obstacle problem describing the equilibrium position of a stretched membrane under the action of an external force function f is the best-studied one. There are a few manuals dedicated to it, see, e.g., Kinderlehrer and Stampacchia [5], Rodrigues [6], Troianiello [7]. The problem has been the subject of investigation for several decades. As a result, today quite a complete theory is available for this obstacle problem.

In finance, an American option price is derived from the underlying assets prices by solving the multidimensional unilateral obstacle problem (see Jaillet, Lamberton and Lapeyre [4]).

An important problem that has arisen in modern finance is to study the sensitivity of an American option price with respect to variations of the volatilities in several underlying assets (see Achdou [1]). In terms of the mathematical theory of variational inequalities this amounts to studying the question whether a solution

The authors acknowledge the financial support by the HEC, Islamabad, for the academic year 2010–2011 during which this research was carried out.

of the multidimensional unilateral obstacle problem is stable with respect to variations of the main coefficients of the corresponding differential operator. Rodrigues [6, Section 5.4, Theorem 4.8] proved a stability estimate with respect to an external force function f only in the case of strictly nondegenerate forces satisfying the conditions

$$\begin{aligned} &f \in L^{\infty}(D), \\ &|f(x)| \geq \lambda, \quad \text{a.e. in } D \text{ for some } \lambda > 0. \end{aligned} \tag{1.1}$$

He assumes that the obstacle function $\psi(x)$ is equal to zero and his estimate has the following form:

$$\begin{aligned} &\|u_2 - u_1\|_{W^{2,p}(D)} \\ &\quad \leq c\big(\lambda, \|f_1\|_{L^\infty(D)}, \|f_2\|_{L^\infty(D)}\big) \cdot \|f_2 - f_1\|_{L^p(D)}^{1/p} \quad \text{for } p \geq \tfrac{2n}{n+2}, \end{aligned} \tag{1.2}$$

where n is the dimension of the Euclidean space, D is a bounded domain in it, $W^{2,p}(D)$ denotes the second order Sobolev space of functions defined in D and integrable together with their first and second order partial derivatives to the p-th power, and $u_i(x)$, $i = 1, 2$, are the solutions of the multidimensional elliptic obstacle problem corresponding to the external forces $f_i(x)$, $i = 1, 2$.

This paper pursues the double aim of, first, deriving a stability result analogous to (1.2) but without restrictions (1.1), and, second, applying it to obtain a stability estimate for a solution of the multidimensional elliptic obstacle problem with respect to variations of the coefficients of the corresponding elliptic differential operator.

The exact formulation of the elliptic obstacle problem reads as follows.

Consider an n-dimensional bounded domain D with a boundary ∂D of the class $C^{1,1}$. Denote by $Lu(x)$ the linear second order elliptic differential operator acting on a function u from the Sobolev space $W^{2,p}(D)$, $p \geq 2$,

$$Lu(x) = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\Big(a_{ij}(x)\,\frac{\partial u(x)}{\partial x_i}\Big) + \sum_{i=1}^{n} b_i(x)\,\frac{\partial u(x)}{\partial x_i} + c(x)u(x), \tag{1.3}$$

with its adjoint operator

$$\begin{aligned} L^* u(x) = &\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\,\frac{\partial u(x)}{\partial x_j}\Big) \\ &- \sum_{i=1}^{n} b_i(x)\,\frac{\partial u(x)}{\partial x_i} + \Big(c(x) - \sum_{i=1}^{n} \frac{\partial b_i(x)}{\partial x_i}\Big) u(x). \end{aligned} \tag{1.4}$$

Assume that

$$\begin{gathered} a_{ij}(x) = a_{ji}(x), \quad a_{ij} \in C^{0,1}(\overline{D}), \quad b_i \in C^{0,1}(\overline{D}), \\ c \in L^\infty(D), \quad c(x) \le 0 \quad \text{a.e. in } D, \end{gathered} \tag{1.5}$$

and that the following uniform ellipticity condition holds:

$$\sum_{i,j=1}^{n} a_{ij}(x) y_i y_j \ge \alpha |y|^2, \quad y = (y_1, \dots, y_n) \in \mathbb{R}^n,\ \alpha > 0. \tag{1.6}$$

Denote by $C_0^\infty(D)$ the space of the infinitely differentiable functions with compact support in D.

Let us consider an obstacle function ψ, $\psi \in W^{2,p}(D)$, $p \ge 2$, such that $\max(0, \psi(x)) \in W_0^{1,p}(D)$, $p \ge 2$. Here $W_0^{1,p}(D)$ is the closure of $C_0^\infty(D)$ in $W^{1,p}(D)$. Throughout the paper the obstacle function $\psi(x)$ is assumed to be fixed.

An external force function is denoted by f, $f \in L^p(D)$, $p \ge 2$.

By [7, Theorem 5.2], there exists a unique solution $u(x)$ of the following unilateral elliptic obstacle problem:

Find $u \in W^{2,p}(D) \cap W_0^{1,p}(D)$, $p \ge 2$, such that

$$\left.\begin{aligned} &u(x) \ge \psi(x), \quad Lu(x) \le f(x), \\ &\big(f(x) - Lu(x)\big)\big(u(x) - \psi(x)\big) = 0, \end{aligned}\right\} \quad \text{a.e. in } D. \tag{1.7}$$

The same theorem establishes the following Lewy–Stampacchia inequality:

$$\min(L\psi, f) \le Lu(x) \quad \text{a.e. in } D. \tag{1.8}$$

The paper is organized as follows. In Section 2, we prove some inequalities which are of independent interest. The proof is based on the relationship between $Lv(x)$ and $L\phi(v(x))$, where $\phi(y)$ is a smooth univariate convex function and $v(x)$ an arbitrary function belonging to the second order Sobolev space $H^2(D) = W^{2,2}(D)$. In Section 3, Rodrigues' stability result with respect to external force functions [6] is generalized to the case without restrictions (1.1). In Section 4, we establish the basic result of this paper, which is the stability estimate with respect to the coefficients of the corresponding differential operator for a solution of the unilateral elliptic obstacle problem.

2 Some auxiliary inequalities

We begin this section with the assertion which is the key to all the estimates derived in this paper.

Theorem 2.1. *Suppose $\phi(u)$, $-\infty < u < \infty$, is a twice continuously differentiable univariate convex function with $\phi(0) = 0$. Then the inequality*

$$\phi'(v(x)) \cdot Lv(x) \le L\phi(v(x)) \tag{2.1}$$

holds for any function $v \in C^2(D)$.

Proof. One can easily check that the following identity is valid:

$$\begin{aligned} L\phi(v(x)) = \phi''(v(x)) \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial v(x)}{\partial x_i} \frac{\partial v(x)}{\partial x_j} \\ + \phi'(v(x))Lv(x) - c(x)\big(v(x)\phi'(v(x)) - \phi(v(x))\big). \end{aligned} \tag{2.2}$$

By the Taylor formula we have

$$\begin{aligned} \phi(u) &= \phi(v) + (u-v)\phi'(v) + \frac{1}{2}(u-v)^2\phi''(\xi) \\ &\ge \phi(v) + (u-v)\phi'(v), \quad \text{as } \phi''(\xi) \ge 0. \end{aligned}$$

Putting $u = 0$ in this inequality, we get

$$v\phi'(v) - \phi(v) \ge 0 \quad \text{for any } v \in \mathbb{R}$$

and hence, by assumption (1.5),

$$-c(x)\big(v(x)\phi'(v(x)) - \phi(v(x))\big) \ge 0.$$

From the latter inequality and the ellipticity condition (1.6) we obtain the estimate (2.1). □

Lemma 2.2. *Let $u_i(x)$, $i = 1, 2$, be a solution of the obstacle problem* (1.7) *corresponding to the external force functions $f_i(x)$, $i = 1, 2$. Take an arbitrary measurable real-valued function $\varphi(u)$ such that*

$$\varphi(u) \ge 0 \text{ if } u > 0, \quad \varphi(u) \le 0 \text{ if } u < 0, \quad \varphi(0) = 0. \tag{2.3}$$

Then the following function is nonnegative:

$$\varphi(u_2 - u_1)\big[(Lu_2 - f_2) - (Lu_1 - f_1)\big] \ge 0 \quad \text{a.e. in } D. \tag{2.4}$$

Proof. By the definition of the obstacle problem (1.7) we have

$$\begin{aligned} &I_{(u_2>u_1)}(f_2 - Lu_2) = 0, \\ &I_{(u_2<u_1)}(f_1 - Lu_1) = 0, \end{aligned} \qquad \text{a.e. in } D, \tag{2.5}$$

where the symbols $I_{(u_2>u_1)}$ and $I_{(u_2<u_1)}$ denote the characteristic function of the sets $\{x : \ u_2(x) > u_1(x)\}$ and $\{x : \ u_2(x) < u_1(x)\}$, respectively.

Therefore we can write

$$\begin{aligned} &\varphi(u_2 - u_1)\big[(Lu_2 - f_2) - (Lu_1 - f_1)\big] \\ &\quad = I_{(u_2>u_1)}\varphi(u_2 - u_1)(f_1 - Lu_1) \\ &\qquad + I_{(u_2=u_1)}\varphi(0)\big[(Lu_2 - f_2) - (Lu_1 - f_1)\big] \\ &\qquad + I_{(u_2<u_1)}\big(-\varphi(u_2 - u_1)\big)(f_2 - Lu_2) \geq 0 \quad \text{a.e. in } D. \qquad \square \end{aligned}$$

Lemma 2.2 leads to our next important assertion.

Theorem 2.3. *The following inequality is fulfilled for a difference of two solutions* $u_i(x)$*,* $i = 1, 2$*, of the obstacle problem* (1.7) *which correspond to the external force functions* $f_i(x)$*,* $i = 1, 2$*,*

$$\big|L(u_2 - u_1)\big| \leq \operatorname{sgn}(u_2 - u_1) \cdot L(u_2 - u_1) + 2|f_2 - f_1| \quad \textit{a.e. in } D, \tag{2.6}$$

where

$$\operatorname{sgn} u = \begin{cases} 1 & \textit{if } u > 0, \\ 0 & \textit{if } u = 0, \\ -1 & \textit{if } u < 0. \end{cases}$$

Proof. Put $\varphi(u) = \operatorname{sgn} u$ in Lemma 2.2, then the expression (2.4) is nonnegative and therefore coincides with its absolute value, i.e.,

$$\begin{aligned} &\operatorname{sgn}(u_2 - u_1)\big[(Lu_2 - f_2) - (Lu_1 - f_1)\big] \\ &\qquad = (1 - I_{(u_2=u_1)})\big|(Lu_2 - f_2) - (Lu_1 - f_1)\big| \quad \text{a.e. in } D. \end{aligned}$$

Thus, we get

$$\begin{aligned} &\big|(Lu_2 - f_2) - (Lu_1 - f_1)\big| = \operatorname{sgn}(u_2 - u_1) \cdot L(u_2 - u_1) \\ &\qquad + \operatorname{sgn}(u_2 - u_1)(f_1 - f_2) + I_{(u_2=u_1)}\big|(Lu_2 - f_2) - (Lu_1 - f_1)\big|. \end{aligned} \tag{2.7}$$

The function $u_2(x) - u_1(x)$ belongs to the second order Sobolev space $W^{2,p}(D)$, $p \geq 2$, and therefore all of its first and second order partial derivatives vanish on the set

$$\{x \in D : u_2(x) - u_1(x) = 0\}. \tag{2.8}$$

The latter fact implies that the function $L(u_2 - u_1)(x)$ also vanishes on the set (2.8), i.e.,

$$I_{(u_2=u_1)} \big|(Lu_2 - f_2) - (Lu_1 - f_1)\big| = I_{(u_2=u_1)} |f_2 - f_1| \quad \text{a.e. in } D.$$

Now from equality (2.7) we easily derive inequality (2.6). □

Let us introduce in a standard manner a bounded continuous bilinear form $a(u, v)$ on the product $H^1(D) \times H^1(D)$, where $H^1(D) = W^{1,2}(D)$:

$$\begin{aligned} a(u, v) = &- \sum_{i,j=1}^{n} \int_D a_{ij}(x) \frac{\partial u(x)}{\partial x_j} \frac{\partial v(x)}{\partial x_i} \, dx \\ &+ \sum_{i=1}^{n} \int_D b_i(x) \frac{\partial u(x)}{\partial x_i} v(x) \, dx + \int_D c(x) u(x) v(x) \, dx. \end{aligned} \tag{2.9}$$

Let us recall Green's classical first formula

$$a(u, w) = \int_D Lu(x) \cdot w(x) \, dx \tag{2.10}$$

for $u \in C^2(\overline{D})$, $w \in H_0^1(D)$, where $H_0^1(D)$ is the closure of the linear space $C_0^\infty(D)$ in the norm of $H^1(D)$.

Theorem 2.4. *Consider the univariate convex function $\phi(x)$ such that*

$$\phi(x) \in C^2(R), \quad |\phi'(x)| \leq k, \quad \phi(0) = 0, \tag{2.11}$$

where $k > 0$ is some positive constant. Then the following inequality holds for any $v \in H^2(D)$ and $w \in H_0^1(D)$ with $w(x) \geq 0$ a.e. in D:

$$\int_D \phi'(v) Lv \, w \, dx \leq a(\phi(v), w). \tag{2.12}$$

Proof. Since the boundary ∂D belongs to the class $C^{1,1}$, the linear space $C^2(\overline{D})$ is dense in the second order Sobolev space $H^2(D)$. Fix a function $v \in H^2(D)$ and choose an approximating sequence $v_m \in C^2(\overline{D})$ such that

$$\|v_m - v\|_{H^2(D)} \underset{m\to\infty}{\longrightarrow} 0. \tag{2.13}$$

Since the sequence of functions $v_m(x)$ converges to a function $v(x)$ in the space $L^2(D)$, we can choose such a subsequence (also denoted by $v_m(x)$) that converges to $v(x)$ a.e. in D.

We write the inequality (2.1) for the functions $v_m(x)$, multiply it by a function $w \in H_0^1(D)$ with $w(x) \geq 0$ a.e. in D, and then integrate it over the domain D. We get

$$\int_D \phi'(v_m) L v_m \, w \, dx \leq \int_D L\phi(v_m) \, w \, dx,$$

which, after applying Green's first formula (2.10), reduces to the inequality

$$\int_D \phi'(v_m) L v_m \, w \, dx \leq a(\phi(v_m), w). \tag{2.14}$$

Now we aim to pass to the limit as $m \to \infty$ in the latter inequality. We have

$$\left|\phi(v(x)) - \phi(0)\right| \leq k \, |v(x)|, \tag{2.15}$$

hence $\phi(v(x)) \in L^2(D)$. By [7, Lemma 1.57] we can write

$$\frac{\partial \phi(v(x))}{\partial x_i} = \phi'(v(x)) \, \frac{\partial v(x)}{\partial x_i},$$

from which it follows that

$$\left| \frac{\partial \phi(v(x))}{\partial x_i} \right| \leq k \left| \frac{\partial v(x)}{\partial x_i} \right| \tag{2.16}$$

and thus $\phi(v(x)) \in H^1(D)$. Moreover, we have

$$\left|\phi(v_m) - \phi(v)\right| \leq k \, |v_m - v| \tag{2.17}$$

and

$$\begin{aligned} &\int_D \Big(\frac{\partial \phi(v_m)}{\partial x_i} - \frac{\partial \phi(v)}{\partial x_i} \Big)^2 dx \\ &\leq 2\bigg(k^2 \int_D \Big(\frac{\partial v_m}{\partial x_i} - \frac{\partial v}{\partial x_i} \Big)^2 dx + \int_D \big(\phi'(v) - \phi'(v_m)\big)^2 \Big(\frac{\partial v}{\partial x_i} \Big)^2 dx \bigg). \end{aligned} \tag{2.18}$$

The right-hand side of the inequality (2.18) tends to zero by the Lebesgue dominated convergence theorem if m tends to infinity. We come to a conclusion that the sequence $\phi(v_m)$ converges to $\phi(v)$ in the first order Sobolev space $H^1(D)$.

Let us estimate the difference

$$\int_D \Big|\big(\phi'(v_m)Lv_m - \phi'(v)Lv\big)w\Big|\,dx \le \big\|\phi'(v_m)Lv_m - \phi'(v)Lv\big\|_{L^2(D)}\|w\|_{L^2(D)}.$$

We have

$$\phi'(v_m)Lv_m - \phi'(v)Lv = \phi'(v_m)L(v_m - v) + \big(\phi'(v_m) - \phi'(v)\big)Lv, \tag{2.19}$$

hence we get the estimate

$$\int_D \big(\phi'(v_m)Lv_m - \phi'(v)Lv\big)^2\,dx \le 2\bigg(k^2\int_D \big(L(v_m - v)\big)^2\,dx + \int_D \big(\phi'(v_m) - \phi'(v)\big)^2(Lv)^2\,dx\bigg). \tag{2.20}$$

By the convergence $\|v_m - v\|_{H^2(D)} \underset{m\to\infty}{\longrightarrow} 0$ and the Lebesgue dominated convergence theorem, the right-hand side of estimate (2.20) tends to zero.

Finally, passing to the limit as $m \to \infty$ in inequality (2.14), we obtain the estimate (2.12). □

3 Stability results with respect to external force functions

Let us first prove the stability of a solution of the obstacle problem (1.7) in the Lebesgue space $L^q(D)$.

Theorem 3.1. *Consider the solutions $u_i(x)$, $i = 1, 2$, of the obstacle problem (1.7) with an obstacle function ψ, $\psi \in W^{2,p}(D)$, $\psi^+ \in W_0^{1,p}(D)$, $p \geq 2$, and external force functions $f_i(x)$, $i = 1, 2$, such that $f_i \in L^p(D)$, $p \geq 2$. Then the following estimate is valid:*

$$\|u_2 - u_1\|_{L^q(D)} \le c\|f_2 - f_1\|_{L^p(D)}, \quad p \ge 2, \tag{3.1}$$

where q is defined by

$$q = \begin{cases} \frac{np}{n-2p} & \text{if } 2 \le p < \frac{n}{2}, \\ \text{any } q < \infty & \text{if } p = \frac{n}{2}, \\ \infty & \text{if } p > \frac{n}{2}. \end{cases} \tag{3.2}$$

Proof. Consider the approximation $\phi_\delta(u)$ to $|u|$ written in the form

$$\phi_\delta(u) = \sqrt{u^2+\delta^2} - \delta, \quad -\infty < u < \infty,\ \delta > 0. \tag{3.3}$$

We have that for $\delta > 0$, $\phi_\delta(u)$ is a smooth function with $\phi_\delta(0) = 0$, $\phi'_\delta(0) = 0$, and

$$\phi_\delta(u) \geq 0, \quad \phi'_\delta(u) = \frac{u}{\sqrt{u^2+\delta^2}}, \quad \phi''_\delta(u) = \frac{\delta^2}{(u^2+\delta^2)^{3/2}} \geq 0. \tag{3.4}$$

Thus, $\phi_\delta(u)$ is a convex function such that

$$|\phi'_\delta(u)| \leq 1, \quad \phi'_\delta(u) < 0 \text{ if } u < 0, \quad \phi'_\delta(u) > 0 \text{ if } u > 0$$

and $\phi'_\delta(0) = 0$, whereas

$$\phi_\delta(u) \text{ tends uniformly to } |u| \text{ if } \delta \to 0$$

(indeed, $\big|\phi_\delta(u) - |u|\big| \leq \delta$)) and

$$\phi'_\delta(u) \text{ tends pointwise to } \operatorname{sgn} u \text{ if } \delta \to 0.$$

If we apply Lemma 2.2 to the function $\varphi(u) = \phi'_\delta(u)$, $\delta > 0$, then we have

$$\phi'_\delta(u_2-u_1)\big[(Lu_2 - f_2) - (Lu_1 - f_1)\big] \geq 0 \quad \text{a.e. in } D,$$

i.e.,

$$\phi'_\delta(u_2-u_1)L(u_2-u_1) \geq -|f_2 - f_1| \quad \text{a.e. in } D. \tag{3.5}$$

Now from inequality (2.12) we obtain

$$\int_D \big(-|f_2-f_1|\big) w\, dx \leq a\big(\phi_\delta(u_2-u_1), w\big) \tag{3.6}$$

for any $w \in H_0^1(D)$ with $w(x) \geq 0$ a.e. in D.

Let us consider a solution $\widetilde{u}(x)$ of the linear second order elliptic partial differential equation

$$L\widetilde{u}(x) = -\big|f_2(x) - f_1(x)\big| \quad \text{a.e. in } D, \tag{3.7}$$

such that $\widetilde{u} \in W^{2,p}(D) \cap W_0^{1,p}(D)$, $p \geq 2$. Then inequality (3.6) can be rewritten as

$$a\big(\phi_\delta(u_2-u_1) - \widetilde{u}, w\big) \geq 0 \tag{3.8}$$

for arbitrary $w \in H_0^1(D)$ with $w(x) \geq 0$ a.e. in D.

We observe that

$$\phi_\delta(u_2 - u_1) - \widetilde{u} \in H_0^1(D) \quad \text{as } \phi_\delta(0) = 0.$$

If we apply the weak maximum principle (see, e.g., Troianiello [7, Theorem 2.4]) to inequality (3.8), then we get

$$0 \le \phi_\delta(u_2 - u_1) \le \widetilde{u} \quad \text{a.e. in } D. \tag{3.9}$$

Making δ tend to zero we have

$$0 \le |u_2 - u_1| \le \widetilde{u} \quad \text{a.e. in } D. \tag{3.10}$$

From this inequality we derive

$$\|u_2 - u_1\|_{L^q(D)} \le \|\widetilde{u}\|_{L^q(D)}, \quad q \ge 1. \tag{3.11}$$

Since the function $\widetilde{u}(x)$ is a solution of (3.7), we can apply to $\widetilde{u}(x)$ standard L^p-estimates (see, e.g., Gilbarg and Trudinger [3, Lemma 9.17]) to get

$$\|\widetilde{u}\|_{W^{2,p}(D)} \le c_1 \|f_2 - f_1\|_{L^p(D)}, \quad p \ge 2. \tag{3.12}$$

We can also use Sobolev's imbedding theorems (Troianiello [7, Theorems 1.33 and 1.41]) and thus obtain the inequality

$$\|\widetilde{u}\|_{L^q(D)} \le c_2 \|\widetilde{u}\|_{W^{2,p}(D)}, \quad p \ge 2, \tag{3.13}$$

where q is defined by (3.2).

Applying successively the inequalities (3.11), (3.13) and (3.12) we come to the estimate (3.1) with the constant $c = c_1 c_2$. □

Theorem 3.2. *Let $\phi(x)$ be a nonnegative univariate convex function satisfying the conditions* (2.11). *Then for arbitrary $v \in H^2(D) \cap H_0^1(D)$ and $h \in C^2(\overline{D})$ with $h(x) \ge 0$ the following estimate holds:*

$$\int_D \phi'(v) L v \, h \, dx \le \int_D \phi(v) L^* h \, dx, \tag{3.14}$$

where L^* is the adjoint operator to L defined in (1.4).

Proof. To prove this theorem we use the same approximation technique as in the proof of Theorem 2.4.

Consider the sequence of functions $v_m \in C^2(\overline{D})$ such that

$$\|v_m - v\|_{H^2(D)} \underset{m\to\infty}{\longrightarrow} 0, \tag{3.15}$$

where $v \in H^2(D) \cap H_0^1(D)$.

Applying Green's classical second formula to the functions $\phi(v_m(x))$ and $h(x)$, both belonging to the space $C^2(\overline{D})$, we have

$$\begin{aligned}\int_D L\phi(v_m)\,h\,dx &= \int_D \phi(v_m)L^*h\,dx\\ &\quad+\int_{\partial D}\sum_{i=1}^n\Bigg[\sum_{j=1}^n\left(ha_{ij}\phi'(v_m)\,\frac{\partial v_m}{\partial x_j}-\phi(v_m)a_{ij}\,\frac{\partial h}{\partial x_j}-\phi(v_m)h\,\frac{\partial a_{ij}}{\partial x_j}\right)n_i\\ &\quad+b_i\phi(v_m)hn_i\Bigg]\,d\sigma, \qquad (3.16)\end{aligned}$$

where the boundary integral is an $(n-1)$-dimensional surface integral and $(n_i)_{i=1,\dots,n}$ is the outer normal vector.

Taking into account the key inequality (2.1), from (3.16) we obtain the estimate

$$\int_D \phi'(v_m)Lv_m\,h\,dx \le \int_D \phi(v_m)L^*h\,dx + I_m(\partial D), \qquad (3.17)$$

where $I_m(\partial D)$ is the boundary integral in (3.16). We have to pass to the limit in inequality (3.17). For the first and second terms this is done in the same manner as in the proof of Theorem 2.4, but for the boundary integral $I_m(\partial D)$ the procedure of passing to the limit is the delicate one, since it requires to consider the trace operator

$$T : H^1(D) \to L^2(\partial D)$$

and the important trace inequality

$$\|Tu\|_{L^2(\partial D)} \le \widehat{c}\,\|u\|_{H^1(D)} \qquad (3.18)$$

(see, e.g., Evans [2, Chapter 5, Section 5.5, Theorem 1]).

For the functions $v_m(x)$ and $\frac{\partial v_m(x)}{\partial x_i}$, $i=1,\dots,n$, the notation of the traces remains the same. We have that the functions

$$v_m(x)-v(x) \quad \text{and} \quad \frac{\partial v_m(x)}{\partial x_i}-\frac{\partial v(x)}{\partial x_i}$$

belong to the Sobolev space $H^1(D)$ and therefore the trace inequality implies

$$\|v_m - Tv\|_{L^2(\partial D)} \le \widehat{c}\,\|v_m - v\|_{H^1(D)}, \qquad (3.19)$$

$$\left\|\frac{\partial v_m}{\partial x_i} - T\,\frac{\partial v}{\partial x_i}\right\|_{L^2(\partial D)} \le \widehat{c}\,\left\|\frac{\partial v_m}{\partial x_i}-\frac{\partial v}{\partial x_i}\right\|_{H^1(D)}. \qquad (3.20)$$

Since $v \in H_0^1(D)$, we have that

$$Tv = 0 \quad \text{in } L^2(\partial D). \tag{3.21}$$

Consider the functions $\phi(v_m(x))$ and $\phi'(v_m(x))$. By the assumption of Theorem 3.2, $\phi(x)$ is a nonnegative twice continuously differentiable convex function with $\phi(0) = 0$. Hence, $u = 0$ is the point of minimum of the function $\phi(x)$, and by the classical theorem of Ferma we have

$$\phi'(0) = 0. \tag{3.22}$$

By the same assumption (2.11),

$$|\phi'(x)| \le k. \tag{3.23}$$

Therefore

$$\left|\phi(v_m) - \phi(Tv)\right| \le k\,|v_m - Tv| \quad \text{on } \partial D,$$

i.e.,

$$\left\|\phi(v_m) - \phi(Tv)\right\|_{L^2(\partial D)} \le k\,\|v_m - Tv\|_{L^2(\partial D)}. \tag{3.24}$$

Taking into account that $Tv = 0$, from (3.19) and (3.24) we obtain

$$\|v_m\|_{L^2(\partial D)} \underset{m\to\infty}{\longrightarrow} 0, \quad \text{and} \quad \|\phi(v_m)\|_{L^2(\partial D)} \underset{m\to\infty}{\longrightarrow} 0. \tag{3.25}$$

Let us choose a subsequence of the sequence $v_m(x)$ (it is also denoted by $v_m(x)$) such that

$$v_m(x) \underset{m\to\infty}{\longrightarrow} 0 \quad \text{a.e. in } \partial D \tag{3.26}$$

(with respect to an $(n-1)$-dimensional Lebesgue measure).

Further, we consider the terms

$$\phi'(v_m)\,\frac{\partial v_m}{\partial x_j}, \qquad j = 1, \dots, n,$$

of the boundary integral $I_m(\partial D)$. We have

$$\left\|\phi'(v_m)\,\frac{\partial v_m}{\partial x_j}\right\|_{L^1(\partial D)} \le \|\phi'(v_m)\|_{L^2(\partial D)} \left\|\frac{\partial v_m}{\partial x_j}\right\|_{L^2(\partial D)}. \tag{3.27}$$

From the trace inequality (3.20) we get that the sequence of norms $\left\|\frac{\partial v_m}{\partial x_j}\right\|_{L^2(\partial D)}$ is bounded. By the continuity of the function $\phi'(u)$, where $\phi'(0) = 0$, and the limit relation (3.26) it is obvious that

$$\phi'(v_m(x)) \underset{m\to\infty}{\longrightarrow} 0 \quad \text{a.e. in } \partial D, \tag{3.28}$$

and since the functions $\phi'(v_m(x))$ are bounded by the constant k, the Lebesgue dominated convergence theorem tells us that

$$\|\phi'(v_m)\|_{L^2(\partial D)} \underset{m\to\infty}{\longrightarrow} 0.$$

Therefore we can write

$$\lim_{m\to\infty} \left\| \phi'(v_m) \frac{\partial v_m}{\partial x_j} \right\|_{L^1(\partial D)} = 0. \tag{3.29}$$

Finally, we conclude that the boundary integral $I_m(\partial D)$ vanishes in the limit, i.e.,

$$\lim_{m\to\infty} I_m(\partial D) = 0.$$

Thus, if in inequality (3.17) we pass to the limit as $m \to \infty$, we will obtain the desired estimate (3.14). □

Let us take the convex functions $\phi_\delta(u)$ defined by (3.3) in inequality (3.14) and make δ tend to zero, then we come to the following assertion.

Corollary 3.3. *Consider an arbitrary function $v(x)$ such that $v(x) \in H^2(D) \cap H_0^1(D)$. Then the following estimate is valid:*

$$\int_D \operatorname{sgn} v(x) L v(x) h(x)\, dx \le \int_D |v(x)| L^* h(x)\, dx, \tag{3.30}$$

where $h(x) \in C^2(\overline{D})$ with $h(x) \ge 0$.

For the particular case $h(x) = 1$, we get the assertion of [6, Section 5.4, Lemma 4.1]. Our next theorem asserts that a solution of the obstacle problem in the second order Sobolev space is stable with respect to external force functions.

Theorem 3.4. *Let the obstacle function $\psi(x)$ belong to $W^{2,p}(D)$ with $\psi^+ \in W_0^{1,p}(D)$, $p \ge 2$. Consider two solutions $u_i(x)$, $i = 1, 2$, of the obstacle problem (1.7) corresponding to external force functions $f_i(x)$, $i = 1, 2$, with $f_i \in L^p(D)$, $p \ge 2$, $i = 1, 2$. Then the following stability estimate is valid:*

$$\begin{aligned} \|u_2 - u_1\|_{W^{2,p'}(D)} &\le c \|f_2 - f_1\|^{\lambda}_{L^p(D)} \\ &\quad \times \left(\|\psi\|_{W^{2,p}(D)} + \|f_1\|_{L^p(D)} + \|f_2\|_{L^p(D)} \right)^{1-\lambda}, \end{aligned} \tag{3.31}$$

where

$$1 < p' < p, \quad \lambda = \frac{p - p'}{(p-1)p'}, \quad 1 - \lambda = \frac{p(p'-1)}{(p-1)p'}, \tag{3.32}$$

and the constant c is independent of $\psi(x)$, $f_1(x)$, $f_2(x)$.

Proof. We start with the interpolation inequality (see Gilbarg and Trudinger [3, Chapter 7, inequality (7.9)]), which is applied to the function $L(u_2 - u_1)(x)$. We get

$$\|L(u_2 - u_1)\|_{L^{p'}(D)} \le \|L(u_2 - u_1)\|^{\lambda}_{L^1(D)} \|L(u_2 - u_1)\|^{1-\lambda}_{L^p(D)}, \tag{3.33}$$

where $p \ge 2$, $1 < p' < p$ and λ is defined in (3.32).

From Theorem 2.3 we have

$$\begin{aligned} &\|L(u_2 - u_1)\|_{L^1(D)} \\ &\qquad \le \int_D \operatorname{sgn}(u_2 - u_1) \cdot L(u_2 - u_1)\, dx + 2\|f_2 - f_1\|_{L^1(D)}, \end{aligned} \tag{3.34}$$

and from Corollary 3.3 we obtain

$$\int_D \operatorname{sgn}(u_2 - u_1) \cdot L(u_2 - u_1)\, dx \le \|(L^*1)^+\|_{L^\infty(D)} \cdot \|u_2 - u_1\|_{L^1(D)}, \tag{3.35}$$

where $(L^*1)^+ = \max(0, L^*1)$.

Applying inequality (3.1) from Theorem 3.1, we have

$$\|u_2 - u_1\|_{L^1(D)} \le c_1 \|f_2 - f_1\|_{L^p(D)}.$$

Hence, we obtain

$$\|L(u_2 - u_1)\|_{L^1(D)} \le c_2 \|f_2 - f_1\|_{L^p(D)}. \tag{3.36}$$

From the Lewy–Stampacchia inequality (1.8) and the formulation of the unilateral obstacle problem (1.7) we have

$$|L(u_2 - u_1)| \le |Lu_2| + |Lu_1| \le 2|L\psi| + |f_1| + |f_2| \quad \text{a.e. in } D. \tag{3.37}$$

Therefore

$$\begin{aligned} \|L(u_2 - u_1)\|_{L^p(D)} &\le 2\|L\psi\|_{L^p(D)} + \|f_1\|_{L^p(D)} + \|f_2\|_{L^p(D)} \\ &\le c_3 \big(\|\psi\|_{W^{2,p}(D)} + \|f_1\|_{L^p(D)} + \|f_2\|_{L^p(D)}\big). \end{aligned} \tag{3.38}$$

The well-known $L^{p'}$-estimates used for solutions of linear second order elliptic partial differential equations (see, e.g., Gilbarg and Trudinger [3, Chapter 9]) have the form

$$\|u_2 - u_1\|_{W^{2,p'}(D)} \le c_4 \|L(u_2 - u_1)\|_{L^{p'}(D)}$$

from which and the inequalities (3.33)–(3.38) we eventually obtain the estimate (3.31). □

4 The stability estimate with respect to the differential operator

Consider an n-dimensional bounded domain D with a boundary ∂D of the class $C^{1,1}$ and introduce two linear second order elliptic differential operators L and $\widehat{L}$, which act on functions u from the Sobolev space $W^{2,p}(D)$, $p \geq 2$:

$$
\begin{aligned}
Lu(x) &= \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial u(x)}{\partial x_i}\right) + \sum_{i=1}^{n} b_i(x)\frac{\partial u(x)}{\partial x_i} + c(x)u(x),\\
\widehat{L}u(x) &= \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(\widehat{a}_{ij}(x)\frac{\partial u(x)}{\partial x_i}\right) + \sum_{i=1}^{n} \widehat{b}_i(x)\frac{\partial u(x)}{\partial x_i} + \widehat{c}(x)u(x).
\end{aligned}
\tag{4.1}
$$

Assume that the following conditions are satisfied:

$$
a_{ij}(x) = a_{ji}(x), \quad \widehat{a}_{ij}(x) = \widehat{a}_{ji}(x), \quad i, j = 1, \ldots, n; \tag{4.2}
$$

$$
\begin{gathered}
\text{all functions } a_{ij}, \widehat{a}_{ij}, b_i \text{ and } \widehat{b}_i \text{ belong to the space } C^{0,1}(\overline{D}),\\
c \text{ and } \widehat{c} \text{ belong to } L^{\infty}(D) \text{ and } c(x) \leq 0, \widehat{c}(x) \leq 0 \text{ a.e. in } D;
\end{gathered}
\tag{4.3}
$$

and that the uniform ellipticity condition

$$
\sum_{i,j=1}^{n} a_{ij}(x) y_i y_j \geq \alpha |y|^2, \quad \sum_{i,j=1}^{n} \widehat{a}_{ij}(x) y_i y_j \geq \alpha |y|^2 \tag{4.4}
$$

is fulfilled for some $\alpha > 0$ and any $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$.

Denote

$$
\begin{aligned}
d = {} & \max_{ij} \|\widehat{a}_{ij}\|_{C^{0,1}(\overline{D})} + \max_{ij} \|a_{ij}\|_{C^{0,1}(\overline{D})} + \max_{i} \|\widehat{b}_i\|_{C^{0,1}(\overline{D})}\\
& + \max_{i} \|b_i\|_{C^{0,1}(\overline{D})} + \|\widehat{c}\|_{L^{\infty}(D)} + \|c\|_{L^{\infty}(D)}.
\end{aligned}
\tag{4.5}
$$

Fix obstacle and external force functions $\psi(x)$ and $f(x)$ such that

$$
\psi \in W^{2,p}(D), \quad \psi^{+} \in W_0^{1,p}(D), \quad f \in L^{p}(D), \quad p \geq 2. \tag{4.6}
$$

Recall that $\widehat{u}$ (respectively u) belonging to $W^{2,p}(D) \cap W_0^{1,p}(D)$, $p \geq 2$, is called a solution of the obstacle problem for the operators $\widehat{L}$ and L if

$$
\left.
\begin{aligned}
&\widehat{u}(x) \geq \psi(x), \quad \widehat{L}\widehat{u}(x) \leq f(x),\\
&\big(f(x) - \widehat{L}\widehat{u}(x)\big)\big(\widehat{u}(x) - \psi(x)\big) = 0
\end{aligned}
\right.
\quad \text{a.e. in } D, \tag{4.7}
$$

respectively,

$$\begin{aligned} &u(x) \geq \psi(x), \quad Lu(x) \leq f(x), \\ &\big(f(x) - Lu(x)\big)\big(u(x) - \psi(x)\big) = 0 \end{aligned} \qquad \text{a.e. in } D. \tag{4.8}$$

We rewrite the obstacle problem (4.7) for the operator $\widehat{L}$ as the obstacle problem for the operator L as follows:

$$\begin{aligned} &\widehat{u}(x) \geq \psi(x), \quad L\widehat{u}(x) \leq \widehat{f}(x), \\ &\big(\widehat{f}(x) - L\widehat{u}(x)\big)\big(\widehat{u}(x) - \psi(x)\big) = 0 \end{aligned} \qquad \text{a.e. in } D, \tag{4.9}$$

where $\widehat{f}(x) = f(x) - (\widehat{L} - L)\widehat{u}(x)$ is the transformed external force function.

According to Troianiello [7, Theorem 5.2], the problem (4.9) has a unique solution if only $\widehat{f} \in L^p(D)$, $p \geq 2$.

Let us now formulate and prove the basic result of this paper.

Theorem 4.1. *Let the conditions* (4.1)–(4.6) *be satisfied. Then the following Hölder type stability estimate is valid for the solutions of the obstacle problem* (4.7), (4.8)*:*

$$\begin{aligned} &\|\widehat{u} - u\|_{W^{2,p'}(D)} \\ &\quad \leq c\Big(\max_{ij}\|\widehat{a}_{ij} - a_{ij}\|_{C^{0,1}(\overline{D})} + \max_i\|\widehat{b}_i - b_i\|_{C^0(\overline{D})} + \|\widehat{c} - c\|_{L^\infty(D)}\Big)^\lambda \\ &\qquad \times \big(\|\psi\|_{W^{2,p}(D)} + \|f\|_{L^p(D)}\big), \end{aligned} \tag{4.10}$$

where

$$p \geq 2, \quad 1 < p' < p, \quad \lambda = \frac{p - p'}{(p-1)p'}, \tag{4.11}$$

the constant c is independent of $\psi(x)$, $f(x)$, but depends on the coefficients of the differential operators L and $\widehat{L}$ through the quantity d in (4.5) *and the ellipticity constant α.*

Proof. We have

$$\widehat{f}(x) - f(x) = -(\widehat{L} - L)\widehat{u}(x). \tag{4.12}$$

Consider the expression

$$\begin{aligned} (\widehat{L} - L)\widehat{u}(x) = &\sum_{i,j=1}^n (\widehat{a}_{ij} - a_{ij})\frac{\partial^2\widehat{u}}{\partial x_i \partial x_j} \\ &+ \sum_{i=1}^n \Bigg(\sum_{j=1}^n \frac{\partial(\widehat{a}_{ij} - a_{ij})}{\partial x_j} + (\widehat{b}_i - b_i)\Bigg)\frac{\partial\widehat{u}}{\partial x_i} + (\widehat{c} - c)\widehat{u}. \end{aligned}$$

We get

$$
\begin{aligned}
&\big\|(\widehat{L}-L)\widehat{u}(x)\big\|_{L^p(D)} \\
&\quad \le c_1(n,p)\big(\max_{ij}\|\widehat{a}_{ij}-a_{ij}\|_{C^{0,1}(\overline{D})}+\max_{i}\|\widehat{b}_i-b_i\|_{C^0(\overline{D})}+\|\widehat{c}-c\|_{L^\infty(D)}\big) \\
&\qquad \times \|\widehat{u}\|_{W^{2,p}(D)}.
\end{aligned}
\tag{4.13}
$$

Applying now the Lewy–Stampacchia inequality (1.8) to the obstacle problem (4.7), we have

$$
|\widehat{L}\widehat{u}(x)| \le |\widehat{L}\psi(x)| + |f(x)| \quad \text{a.e. in } D \tag{4.14}
$$

and hence

$$
\|\widehat{L}\widehat{u}\|_{L^p(D)} \le c_2(n,p)d\,\big(\|\psi\|_{W^{2,p}(D)}+\|f\|_{L^p(D)}\big). \tag{4.15}
$$

Again using the well-known L^p-estimates, we get

$$
\|\widehat{u}\|_{W^{2,p}(D)} \le c_3(n,p,\alpha,d,D)\big(\|\psi\|_{W^{2,p}(D)}+\|f\|_{L^p(D)}\big). \tag{4.16}
$$

Taking into account the inequalities (4.13) and (4.16), we obtain the bound

$$
\begin{aligned}
\|\widehat{f}-f\|_{L^p(D)} &= \big\|(\widehat{L}-L)\widehat{u}\big\|_{L^p(D)} \\
&\le c_4(n,p,\alpha,d,D)\big(\max_{ij}\|\widehat{a}_{ij}-a_{ij}\|_{C^{0,1}(\overline{D})}+\max_{i}\|\widehat{b}_i-b_i\|_{C^0(\overline{D})} \\
&\qquad + \|\widehat{c}-c\|_{L^\infty(D)}\big)\big(\|\psi\|_{W^{2,p}(D)}+\|f\|_{L^p(D)}\big).
\end{aligned}
\tag{4.17}
$$

Now, if we take in inequality (3.31) of Theorem 3.4 (applied to the obstacle problem (4.8), (4.9))

$$
f_1(x) = f(x) \quad \text{and} \quad f_2(x) = \widehat{f}(x), \tag{4.18}
$$

then by the bound (4.17) we obtain the desired stability estimate (4.10). □

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Received April 2, 2011.

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