Stochastic integral representation of nonsmooth Wiener functionals

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After Clark [1][1] obtained the formula for the stochastic integral representation for Wiener functionals, which asserted only the existence of this representation, many authors tried to find the integrand explicitly, and the corresponding results were obtained when the functionals were smooth in some sense. When the functional F belongs to the Hilbert space $D_{2,1}$ (where $D_{2,1}$ denotes the space of square integrable functionals having the first order stochastic derivative[2]) Ocone [3] proved that the integrand in the Clark representation is $E[D_tF|\Im_t^W]$ (the optional projection of the stochastic (Malliavin) derivative of F). Shiryaev, Yor and Graversen [4,5] proposed a method to find representation of the running maximum of Wiener process. Later on, using the Clark-Ocone formula, Renaud and Remillard [6] have established explicit martingale representations for path-dependent Wiener functionals.

[1] If F is a square integrable $\Im_T^W := \sigma\{W_s : 0 \leq s \leq T\}$-measurable random variable, then (due to the Clark formula) there exist a square integrable $\Im_t^W := \sigma\{W_s : 0 \leq s \leq t\}$-adapted random process $\varphi(t,\omega)$ such that

$$F = E[F] + \int_0^T \varphi(t,\omega)dW_t(\omega).$$

In fact, this is the inverse statement of one important property of the Ito stochastic integral: if f is square-integrable $\Im_t^W$-adapted random process, then the process

$$M_t = \int_0^t f(s,\omega)dW_s(\omega)$$

is a martingale with respect to the filtration $\{\Im_t^W\}_{t\geq 0}$.

To be convinced of this, it is sufficient to take the conditional mathematical expectation on both sides of the Clark representation. Indeed, in this way, we obtain that for the associated to F Levy's martingale $M_t = E[F|\Im_t^W]$ the following stochastic integral representation is true

$$M_t = M_0 + \int_0^t \varphi(s,\omega)dW_s(\omega).$$

[2] Let us recall some definitions from [2].

The class of smooth Wiener functionals S is the class of a random variables which has the form

$$F = f(W_{t_1}, ..., W_{t_n}),\ f \in C_p^\infty(R^n),\ t_i \in [0,T],\ n \geq 1,$$

where $C_p^\infty(R^n)$ is the set of all infinitely continuously differentiable functions $f : R^n \to R$ such that f and all of its partial derivatives have polynomial growth.

The stochastic (Malliavin) derivative of a smooth random variable $F \in S$ is the stochastic process D_tF given by

$$D_tF = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(W_{t_1}, ..., W_{t_n})I_{[0,t_i]}(t).$$

Denote by $D_{2,1}$ the Hilbert space that is the closure of the class of smooth Wiener functionals with the following Sobolev type norm:

$$||F||_{2,1} = ||F||_{L_2(\Omega)} + |||D.F|||_{L_2(\Omega;L_2([0,T]))}.$$

In fact, we have defined the Malliavin derivative as an "inverse"of the Ito stochastic integral (with deterministic integrand) in the sense that $D^W W(h) = h$ (where $W(h) := \int_0^T h(s)dW_s$ and $D_t^W \int_0^T h(s)dW_s = h(t)$, as well as it's clear that $W_\theta = W(I_{[0,\theta]}(\cdot))$ and $D_t^W W_\theta = I_{[0,\theta]}(t)$).

In all cases described above investigated functionals, were stochasticaly (in Malliavin sense) smooth. Our approach with prof. Jaoshvili (2005-2009) in the framework of the classical Ito calculus, on the basis of the standard L_2 theory and the theory of weighted Sobolev spaces, made it possible to construct an explicit formula for the itegrand when the functional does not have the mentioned smoothness (see, for example [7]). Here we will explore the stochastically nonsmooth Wiener functionnals that can be considered in the future as a payoff function of a certain exotic European Option and study the issues of their stochastic integral representation, which it is known to play a significant role in the hedging problem of European Options. It has turned out that the requirement of smoothness of functional can be weakened by the requirement of smoothness only of its conditional mathematical expectation. We (with prof. O. Glonti, 2014) considered Wiener functionals which are not stochastically differentiable. In particular, we (see [8]) generalized the Clark-Ocone formula in case, when functional is not stochastically smooth, but its conditional mathematical expectation is stochastically differentiable and established the method of finding of integrand. Next, we have considered functionals which didn't satisfy even these weakened conditions. To such functionals belong, for example, Lebesgue integral (with respect to time variable) from stochastically non smooth square integrable processes.

In the 80th of the past century, it turned out (see, [9]) that the stochastic integral representation theorems (along with the Girsanov's measure change theorem) play an important role in the modern financial mathematics. In particular, using the integrand of the stochastic integral appearing in the integral representation, one can construct hedging strategies in the European options of different type. In contrast to the standard European Option payoff function (i. e. $(S_T - K)^+$), which is stochastically (in Malliavin sense) differentiable, we will discuss European type options with nonsmooth payoff functions. The payoff functions of derivative securities with more complicated forms than standard European or American call and put options are known as exotic options.

One of such kind exotic option is so-called Binary Option. It is an option with discontinuous payoff function. The simplest examples of the Binary Options are call and put options "cash or nothing". The payoff function of the call option has the form $BC_T = QI_{\{S_T > K\}}$, and for the put option – $BC_T = QI_{\{S_T < K\}}$, where K is the strike price at the time of execution T (it should be noted that indicator of event A is Malliavin differentiable if and only if probability $P(A)$ is equal to zero or one [2,3]). Moreover, so-called Asian Options also are type of Exotic Option.

Despite that application of the Clark-Ocone formula needs as a rule essential efforts it is possible in many cases to determine the form of the representation using Malliavin calculus, if a functional is Malliavin differentiable. We consider nonsmooth (in Malliavin sense) functionals and have developed some methods of obtaining of constructive martingale representation theorems. The obtained results can be used to establish the existence of a hedging strategy in various European Options with corresponding payoff functions.

Theorem (Ocone [3]). If F is differentiable in Malliavin sense, $F \in D_{2,1}$, then the stochastic integral representation is fulfilled

$$F = E[F] + \int_0^T E[D_t F | \Im_t^W] \mathrm{d}W_t \quad (P - a.s.).$$

A different method for finding the integrand of stochastic integral was proposed by Shiryaev, Yor and Graversen [4,5], which was based on the Ito (generalized) formula and the Levy theorem for the Levy martingale $M_t = E[F|\Im_t]$ associated with F.

Theorem (Shiryaev, Yor [4]). Let $M_T = \sup_{0\le t\le T} W_t$. Then the following stochastic integral representation holds

$$M_T = EM_T + 2\int_0^T [1-\Phi(\frac{M_t - W_t}{\sqrt{T-t}})]dW_t,$$

where Φ is standard normal distribution function.

Theorem (Graversen, Shiryaev and Yor [5]). Let

$$g_T = sup\{0 < t \le T : W_t = 0\},\quad M_u = \max_{t\le u} W_t \quad and \quad M_{g_T} = \max_{t\le g_T} W_t.$$

Then we have

$$M_{g_T} = \frac{1}{2}EM_T + \int_0^T [\frac{1}{2}\Psi(\frac{2M_u - W_u}{\sqrt{T-u}}) - (M_u - M_{g_u})\varphi_{T-u}(W_u)]dW_u,$$

where

$$EM_T = \sqrt{2T/\pi},\quad \Psi(x) = 2[1-\Phi(x)]$$

and

$$\varphi_{T-u}(x) = \frac{1}{\sqrt{T-u}}\varphi(\frac{x}{\sqrt{T-u}}),$$

where φ is standard normal distribution density function.

Let $\mathcal{B}(R)$ be a Borel $\sigma-$ algebra on R, λ be a Lebesgue measure, and $\rho(x,T) := \exp\{-\frac{x^2}{2T}\}$.

Theorem (Jaoshvili, Purtukhia [7]). Let the function $f \in L_{2,T/\alpha}, 0 < \alpha < 1$, and it has the generalized derivative of the first order $\partial f/\partial x$, such that $\partial f/\partial x \in L_{2,T/\beta}$, $0 < \beta < 1/2$, then the following integral representation holds

$$f(W_T) = E[f(W_T)] + \int_0^T E\left[\frac{\partial f}{\partial x}(W_T)|\Im_t^W\right] \mathrm{d}W_t \quad (P-a.s.),$$

where $L_{2,T}$ denotes the set of measurable functions $u : R \to R$, such that $u(\cdot)\rho(\cdot,T) \in L_2 := L_2(R,\mathcal{B}(R),\lambda)$.

Theorem (Glonti, Purtukhia [8]). Suppose that $g_t := E[F|\Im_t^W]$ is Malliavin differentiable ($g_t \in D_{2,1}^W$)[3] for almost all $t \in [0,T)$. Then we have the stochastic integral representation

$$g_T = F = E[F] + \int_0^T \nu_s dW_s \quad (P-a.s.),$$

where

$$\nu_s = \lim_{t\uparrow T} E[D_s^W g_t|\Im_s^W] \quad in\ the \quad L_2([0,T]\times\Omega).$$

Let us now fix the constants $C_2 \le 0$ and $C_1 \ge C_2$ and consider the following nonsmooth path-dependent Wiener functional

$$F = (W_T - C_1)^- I_{\{\inf\limits_{0\le t\le T} W_t \le C_2\}}. \qquad (1)$$

[3]It is well-known, that if random variable is stochastically differentiable in Malliavin sense, then its conditional mathematical expectation is differentiable too ([2]). On the other hand, it is possible that conditional expectation can be smooth even if random variable is not stochastically smooth. For example, it is well-known that $I_{\{W_T\le x\}} \notin D_{2,1}$, but for all $t \in [0,T)$:

$$E[I_{\{W_T\le x\}}|\Im_t^W] = \Phi\left(\frac{x-W_t}{\sqrt{T-t}}\right) \in D_{2,1}.$$

Then we have.

Theorem 1. For the Wiener functional (1) the following stochastic integral representation holds

$$F = EF + \int_0^T \Phi\Big(\frac{2C_2 - C_1 - W_t}{\sqrt{T-t}}\Big)dW_t \quad (P - a.s.).$$

Next, we have considered functionals which didn't satisfy even the weakened conditions from Glonti, Purtukhia [8]. To such functionals belong, for example, Lebesgue integral (with respect to time variable) from stochastically nonsmooth square integrable processes[4].

Theorem 2. If the deterministic function

$$V(t,x) := E\big[\int_t^T f(s,W_s)ds \big| W_t = x\big]$$

satisfies the requirements of the generalized Ito theorem, then the following stochastic integral representation is fulfilled

$$f(t,W_t) = E\big[f(t,W_t)\big] + \int_0^T \frac{\partial}{\partial x}V(t,W_t)dW_t \quad (P - a.s.).$$

Let us now fix the deterministic functions $h_1(t) \le h_2(t)$ and consider the following integral type, nonsmooth Wiener functional

$$\int_0^T I_{\{h_1(t)\le W_t \le h_2(t)\}}dt.$$

Then we have.

Theorem 3. The following stochastic integral representation is fulfilled

$$\int_0^T I_{\{h_1(t)\le W_t \le h_2(t)\}}dt = \int_0^T \Phi\Big(\frac{x}{\sqrt{t}}\Big)\Big|_{x=h_1(t)}^{h_2(t)} +$$

$$+ \int_0^T \int_t^T \varphi\Big(\frac{x - W_t}{\sqrt{s-t}}\Big)\Big|_{x=h_1(t)}^{h_2(t)} ds dW_t \quad (P - a.s.).$$

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[4]In particular, to such functional belongs the integral type functional $\int_0^T u_s(\omega)ds$ with nonsmooth integrand $u_s(\omega)$. It is well-known that if $u_s(\omega) \in D_{2,1}$ for all s, then $\int_0^T u_s(\omega)ds \in D_{2,1}$ and $D_t\{\int_0^T u_s(\omega)ds\} = \int_0^T D_t u_s(\omega)ds$. But if $u_s(\omega)$ is not differentiable in Malliavin sense, then the Lebesgue average (with respect to ds) also is not differentiable in Malliavin sense (see, for example, [10]). Indeed, in this case the conditional mathematical expectation is not stochastically smooth, because we have:

$$E[\int_0^T u_s(\omega)ds|\Im_t^W] = \int_0^t u_s(\omega)ds + \int_t^T E[u_s(\omega)|\Im_t^W]ds,$$

where the first summand (integral) is analogous that the initial integral and therefore it is not Malliavin differentiable, but the second summand is differentiable in Malliavin sense when u_s satisfied our weakened condition. It should be noted that such type integral functionals have been considered in our previous works (Glonti, Purtukhia [10]) and (Glonti, Jaoshvili and Purtukhia [11]).

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Sequential statistical testing of several simple hypotheses under distortion of the observations probability distribution

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Introduction

In applied problems of modern data analysis we often face with a problem of statistical testing of several simple hypotheses on the parameter value that controls the probability distribution of an observed random sequence. Those simple hypotheses, as usual, correspond to some typical modes of the process observed: for example, one value corresponds to the growth of the incidence level, another one - to the recession, and the third one - to the “no-changes” mode.

Especially in the situations, where data come one after another, and are not available all together simultaneously, it is natural to use sequential tests [1] to discriminate between those hypotheses. Sequential tests are known to hold some optimal properties [2], e.g. the sequential probability ratio test minimizes the expected sample size provided the upper bounds for error type I and II probabilities are satisfied. Exact calculation of the performance characteristics for sequential tests is a complicated problem even for