Proceedings of A. Razmadze
Mathematical Institute
Vol. **143** (2007), 37–60

STOCHASTIC INTEGRAL REPRESENTATION OF FUNCTIONALS OF POISSON PROCESSES

V. JAOSHVILI AND O. PURTUKHIA

Abstract. In this paper we suggest the method which allows to construct explicit expressions for integrands taking part in the stochastic integral representation of functionals of Poisson processes and for these functionals the formulas for calculation of the predictable projection of their stochastic derivatives are given.

რეზიუმე. ნაშრომში გადმოცემულია ერთი მეთოდი, რომელიც საშუალებას იძლევა ცხადი სახით ავაგოთ პუასონის ფუნქციონალების ინტეგრალურ წარმოდგენაში მონაწილე სტოქასტური ინტეგრალის ინტეგრანდი და მოყვანილია მათი სტოქასტური წარმოებულის ჭვრეტადი პროექციების გამოსათვლელი ფომულები.

0. Introduction

As is known, in the theory of standard integration, the requirement for the integrand to be measurable is a very small restriction as compared to the condition of integrability which implies the boundedness in a certain sense of an absolute integrand value. As for the stochastic Ito's integral $\int_0^T f(s,\omega)dw(s)$, the situation here is opposite. Besides the fact that the integrand $f(s,\omega)$ is the measurable function of two variables, it should be the adapted (nonanticipated) process, i.e. for any $s \in [0,T]$ the random variable $f(s,\cdot)$ should be measurable with respect to the $\mathcal{F}_s^w := \sigma\{w(t),\, t \in [0,s]\}$ – σ-algebra (in other words, it should be independent of the future increments of the Wiener process). On the one hand, this requirement is natural for many situations, when filtration (the flow of σ-algebras $\mathcal{F}_s^w$) shows possible evolution of information. On the other hand, over a long period of time this requirement restricted both the development of the theory and the application of stochastic calculus.

Starting from the 70th of the past century, many attempts were made to weaken the requirement to be adapted for the integrand of the stochastic Ito's integral as well as in the theory of "the extension of filtration".

2000 *Mathematics Subject Classification.* 60G51, 60H07, 62P05, 91B28.
Key words and phrases. Ocone-Haussmann-Clark's formula, compensated Poisson process, stochastic derivative, predictable projection.

Skorokhod (1975) suggested absolutely different method, symmetric with respect to the time inversion, i.e., it generalized the direct and inverse Ito's integrals and did not require for the integrand to be independent of the future Wiener process. Towards this end, he required for the integrand to be smooth in a certain sense, i.e., its stochastic differentiability. This idea was later on developed in the works of Protter, Malliavin (1979), Gaveau-Trauber (1982), Pardoux (1982), Nualart, Zakai (1986), etc. In particular, Gaveau-Trauber have proved that the Skorokhod operator of stochastic integration coincides with the conjugate operator of a stochastic derivative (with the so-called Malliavin's) operator.

On the other hand, in the theory of random processes special place take the so-called martingale representation theorems which, for example, implies the representation of the Wiener and Poisson functionals in the form of stochastic integrals. In the 80th of the past century, it turned out (see Harison and Pliska (1981)) that the martingale representation theorems (along with the Girsanov's measure change theorem) play an important role in the modern financial mathematics. In particular, using the integrand of the stochastic integral appearing in the integral representation, one can construct hedging strategies in the European options of different type.

According to the well-known result obtained by Clark (1970), if ξ is a $\mathcal{F}_T^w$-measurable random variable with $E\xi^2 < \infty$, then there exists the adapted process $\varphi(t,\omega) \in L_2([0,T]\times\Omega)$, such that the integral representation: $\xi = E\xi + \int_0^T \varphi(t,\omega)dw(t)$ (P-a.s.) holds. However, this result says nothing on finding the process $\varphi(t,\omega)$ explicitly. In this direction we are familiar with one sufficiently general result, the so-called Ocone-Clark's formula by which for the Wiener functionals: $\varphi(t,\omega) = E[D_t^w\xi|\mathcal{F}_t^w](\omega)$, where $D_t^w\xi$ is the stochastic derivative of the functional ξ. Application of the above expression needs as a rule, on the one hand, essential efforts, and, on the other hand, in the cases if the functional ξ has no stochastic derivative, its application is impossible. Another distinct method of finding an integrand $\varphi(t,\omega)$ belongs to Shyryaev, Yor (2003), when the functional ξ is of "maximal" type. With the above-mentioned functional they linked the associated Lewy's martingale and used the generalized Ito's formula. Our approach within the classical Ito's calucus allows one to construct $\varphi(t,\omega)$ explicitly by using both the standard L_2 theory and the theory of weighted Sobolev spaces, if the functional ξ has no stochastic derivative (see Jaoshvili, Purtukhia [13]). It is known that the events indicator has, in general, no stochastic derivative: $\exists I_A \Longleftrightarrow P(A) = 0$. Consequently, one cannot apply the Ocone-Clark formula for the indicator $I_{\{w(T)>0\}}$, whereas our approach allows one to write the following representation:

$$I_{\{w(T)>0\}} = 1/2 + \int_0^T \Phi_{0,T-t}(w(t))dw(t),$$

where $\Phi_{0,r-t}(\cdot)$ is the function of a normal distribution with parameters: 0 and $T-t$.

The subsequent generalization of the Ocone-Clark formula to the so-called normal martingales (the martingale is said to be normal, if $\langle M, M\rangle_t = t$) is due to Ma, Protter, Martin (1998). According to this formula, if $F \in D_{2,1}^M$, then the Ocone-Haussmann-Clark's representation

$$F = E(F) + \int_0^T {}^p(D_t^M F)dM(t)$$

is valid; here $D_{2,1}^M$ denotes the space of square integrable functionals having the first order stochastic derivative, and ${}^p(D_t^M F)$ is the predictable projection of the stochastic derivative $D_t^M F$ of the functional F. It is obvious, that in the Wiener process case this formula is coincide with Ocone-Clark's formula, because the flow of σ-algebras $\mathcal{F}_s^w$ is continuous and therefore the predictable projection is equal to the corresponding conditional mathematical expectation. As is seen, this representation likewise needs the existence of a stochastic derivative. On the other hand, in this case, unlike the Wiener's one, it is impossible to define in a generally adopted manner an operator of stochastic differentiation to obtain the structure of Sobolev space $D_{2,1}^M$. Here, the determination of the stochastic derivative is based on the expansion in series of multiple stochastic integrals of the functional, whereas the Wiener case involves, besides the above-mentioned approach, the structure of Sobolev spaces, which allows one to construct explicitly the stochastic derivative operator in many cases.

For a class of normal martingales one fails to define the space $D_{p,1}^M$ $(1 < p < 2)$ in a commonly adopted manner (i.e., by closing a class of smooth functionals with respect to the corresponding norm). In his work, Purtukhia (2003) defined the space $D_{p,1}^M$ $(1 < p < 2)$ for the normal martingales and generalized the Ocone-Hausmann-Clark formula to the functionals of that space.

Ma, Protter, Martin gave an example showing that two possible ways of determination of a stochastic derivative coincide if and only if the quadratic martingale characteristic $[M, M]$ is the deterministic function (as, for example, in the Wiener's case $[w, w]_t = t$). Consequently, the Ocone-Haussmann-Clark's formula makes it impossible to construct explicitly the operator of the stochastic derivative of the functionals of the Compensated Poisson process (which, obviously, belongs to a class of normal martingales $\langle M, M\rangle_t = t$, but its quadratic variation is not deterministic, $[M,M]_t = N(t) = M(t)+t$, saying nothing on the construction of its predictable projection.

Our approach within the framework of nonanticipative stochastic calculus of semimartingales allows one to construct explicitly the expression for the integrand of the stochastic integral in the theorem of martingale representation for power functionals of the Compensated Poisson process, and to derive the formu la allowing one to construct explicitly predictable projections of their stochastic derivatives (see Jaoshvili, Purtukhia [14]). In this work we will extend the aforementioned results for polynomial and square integrable Poisson functionals, in particular, we will generalize the Ocone-Haussmann-Clark's formula for Poisson functionals.

1. Auxiliary Notations and Results

Below we present (without proof) some properties of Poisson and compensated Poisson processes which (aside of the fact that moments occur in the representation theorem) are not directly used in obtaining the main results (except for Proposition 1.5), but, in our opinion, are of independent interest. Therfore we will reproduce proofs of these results in appendix. As for Proposition 1.5 - its proof with the aid of stochastic calculus is given in Remark 2.1, whereas in the appendix will be given a proof in the framework of classical differential calculus.

Let $\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{0\le t\le\infty}$ be a filtered probability space satisfying the usual conditions. Assume that the standard Poisson process N_t is given on it ($P(N_t = n) = \frac{t^n}{n!}e^{-t}$, $n = 0, 1, 2, \ldots$) and that $\mathcal{F}_t$ is generated by N ($\mathcal{F}_t = \mathcal{F}_t^N$), $\mathcal{F} = \mathcal{F}_T$. Denote by $\mu_n(t)$ the n-th order moment of the Poisson process ($\mu_n(t) := E(N_t^n)$).

Proposition 1.1. *The n-th order moment $(n \ge 1)$ of the Poisson process satisfies the following recurrence representation:*

$$\mu_n(t) = t\sum_{i=0}^{n-1} C_{n-1}^i \mu_i(t). \tag{1.1}$$

Corollary 1.1. *The n-th order moment $(n \ge 1)$ of the Poisson process N_t is a polynomial of degree n with respect to t.*

Proposition 1.2. *For any $n \ge 1 : \mu_n(t) = \sum_{k=0}^n a_k t^k$, where $a_0 = 0$, $a_1 = 1$, and for any $k \ge 2$ the coefficients are calculated from the recurrence relations:*

$$a_k = \frac{k^n}{k!} - \sum_{i=0}^{k-1} \frac{a_i}{(k-i)!} \tag{1.2}$$

Proposition 1.3. *The n-th order moment $(n \geq 1)$ of the Compensated Poisson process satisfies the following recurrence equations:*

$$\nu_n(t) = t \sum_{i=1}^{n-2} C_{n-1}^i \nu_i(t), \quad n \geq 2; \quad \nu_0(t) = 1; \quad \nu_1(t) = 0.$$

Proposition 1.4. *The n-th order moment $(n \geq 1)$ of the Compensated Poisson process is a polynomial of degree $[n/2]$ with respect to t:*

$$\nu_n(t) = \sum_{k=0}^{[n/2]} a_k^n t^k, \tag{1.3}$$

where the coefficients satisfy the following recurrence equalities:

$$a_k^n = \sum_{i+j=k} (-1)^i C_n^i \frac{j^{n-i}}{j!} - \sum_{i=0}^{k-1} \frac{a_i^n}{(k-i)!}, \quad 2 \leq k \leq [n/2]; \tag{1.4}$$

$$a_0^n = 1, \quad a_1^n = 0,$$

(*here* $[s]$ *- denotes the integer part of* s).

Corollary 1.2. *For $n/2 < k \leq n$ the solutions of recurrence equations:*

$$a_k^n = \sum_{i+j=k} (-1)^i C_n^i \frac{j^{n-i}}{j!} - \sum_{i=0}^{n-1} \frac{a_i^n}{(k-i)!}$$

are $a_k^n = 0$.

Proposition 1.5. *The n-th order moment $\nu_n(t)$ of the Compensated Poisson process satisfies the following differential equation:*

$$\frac{d\nu_n(t)}{dt} = \sum_{k=0}^{n-2} C_n^k \nu_k(t).$$

2. Integral Representation for Polynomial Functionals of Poisson Processes

Proposition 2.1. *For any natural power n of the Compensated Poisson process M_t the following representation*

$$M_t^n = \int_{(0,t]} n M_{s-}^{n-1} dM_s + \sum_{i=2}^{n} \int_{(0,t]} C_n^i M_{s-}^{n-i} dN_s \quad (P\text{-a.s.}) \tag{2.1}$$

is valid.

Proof. According to the Ito's formula, we have:

$$M_t^n = \int_{(0,t]} n M_{s-}^{n-1} dM_s + \sum_{s \leq t} (M_s^n - M_{s-}^n - n M_{s-}^{n-1} \Delta M_s).$$

Further, using the relation:

$$a^n - b^n - n(a-b)b^{n-1} = \sum_{i=2}^{n} C_n^i (a-b)^i b^{n-i}, \tag{2.2}$$

it is not difficult to see, that:

$$M_t^n = n \int\limits_{(0,t]} M_{s-}^{n-1} dM_s + \sum_{s\le t}\sum_{i=2}^{n} C_n^i \Delta M_s^i M_{s-}^{n-i} =$$

$$= n \int\limits_{(0,t]} M_{s-}^{n-1} dM_s + \sum_{s\le t}\sum_{i=2}^{n} C_n^i M_{s-}^{n-i} \Delta N_s =$$

$$= n \int\limits_{(0,t]} M_{s-}^{n-1} dM_s + \int\limits_{(0,t]} \sum_{i=2}^{n} C_n^i M_{s-}^{n-i} dN_s. \qquad \square$$

Remark 2.1. It must be noted that to obtain Proposition 1.5 it suffices in expression (2.1) to replace dN_s by $dM_s + ds$ and take mathematical expectation of both sides. Indeed, we will then obtain the equality

$$\nu_n(t) = \int\limits_0^t \sum_{i=2}^{n} C_n^i \nu_{n-i}(s) ds = \int\limits_0^t \sum_{i=0}^{n-2} C_n^i \nu_i(s) ds,$$

which is equivalent to the claim of Proposition 1.5.

Before formulating the general representation theorem, let us give two examples.

Example 2.1. The random variable M_T^2 has the following stochastic integral representation:

$$M_T^2 = E[M_T^2] + \int\limits_{(0,t]} (1 + 2M_{s-}) dM_s \quad (P\text{-a.s.}).$$

Let us denote $X_t := E[M_T^2|\mathcal{F}_t]$. According to the well-known properties of the Compensated Poisson process, we obtain the following representation:

$$X_t = E[(M_T - M_t + M_t)^2|\mathcal{F}_t] = E[(M_T - M_t)^2] +$$
$$+2M_t E(M_T - M_t) + M_t^2 = T - t + M_t^2.$$

On the other hand, due to the Proposition 2.1, (P-a.s.) we have:

$$M_t^2 = \int\limits_{(0,t]} 2M_{s-} dM_s + \int\limits_{(0,t]} dN_s = \int\limits_{(0,t]} 2M_{-s} dM_s + N_t.$$

Substituting now the last expression in the above relation we ascertain that:

$$X_t = T - t + \int_{(0,t]} 2M_{s-}dM_s + N_t = T + M_t + \int_{(0,t]} 2M_{s-}dM_s = T + \int_{(0,t]} (1+2M_{s-})dM_s,$$

whence, taking into account the relations: $X_T = E[M_T^2|\mathcal{F}_T] = M_T^2$ and $T = E[M_T^2]$, we obtain the desired representation, which is an analog of the Ocone-Clark's formula written for the power functionals of the Compensated Poisson process in cases $n = 2$ (note that in the Wiener process cases the Ocone-Clark's formula gives us that:

$$w_T^2 = E[w_T^2] + \int_{(0,t]} 2w_t dw_t \quad (P\text{-a.s.})).$$

Remark 2.2. It must be noted that in the case of square stochastic integral representation can be obtained more easily using the Ito's formula. We present the above method to illustrate that approach which we use in the general case.

Example 2.2. Using the arguments similar to those presented above, we obtain the following stochastic integral representation for the third power of the Compensated Poisson process:

$$M_T^3 = E[M_T^3] + \int_{(0,T]} [1 + 3(T-s) + 3M_{s-} + 3M_{-s}^2]dM_s \quad (P\text{-a.s.})$$

(note that in the Wiener process cases the Ocone-Clark's formula gives us that:

$$w_T^3 = E[w_T^3] + \int_{(0,t]} 3w_t^2 dw_t \quad (P\text{-a.s.})).$$

Theorem 2.1. *For any natural $n \geq 1$ the following integral representation is valid:*

$$M_T^n = E[M_T^n] + \sum_{k=1}^{n} C_n^k \sum_{i=1}^{k} C_k^i \int_{(0,T]} M_{s-}^{k-i} v_{n-k}(T-s)dM_s \quad P\text{-}a.s. \tag{2.3}$$

Proof. It is obvious that the representation (2.3) holds for $n = 1$. Due to the Examples 2.1 and 2.2 it is true for $n = 2$ and $n = 3$. Therefore we consider the case $n \geq 4$. For that let us calculate the conditional mathematical expectation $X_t := E[M_T^n|\mathcal{F}_t]$. Using the well-known properties of the Compensated Poisson process and the relation $V_1(T-t) = 0$, one can easily ascertain that:

$$X_t := E[M_T^n|\mathcal{F}_t] = E[(M_T - M_t + M_t)^n|\mathcal{F}_t] =$$
$$= E\Big[\sum_{k=0}^{n} C_n^k (M_T - M_t)^{n-k} M_t^k|\mathcal{F}_t\Big] = \sum_{k=0}^{n} C_n^k \nu_{n-k}(T-t) M_t^k =$$
$$= \nu_n(T-t) + nM_t\nu_n(T-t) + \sum_{k=1}^{n-2} C_n^k M_t^k \nu_{n-k}(T-t) + M_t^n.$$

Further, according to the Ito's formula, taking into account the Propositions 1.5 and 2.1, it is not difficult to see that:

$$X_t = X_0 - \sum_{i=0}^{n-2} C_n^k \int\limits_{(0,t]} \nu_i(T-s)ds+$$
$$+n \int\limits_{(0,t]} \nu_{n-1}(T-s)dM_s - \sum_{i=0}^{n-3} nC_{n-1}^i \int\limits_{(0,t]} M_{s-}\nu_i(T-s)ds-$$
$$-\sum_{k=1}^{n-2} C_n^k \sum_{i=0}^{n-k-2} C_{n-k}^i \int\limits_{(0,t]} M_{s-}^k \nu_i(T-s)ds+$$
$$+\sum_{k=1}^{n-2} C_n^k \sum_{i=2}^{k} C_k^i \int\limits_{(0,t]} M_{s-}^{k-i}\nu_{n-k}(T-s)dN_s+$$
$$+\sum_{k=1}^{n-2} C_n^k k \int\limits_{(0,t]} M_{s-}^{k-1}\nu_{n-k}(T-s)dM_s+$$
$$+\sum_{k=2}^{n} C_n^k \int\limits_{(0,t]} M_{s-}^{n-k}dN_s + n\int\limits_{(0,t]} M_{s-}^{n-1}dM_s := \nu_n(T) - I_t + I_{N_t} + I_{M_t}.$$

Studying carefully the obtained relation, one can easily notice (as well as in Examples 2.1 and 2.2) that for any Lebesgue integral with respect to ds there exists the corresponding stochastic integral with respect to dN_s with the same integrand of opposite sign.

Indeed, due to relations $\nu_0(T-s) = 1$, $\nu_1(T-s) = 0$, we have:

$$I_t := \sum_{i=0}^{n-2} C_n^i \int\limits_{(0,t]} \nu_i(T-s)ds + \sum_{i=0}^{n-3} nC_{n-1}^i \int\limits_{(0,t]} M_{s-}\nu_i(T-s)ds+$$
$$+\sum_{k=2}^{n-2} C_n^k \sum_{i=0}^{n-k-2} C_{n-k}^i \int\limits_{(0,t]} M_{s-}^k \nu_i(T-s)ds =$$

$$= \sum_{k=0}^{n-2} C_n^k \sum_{i=0}^{n-k-2} C_{n-k}^i \int\limits_{(0,t]} M_{s-}^k \nu_i(T-s)ds$$

and

$$I_{N_t} := \sum_{k=2}^{n-2} C_n^k \sum_{i=2}^{k} C_k^i \int\limits_{(0,t]} M_{s-}^{k-i} \nu_{n-k}(T-s)dN_s + \sum_{k=2}^{n} C_n^k \int\limits_{(0,t]} M_{s-}^{n-k} dN_s =$$

$$= \sum_{k=2}^{n-2} C_n^k \sum_{i=2}^{k} C_k^i \int\limits_{(0,t]} M_{s-}^{k-i} \nu_{n-k}(T-s)dN_s +$$

$$+ C_n^{n-1} \sum_{i=2}^{n-1} C_{n-1}^i \int\limits_{(0,t]} M_{s-}^{n-1-i} \nu_{n-(n-1)}(T-s)dN_s +$$

$$+ C_n^n \sum_{i=2}^{n} C_n^i \int\limits_{(0,t]} M_{s-}^{n-i} \nu_{n-n}(T-s)dN_s =$$

$$= \sum_{k=2}^{n} C_n^k \sum_{i=2}^{k} C_k^i \int\limits_{(0,t]} M_{s-}^{k-i} \nu_{n-k}(T-s)dN_s.$$

Compare now members with the same powers m of M_{s-} in representations of I_t and I_{N_t}. Denote these members by $I_t(m)$ and $I_{N_t}(m)$ accordingly. Then we have:

$$I_t(m) = C_n^m \sum_{i=0}^{n-m-2} C_{n-m}^i \int\limits_{(0,t]} M_{s-}^m \nu_i(T-s)ds$$

and

$$I_{N_t} = C_n^{m+2} C_{m+2}^2 \int\limits_{(0,t]} M_{s-}^m \nu_{n-m-2}(T-s)dN_s +$$

$$+ C_n^{m+3} C_{m+3}^3 \int\limits_{(0,t]} M_{s-}^m \nu_{n-m-3}(T-s)dN_s +$$

$$+ \cdots + C_n^{m+n-m} C_{m+n-m}^{n-m} \int\limits_{(0,t]} M_{s-}^m \nu_{n-m-(n-m)}(T-s)dN_s =$$

$$= \sum_{i=2}^{n-m} C_n^{m+i} C_{m+i}^i \int\limits_{(0,t]} M_{s-}^m \nu_{n-m-i}(T-s)dN_s.$$

Denote by $I_t^j(m)$ (accordingly by $I_{N_t}^{n-m-j}(m)$) the j-th member of $I_t(m)$ (accordingly the $(n-m-j)$-th member of $I_{N_t}(m)$). Then it is obvious that:

$$I_t^j(m) = C_n^m C_{n-m}^{n-m-j-1} \int\limits_{(0,t]} M_{s-}^m \nu_{n-m-j-1}(T-s)ds =$$

$$= \frac{n!}{m!(n-m-j-1)!(j+1)!} \int\limits_{(0,t]} M_{s-}^m \nu_{n-m-j-1}(T-s)ds$$

and

$$I_{N_t}^{n-m-j}(m) = C_n^{m+j+1} C_{m+j+1}^{j+1} \int\limits_{(0,t]} M_{s-}^m \nu_{n-m-j-1}(T-s)dN_s =$$

$$= \frac{n!}{m!(n-m-j-1)!(j+1)!} \int\limits_{(0,t]} M_{s-}^m \nu_{n-m-j-1}(T-s)dN_s$$

Therefore, summing up the aforementioned integrals, due to relation $M_s = N_s - s$, we obtain the stochastic integral with respect to dM_s. Hence, using the equalities $X_T = M_T^n$, $X_0 = E[M_t^n]$ and $\nu_1(T-s) = 0$ it is not difficult to see that the following representation is valid:

$$M_T^n = E[M_T^n] + \sum_{k=1}^{n} C_n^k \int\limits_{(0,T]} M_{s-}^{n-k} dM_s +$$

$$+ \sum_{k=2}^{n-2} C_n^k \sum_{i=1}^{k} C_k^i \int\limits_{(0,T]} M_{s-}^{k-i} \nu_{n-k}(T-s)dM_s + n \int\limits_{(0,T]} \nu_{n-1}(T-s)dM_s.$$

The last representation can be rewritten in the following form:

$$M_T^n = E[M_T^n] + \sum_{k=1}^{n} C_n^k \sum_{i=1}^{k} C_k^i \int\limits_{(0,T]} M_{s-}^{k-i} \nu_{n-k}(T-s)dM_s,$$

which completes the proof of the theorem. □

Corollary 2.1. *For any natural $n \geq 1$ the following integral representation holds:*

$$M_T^n = E[M_T^n] +$$

$$+ \int\limits_{(0,T]} E[(1 + M_{s-} + M_T - M_s)^n - (M_{s-} + M_T - M_s)^n | \mathcal{F}_s] dM_s \ (P\text{-}a.s.). \tag{2.4}$$

Proof. Due to the relation (2.2) (from the Proposition 2.1), we have:

$$(M_{s-}+1)^k - M_{s-}^k = \sum_{i=1}^{k} C_k^i M_{s-}^{k-i}.$$

Therefore, according to the Theorem 2.1, we can write:

$$M_T^n = E[M_T^n] + \int\limits_{(0,T]} \sum_{k=1}^{n} C_n^k [(M_{s-}+1)^k - M_{s-}^k]\nu_{n-k}(T-s)dM_s \quad (P\text{-a.s.}).$$

On the other hand, using the Newton's binomial formula and the well-known properties of the Compensated Poisson process, one can conclude that:

$$E[(1+M_{s-}+M_T-M_s)^n|\mathcal{F}_s] = \sum_{k=0}^{n} E[C_n^k(1+M_{s-})^k(M_T-M_s)^{n-k}|\mathcal{F}_s] =$$

$$= \sum_{k=0}^{n} C_n^k(1+M_{s-})^k E[(M_T-M_s)^{n-k}|\mathcal{F}_s] = \sum_{k=0}^{n} C_n^k(1+M_{s-})^k E(M_T-M_s)^{n-k} =$$

$$= \sum_{k=0}^{n} C_n^k(1+M_{s-})^k EM_{T-s}^{n-k} =$$

$$= \sum_{k=0}^{n} C_n^k(1+M_{s-})^k \nu_{n-k}(T-s).$$

Analogously, one can easily obtain that:

$$E[(M_{s-}+M_T-M_s)^n|\mathcal{F}_s] = \sum_{k=0}^{n} C_n^k M_{s-}^k \nu_{n-k}(T-s).$$

Combining the relations obtained above, we ascertain that the representation (2.4) is valid. □

Let us denote by $P_n(x)$ a polynomial of order with respect to x.

Corollary 2.2. *The following representation is valid* (P*-a.s.*)*:*

$$P_n(M_T) = E[P_n(M_T)] +$$

$$+ \int\limits_{(0,T]} E[P_n(1+M_{s-}+M_T-M_s) - P_n(M_{s-}+M_T-M_s)|\mathcal{F}_s]dM_s.$$

Proof. The proof is easily obtained from the linearity of stochastic integrals and conditional mathematical expectation. □

Let us denote $\Delta f(x) := f(x+1) - f(x)$, $(\Delta P_n(M_T) = \Delta P_n(x)|_{x=M_T})$.

Theorem 2.2. *For any natural* $n \geq 1$ *the following representation holds:*

$$P_n(M_T) = E[P_n(M_T)] + \int\limits_{(0,T]} E[\Delta P_n(M_T)|\mathcal{F}_{t-}]dM_t \quad (P\text{-}a.s.). \tag{2.5}$$

Proof. At first we prove the representation (2.5) for power functionals. Due to the well-known properties of the Compensated Poisson process, we can write:

$$E[\Delta(M_T)^n|\mathcal{F}_{s-}] = E[(M_T+1)^n - M_T^n|\mathcal{F}_{s-}] = E\Big[\sum_{k=0}^{n-1} C_n^k M_T^k|\mathcal{F}_{s-}\Big] =$$
$$= E\Big[\sum_{k=0}^{n-1} C_n^k (M_T - M_{s-} + M_{s-})^k|\mathcal{F}_{s-}\Big] =$$
$$= E\Big[\sum_{k=0}^{n-1} C_n^k \sum_{i=0}^{k} C_k^i (M_T - M_{s-})^i M_{s-}^{k-i}|\mathcal{F}_{s-}\Big] =$$
$$=\sum_{k=0}^{n-1} C_n^k \sum_{i=0}^{k} C_k^i [E(M_T - M_{s-})^i] M_{s-}^{n-i} = \sum_{k=0}^{n-1} C_n^k \sum_{i=0}^{k} C_k^i \nu_i(T-t) M_{s-}^{k-i}, \tag{2.6}$$

where in the last equality we used the stochastic continuity of the Compensated Poisson process.

If now we consider both the right side of the last relation and the second summand as the polynomial of order $n-1$ with respect to M_{s-}, using the properties of Newton's binomial, it is not difficult to see that these polynomials are just the same. Therefore, we conclude that:

$$M_T^n = E[M_T^n] + \int\limits_{(0,T]} E[\Delta M_T^n|\mathcal{F}_{t-}]dM_t \quad (P\text{-a.s.}).$$

From this, using the linearity of the stochastic integrals, the conditional mathem-atical expectation and of the Δ operator, we easily obtain the desired integral represent-tation for the polynomial functionals of the Compensated Poisson process. □

Corollary 2.3. *Because, for any fixed* T *the polynomial* $P_n(x-T)$ *again is a polynomial of order* n *with respect to* x *but with other coefficients, therefore it is clear that the representation of type* (2.5) *is valid for the polynomial functionals of the Poisson process:*

$$P_n(N_T) = E[P_n(N_T)] + \int\limits_{(0,T]} E[\Delta P_n(N_T)|\mathcal{F}_{t-}]dM_t \quad (P\text{-}a.s.).$$

Theorem 2.3. *For any polynomial function $P_n(x)$ the following representation is valid:*

$$^{p}[D_t^M P_n(M_T)] = E[\Delta P_n(M_T)|\mathcal{F}_{t-}] \quad (dP \otimes ds\text{-}a.s.). \tag{2.7}$$

where $^{p}[D_t^M P_n(M_T)]$ denotes the predictable projection of the stochastic derivative of the Compensated Poisson process.

Proof. According to the well-known result of Ma, Protter and Martin (1998) we have:

$$P_n(M_T) = E[P_n(M_T)] + \int\limits_{(0,T]} \{^{p}[D_t^M P_n(M_T)]\} dM_t \quad (P\text{-a.s.}).$$

Consider the difference:

$$Y_T = \int\limits_{(0,T]} \{E[\Delta P_n(M_T)|\mathcal{F}_{t-}] - {}^{p}[D_t^M P_n(M_T)]\} dM_t := \int\limits_{(0,T]} \eta_t dM_t.$$

In the one hand, it is clear, due to the Theorem 2.1, that $Y_t = 0$ (P-a.s.). On the other hand, according to the Ito's formula, we have:

$$Y_t^2 = 2 \int\limits_{(0,T]} Y_{t-}\eta_t dM_t + \int\limits_{(0,T]} \eta_t^2 d[M,M]_t.$$

If now we take the mathematical expectation from the both sides of the last relation, using the well-known properties of the square and predictable characteristics of the martingale, we ascertain that:

$$0 = E \int\limits_{(0,T]} \eta_t^2 d[M,M]_t = E \int\limits_{(0,T]} \eta_t^2 d\langle M,M\rangle_t = E \int\limits_{(0,T]} \eta_t^2 dt,$$

whence we conclude that the relation (2.7) is true. □

3. Integral Representation for Square Integrable Functionals

Below we will see that the results obtained above are true for more wide classes of functionals than the polynomial functionals. In this section we prove the integral representation for some class of functionals from the space L_2.

Let $Z^+ = \{0, 1, 2, \dots\}$ and $P = \{P_1, P_2, P_3, \dots\}$ – be the Poisson distribution: $P_x = \frac{e^{-T}T^x}{x!}$, $x = 0, 1, 2, \dots$. Let us denote $\triangledown f(x) = f(x) - f(x-1)$ ($f(x) = 0, x < 0$) and define the Poisson-Sharle's polynomials: $\Pi_n(x) = \frac{(-1)^n \triangledown^n P_x}{P_x}$, $n \geq 1$; $\Pi_0 = 1$.

It is wellknown from the course of Functional Analysis [15] that the sequence $\{\pi_n(x)\}_{n\geq 0}$ $\left(\pi_n(x) = \frac{\Pi_n(x)}{c_n}\right)$ is a basis in the space $L_2(Z^+)(L_2(Z^+) = \{f : \sum_{x=0}^{\infty} f^2(x) < \infty\})$.

Let $\rho(x,T) := \frac{T^x}{x!}e^{-T}$ and denote by $L_2^T := L_2(Z^+;\rho(x,T))$ the functional space on Z^+ with the finite norm $\|g\|_{2,T} := \|g\rho^{1/2}(T)\|_{L_2}$.

Proposition 3.1. *The space L_2^T is a Banach space with basis $\{x^n\rho(x,T)\}$.*

Proof. The proof is based on the well-known result of Functional Analysis. By the statement VIII.4.3. [15], it is not difficult to ensure this, since:

$$\sum_{x=0}^{\infty} e^{c|x|}\frac{T^x}{x!}e^{-T} = e^{-T}\sum_{x=0}^{\infty}\frac{(e^cT)^x}{x!} = e^{-T}e^{e^cT} < \infty. \qquad \square$$

Proposition 3.2. *If $f(\cdot - T) \in L_2^T$, then the stochastic integral*

$$\int_{(\sigma,T)} E[f(M_T)|\mathcal{F}_{t-}]dM_t$$

is well defined.

Proof. According to the well-known properties of the stochastic integral, using the Lyapunov's and Jensen's inequalities and by the Fubini's theorem, it is not difficult to see that:

$$\|E[f(M_T)|\mathcal{F}_{\cdot -}]\| := E\Big\{\int_{(0,T]}(E[f(M_T)|\mathcal{F}_{t-}])^2 d[M,M]_t\Big\}^{1/2} \le$$

$$\le \Big\{E\int_{(0,T]}(E[f(M_T)|\mathcal{F}_{t-}])^2 d[M,M]_t\Big\}^{1/2} =$$

$$= \Big\{E\int_{(0,T]}(E[f(M_T)|\mathcal{F}_{t-}])^2 d\langle M,M\rangle_t\Big\}^{1/2} =$$

$$= \Big\{E\int_{(0,T]}(E[f(M_T)|\mathcal{F}_{t-}])^2 dt\Big\}^{1/2} \le \Big\{E\int_{(0,T]}E[\{f(M_T)\}^2|\mathcal{F}_{t-}]dt\Big\}^{1/2} =$$

$$= \Big\{\int_{(0,T]}E[f(M_T)]^2 dt\Big\}^{1/2} = \{T\cdot E[f(M_T)]^2\}^{1/2} = \{T\cdot\|f(\cdot - T)\|_{2,T}^2\}^{1/2}.$$

For this, due to the statements II.2 [16], $E[f(M_T)|\mathcal{F}_{t-}] \in L[F,[M,M]]$ and, hence, aforementioned stochastic integral is well defined. $\square$

Theorem 3.1. *If $f \in L_2^T$ and for some $0 < \alpha < 1$ number $\Delta f(\cdot - T) \in L_2^{T/\alpha}$, then the stochastic integral below is well defined and for the functional $f(M_T)$ the following representation is valid:*

$$f(M_T) = E[f(M_T)] + \int_{(0;T]} E[\Delta f(M_T)|\mathcal{F}_{t-}]dM_t \quad (P\text{-}a.s.). \tag{3.1}$$

Proof. Since, it is obvious that: $\|g\|^2_{2,T} \le \exp\{(1-\alpha)T/\alpha\}\|g\|^2_{2,T/\alpha}$, according to the Proposition 3.2, under the conditions of the theorem, we conclude that the stochastic integral in (3.1) is well defined.

Denote by: $\widetilde{f}(x) := f(x-t)$. Due to the Proposition 3.1 there exists a sequence of polynomials $Q_n(x)$ such that the relation

$$\lim_{n\to\infty} \|Q_n(x) - \Delta\widetilde{f}(x)\|_{2,T/\alpha} = 0$$

holds.

Let us define $\widetilde{P}_n(x) := \widetilde{f}(0) + \sum_{i=0}^{x-1} Q_n(i)$, then according to the Corollary 2.3 the following representation is fulfilled:

$$\widetilde{P}_n(N_T) = E[\widetilde{P}_n(N_T)] + \int\limits_{(0,T]} E[\Delta\widetilde{P}_n(N_T)|\mathcal{F}_{t-}]dM_t \quad (P-\text{a.s.}). \tag{3.2}$$

It is obvious that:

$$\widetilde{P}_n(N_T) = \widetilde{f}(0) + \sum_{i=0}^{N_T-1} Q_n(i),$$

$$\widetilde{f}(N_T) = \widetilde{f}(0) + \sum_{i=0}^{N_T-1}[\widetilde{f}(i+1) - \widetilde{f}(i)] = \widetilde{f}(0) + \sum_{i=0}^{N_T-1}\Delta\widetilde{f}(i)$$

and

$$\widetilde{P}_n(N_T) - \widetilde{f}(N_T) = \sum_{i=0}^{N_T-1}[Q_n(i) - \Delta\widetilde{f}(i)].$$

Further, using the elementary inequality $\Big(\sum_{i=1}^n a_i\Big)^2 \le n\sum_{i=1}^n a_i^2$, one can easily see that:

$$\|\widetilde{P}_n(N_T) - \widetilde{f}(N_T)\|^2_{2,T} \le \sum_{k=0}^{\infty} k\sum_{i=0}^{k-1}[Q_n(i) - \Delta\widetilde{f}(i)]^2\frac{T^k}{k!}e^{-T} \le$$
$$\le T\sum_{k=0}^{i_0+1}\sum_{i=0}^{k-1}[Q_n(i) - \Delta\widetilde{f}(i)]^2\frac{(T/\alpha)^i}{i!}e^{-\frac{T}{\alpha}}e^{\frac{T}{\alpha}}\frac{T^{k-1}}{(k-1)!}e^{-T} +$$
$$+T\sum_{k=i_0+1}^{\infty}\sum_{i=0}^{k-1}[Q_n(i) - \Delta\widetilde{f}(i)]^2\frac{(T/\alpha)^i}{i!}e^{-\frac{T}{\alpha}}\frac{(k-1)!}{(T/\alpha)^{k-1}}e^{\frac{T}{\alpha}}\frac{T^{k-1}}{(k-1)!}e^{-T},$$

where by $i_0 = i_0(T/\alpha)$ is denoted the natural number for which $\frac{i_0!}{(T/\alpha)^{i_0}} \le 1$ and $\frac{(i_0+1)!}{(T/\alpha)^{i_0+1}} \ge 1$. Then, it is evident that for such number, we have:

$$\max\left\{\frac{i!}{(T/\alpha)^i}\right\} = \begin{cases} 1, & \text{if } k-1 \le i_0(T/\alpha) \\ \frac{(k-1)!}{(T/\alpha)^{k-1}}, & \text{if } k-1 \ge i_0(T/\alpha) \end{cases}$$

Taking into account the aforementioned arguments, we can write:

$$\|\widetilde{P}_n(N_T)-\widetilde{f}(N_T)\|^2_{2,T}\leq\|Q_n(N_T)-\widetilde{f}(N_T)\|^2_{2,T/\alpha}\times$$
$$\times\bigg(T\sum_{k=0}^{i_0+1}\frac{T^{k-1}}{(k-1)!}e^{-T}e^{T/\alpha}+\sum_{k=i_0+1}^{\infty}\alpha^{k-1}e^{T/\alpha}e^{-T}\bigg)\to 0,\quad\text{as}\quad n\to\infty.$$

Therefore, we can ascertain that: $E[\widetilde{P}_n(N_T)]\to E[\widetilde{f}(N_T)]$, as $n\to\infty$. On the other hand, one can verify that:

$$\int\limits_{(0,T]}E[\Delta\widetilde{P}_n(N_T)|\mathcal{F}_{t-}]dM_t\xrightarrow{L_2}\int\limits_{(0,T]}E[\Delta\widetilde{f}(N_T)|\mathcal{F}_{t-}]dM_t,\quad\text{as}\quad n\to\infty.$$

Indeed, according to the well-known properties of the stochastic integral, using the Jensen's inequality, it is not difficult to see that:

$$E\bigg\{\int\limits_{(0,T]}E[\Delta\widetilde{P}_n(N_T)|\mathcal{F}_{t-}]-E[\Delta\widetilde{f}(N_T)|\mathcal{F}_{t-}]dM_t\bigg\}^2=$$
$$=E\int\limits_{(0,T]}\{E[\Delta\widetilde{P}_n(N_T)|\mathcal{F}_{t-}]-E[\Delta\widetilde{f}(N_T)|\mathcal{F}_{t-}]\}^2d[M,M]_t=$$
$$=E\int\limits_{(0,T]}\{E[\Delta\widetilde{P}_n(N_T)-\Delta\widetilde{f}(N_T)|\mathcal{F}_{t-}]\}^2d\langle M,M\rangle_t=$$
$$=E\int\limits_{(0,T]}\{E[\Delta\widetilde{P}_n(N_T)-\Delta\widetilde{f}(N_T)|\mathcal{F}_{t-}]\}^2dt\leq\int\limits_{(0,T]}E[\Delta\widetilde{P}_n(N_T)-\Delta\widetilde{f}(N_T)]^2dt=$$
$$=\int\limits_{(0,T]}E[Q_n(N_T)-\Delta\widetilde{f}(N_T)]^2dt=\int\limits_{(0,T]}\|Q_n(x)-\Delta\widetilde{f}(x)\|^2_{2,T}dt\leq$$
$$\leq T\cdot\exp\{(1-\alpha)T/\alpha\}\cdot\|Q_n(x)-\Delta\widetilde{f}(x)\|^2_{2,T/\alpha}\to 0,\quad\text{as}\quad n\to\infty.$$

Passing now to the limit in the equality (3.2), as $n\to\infty$ and taking into account all the above-obtained relations, we conclude that:

$$\widetilde{f}(N_T)=E[\widetilde{f}(N_T)]+\int\limits_{(0,T]}E[\Delta\widetilde{f}(N_T)|\mathcal{F}_{t-}]dM_t\quad(P-\text{a.s.})\tag{3.3}$$

whence, according to the equalities $\widetilde{f}(x)=f(x-T)$ and $\Delta\widetilde{f}(x)=\Delta f(x-T)$, we obtain the desired representation. □

Using the arguments similar to those presented in the proof of the Theorem 2.3, one can prove that:

Theorem 3.2. *If $f \in L_2^T$ and for some $0<\alpha<1$ number $\Delta f(\cdot - T) \in L_2^{T/\alpha}$, then for any $t \in [0,T]$ the following relation holds:*

$$^{p}(D_t^M f(M_T)) = E[\Delta f(M_T)|\mathcal{F}_{t-}] \quad (P\text{-}a.s.). \tag{3.4}$$

4. Integral Representation for Functionals from the space $W_{p,1,\alpha}$

Let us denote by $W_{2,1,\alpha} := \{f : f \in L_2^T, \Delta\widetilde{f}_T \in L_2^{T/\alpha}\}$. It is obvious that $W_{2,1,\alpha}$ is the Hilbert space whose scalar product is given by the norm:

$$\|f\|_{2,1,\alpha} := \|f\|_{L_2^T} + \|\Delta\widetilde{f}_T\|_{L_2^{T/\alpha}}.$$

Fix $1 \le p \le 2$ and define: $L_p^T := \{f : f\rho^{1/p} \in L_2\}$ and $\|f\|_{L_p^T} := \|f\rho^{1/p}\|_{L_2}$. Let us introduce also the norm:

$$\|f\|_{p,1,\alpha} := \|f\|_{L_p^T} + \|\Delta\widetilde{f}_T\|_{L_p^{T/\alpha}}.$$

Definition 4.1. Denote by $W_{p,1,\alpha}$ $(1 \le p < 2)$ the Banach space which is the closure of $W_{2,1,\alpha}$ under the norm $\|\cdot\|_{p,1,\alpha}$.

Theorem 4.1. *If $f \in W_{p,1,\alpha}$, where $1<p<2$, then the following representation is valid:*

$$f(M_T) = E[f(M_T)] + \int\limits_{(0,T]} E[\Delta f(M_T)|\mathcal{F}_{t-}]dM_t \quad (P\text{-}a.s.).$$

Proof. According to the Definition 4.1 there exists a sequence $\{f_n\}_{n=1}^{\infty} \subset W_{p,1,\alpha}$, such that:

$$\lim_{n\to\infty} \|f_n - f\|_{p,1,\alpha} = 0. \tag{4.1}$$

Let us introduce the following martingales $m(t) := E[f(M_T)|\mathcal{F}_t]$ and $m_n(t) := E[f_n(M_T|\mathcal{F}_t)]$.

Due to the the well-known martingale representation theorem one has:

$$m(t) = E[f(M_T)] + \int\limits_{(0,T]} g(s)dM_s.$$

On the other hand, under the conditions of the theorem, from the Theorem 3.1, for any n we can write:

$$f_n(M_T) = E[f_n(M_T)] + \int\limits_{(0,T]} E[\Delta f_n(M_T)|\mathcal{F}_{t-}]dM_t,$$

whence, if now we take the conditional mathematical expectation with respect to $\mathcal{F}_t$ from the both sides of the last relation, we obtain that:

$$m_n(t) = E[f_n(M_T)] + \int\limits_{(0,T]} E[\Delta f_n(M_T)|\mathcal{F}_{s-}]dM_s \quad (P\text{-a.s.}). \tag{4.2}$$

According to the Doob's maximal inequality and the Iensen's inequality, taking into account the relation (4.1), it is not difficult to see that:

$$\Big\{E[\sup_{0\le t\le T}|m_n(t)-m(t)|]^p\Big\}^{1/p}\le\frac{p}{1-p}\sup_{0\le t\le T}[E|m_n(t)-m(t)|^p]^{1/p}=$$
$$=\frac{p}{1-p}[E|m_n(T)-m(T)|^p]^{1/p}=\frac{p}{p-1}\|f_n(M_T)-f(M_T)\|_{L_p}\le$$
$$\le\frac{p}{p-1}\|f_n-f\|_{p,1,\alpha}\to 0,\quad\text{as}\quad n\to\infty.$$

Further, using the Bukholder-Davis-Gandy's inequality, due to the estimate obtained above, we have:

$$E[m_n-m,m_n-m]_T^{p/2}=E\Big\{\int\limits_{(0,T]}|E[\Delta f_n(M_T)|\mathcal{F}_{t-}]-g(t)|^2d[M,M]_t\Big\}^{p/2}\le$$
$$\le cE\{\sup_{0\le t\le T}|m_n(t)-m(t)|\}^p\to 0,\quad\text{as}\quad n\to\infty.$$

Moreover, according to the Lyapunov's inequality, from the previous relation we conclude that:

$$\lim_{n\to\infty}E\Big\{\int\limits_{(0,T]}|E[\Delta f_n(M_T)|\mathcal{F}_{t-}]-g(t)|^2d[M,M]_t\Big\}^{1/2}\le$$
$$\le\lim_{n\to\infty}\bigg(E\Big\{\int\limits_{(0,T]}|E[\Delta f_n(M_T)|\mathcal{F}_{t-}]-g(t)|^2d[M,M]_t\Big\}^{p/2}\bigg)^{1/p}=0.$$

From here, due to the Chebyshev's inequality, we see that:

$$P-\lim_{n\to\infty}\int\limits_{(0,T]}|E[\Delta f_n(M_T)|\mathcal{F}_{t-}]-g(t)|^2d[M,M]_t=0,$$

where $P-\lim_{n\to\infty}-$ denotes the limit in probability.

Therefore, according to the Kunita-Watanebe's inequality, one can ascertain that:

$$P-\lim_{n\to\infty}\int\limits_{(0,T]}|E[\Delta f_n(M_T)|\mathcal{F}_{t-}]-g(t)|d[M,M]_t\le$$
$$\le P-\lim_{n\to\infty}\bigg\{\Big(\int\limits_{(0,T]}|E[\Delta f_n(M_T)|\mathcal{F}_{t-}]-g(t)|^2d[M,M]_t\Big)^{1/2}\times$$
$$\times([M,M]_T)^{1/2}\bigg\}=0.\tag{4.3}$$

On the other hand, by linearity of the conditional mathematical expectation and of the Δ operator, we can write:

$$E \int_{(0,T]} |E[\Delta f_n(M_T)|\mathcal{F}_{t-}] - E[\Delta f(M_T)|\mathcal{F}_{t-}]dt =$$
$$= E \int_{(0,T]} |E\{\Delta[f_n(M_T) - f(M_T)]\mathcal{F}_{t-}\}|dt.$$

Therefore, due to the Jensen's, Cauchy-Bunyakovski' s and Lyapunov's inequalities and by the Fubini's theorem, under the conditions of the theorem, we conclude that:

$$E \int_{(0,T]} |E[\Delta f_n(M_T)|\mathcal{F}_{t-}] - E[\Delta f(M_T)|\mathcal{F}_{t-}]|dt \leq$$
$$\leq E \int_{(0,T]} E\{|\Delta[f_n(M_T) - F(M_T)]||\mathcal{F}_{t-}\}dt =$$
$$= E \int_{(0,T]} |\Delta[f_n(M_T) - f(M_T)]|dt \leq$$
$$\leq \sqrt{T} E\Big\{ \int_{(0,T]} |\Delta[f_n(M_T) - f(M_T)]|^2 dt \Big\}^{1/2} \leq$$
$$\leq \sqrt{T}\Big\{ \int_{(0,T]} E|\Delta[f_n(M_T) - f(M_T)]|^2 dt \Big\}^{1/2} \leq$$
$$\leq \sqrt{T}\|f_n - f\|_{p,1,\alpha} \to 0, \quad \text{as} \quad n \to \infty. \tag{4.4}$$

Since $\langle M, M\rangle_t = t$ and for any predictable h process:

$$E \int_{(0,T]} |h(t)|dt = E \int_{(0,T]} |h(t)|d\langle M, M\rangle_t = E \int_{(0,T]} |h(t)|d[M, M]_t,$$

from the relation (4.4) we ascertain that:

$$\lim_{n\to\infty} E \int_{(0,T]} |E[\Delta f_n(M_T)|\mathcal{F}_{t-}] - E[\Delta f(M_T)|\mathcal{F}_{t-}]|d[M, M]_t = 0, \tag{4.5}$$

and, moreover:

$$E \int_{(0,T]} |E[\Delta f(M_T)|\mathcal{F}_{t-}]|d[M, M]_t = E \int_{0,T]} |E[\Delta f(M_T)|\mathcal{F}_{t-}]|dt < \infty.$$

Due to the relations (4.3) and (4.5) one can choose a subsequence $\{n_k\}_{k\geq 1}$, such that (P -a.s.) the following relations are fulfilled:

$$\lim_{k\to\infty} \int\limits_{(0,T]} |E[\Delta f_{n_k}(M_T)|\mathcal{F}_{t-}] - g(t)|d[M,M]_t = 0$$

and

$$\lim_{k\to\infty} \int\limits_{(0,T]} |E[\Delta f_{n_k}(M_T)|\mathcal{F}_{t-}] - E[\Delta f(M_T)|\mathcal{F}_{t-}]|d[M,M]_t = 0.$$

From this, one can conclude that for almost all ω:
$g(\cdot,\omega) = E[\Delta f(M_T)|\mathcal{F}_{\cdot-}](\omega)\ d[M,M]$-almost everywhere.
Thus, for almost all ω we have:

$$\int\limits_{(0,T]} |g(t) - E[\Delta f(M_T)|\mathcal{F}_{t-}]|d[M,M]_t = 0.$$

Therefore, we obtain that:

$$E\int\limits_{(0,T]} |g(t) - E[\Delta f(M_T)|\mathcal{F}_{t-}]|d[M,M]_t = 0,$$

whence, we conclude that:

$$g(t,\omega) = E[\Delta f(M_T)|\mathcal{F}_{t-}](\omega) \quad dP\otimes d[M,M]\text{-a.s.},$$

and, hence, dP-almost surely:

$$\int\limits_{(0,T]} g(t)dM_t = \int\limits_{(0,T]} E[\Delta f(M_T)|\mathcal{F}_{t-}]dM_t.$$

This complete the proof of the theorem. □

From the proof of the Theorem 4.1 one can ascertain that:

Corollary 4.1. *If $f\in W_{p,1,\alpha}$ $(1<p<2)$, then:*

$$\int\limits_{(0,T]} |E[\Delta f_n(M_T)|\mathcal{F}_{t-}]|^2 d[M,M]_t < \infty \quad (P-a.s.).$$

Following the scheme of the proof of the Theorem 2.3 one can prove:

Theorem 4.2. *If $f\in W_{p,1,\alpha}$, where $1<p<2$, then for any $t\in[0,T]$ the following relation is valid:*

$$^p[D_t^M f(M_T)] = E[\Delta f(M_T)|\mathcal{F}_{t-}] \quad (dP\otimes ds - a.s.).$$

Appendix

In this appendix we prove results which are given in section 1.

Proof of Proposition 1.1. Due to the Newton's binomial formula, we have:

$$\mu_n(t)=\sum_{x=0}^{\infty}x^n\frac{t^x}{x!}e^{-t}=\sum_{x=1}^{\infty}x^{n-1}\frac{t^x}{(x-1)!}e^{-t}=$$

$$=t\sum_{x=1}^{\infty}(x-1+1)^{n-1}\frac{t^{x-1}}{(x-1)!}e^{-t}=t\sum_{x=1}^{\infty}\sum_{i=0}^{n-1}C_{n-1}^{i}(x-1)^i\frac{t^{x-1}}{(x-1)!}e^{-t}=$$

$$=t\sum_{i=0}^{n-1}C_{n-1}^{i}\sum_{x=1}^{\infty}(x-1)^i\frac{t^{x-1}}{(x-1)!}e^{-t}=t\sum_{i=0}^{n-1}C_{n-1}^{i}\mu_i(t).$$ □

Proof of Corollary 1.1. Since, $E[N_t]=\mu_1(t)=t$ taking into account the Proposition 1.1, by virtue of the Mathematical Induction Method, one can easily ascertain that the assertion of the corollary is true. □

Proof of Proposition 1.2. Due to the Corollary 1.1: $\mu_n(t)=\sum_{k=0}^{n}a_kt^k$.

On the other hand, we have:

$$\mu_n(t)=\sum_{x=0}^{\infty}x^n\frac{t^x}{x!}e^{-t}=\Big(\sum_{x=0}^{\infty}x^n\frac{t^x}{x!}\Big)\Big/\Big(\sum_{i=0}^{\infty}\frac{t^i}{i!}\Big).$$

Therefore, we can write: $\sum_{x=0}^{\infty}x^n\frac{t^x}{x!}=\sum_{k=0}^{n}a_kt^k\cdot\sum_{i=0}^{\infty}\frac{t^i}{i!}$.

Using now the rule of multiplication of series, equating with each other the coefficients at the same degrees of t in the both sides of the equality, we ascertain that:

$$a_0=0;$$

$$a_0+a_1=1;$$

$$\frac{a_0}{2!}+a_1+a_2=\frac{2^n}{2!};$$

$$\dots\dots\dots\dots\dots\dots$$

$$\frac{a_k}{0!}+\frac{a_{k-1}}{1!}+\cdots+\frac{a_1}{(k-1)!}+\frac{a_0}{k!}=\frac{k^n}{k!},\quad k=1,2,\dots,n.$$

Whence we obtain that:

$$a_1=1;$$

$$a_2=\frac{2^n}{n!}-a_1;$$

$$\dots\dots\dots\dots\dots\dots$$

$$a_k=\frac{k^n}{k!}-\sum_{i=0}^{k-1}\frac{a_i}{(k-1)!},\quad k=1,2,\dots,n.$$

Let us denote by M_t-the Compensated Poisson process $M_t := N_t - t$ and by $\nu_n(t)$ –its n-th order moment ($\nu_n(t) := E[M_t^n], n \geq 1$). □

Proof of Proposition 1.3. It is clear that: $\nu_0(t) = E[M_t^0] = E[1] = 1$ and $\nu_1(t) = E[M_t] = E[N_t - t] = t - t = 0$.

On the other hand, it is not difficult to see that (for any $n \geq 2$), we have:

$$\nu_n(t) = E[M_t^n] = E[(N_t - t)^n] = \sum_{k=0}^{\infty}(k-t)^n \frac{t^k}{k!}e^{-t} =$$

$$= \sum_{k=0}^{\infty}(k-t)^{n-1}(k-t)\frac{t^k}{k!}e^{-t} = \sum_{k=1}^{\infty}(k-t)^{n-1}k\frac{t^k}{k!}e^{-t} - t\sum_{k=0}^{\infty}(k-t)^{n-1}\frac{t^k}{k!}e^{-t} =$$

$$= \sum_{k=1}^{\infty}(k-1-t+1)^{n-1}\frac{t^k}{(k-1)!}e^{-t} - t\nu_{n-1}(t) =$$

$$= t\sum_{k=1}^{\infty}\sum_{i=0}^{n-1}C_{n-1}^i(k-1-t)^i\frac{t^{k-1}}{(k-1)!}e^{-t} - t\nu_{n-1}(t) =$$

$$= t\sum_{i=0}^{n-1}C_{n-1}^i\sum_{m=0}^{\infty}(m-t)^i\frac{t^m}{m!}e^{-t} - t\nu_{n-1}(t) =$$

$$= t\sum_{i=0}^{n-1}C_{n-1}^i\nu_i(t) - t\nu_{n-1}(t) = t\sum_{i=0}^{n-2}C_{n-1}^i\nu_i(t).$$ □

Proof of Proposition 1.4. Because $\nu_1 = 0$ and $\nu_2 = \nu_3 = t$, therefore proposition is fulfilled in cases $n = 1, 2, 3$. Assume that the assertion is true for $n = m$ and verify that is true for $n = m+1$. We have: $\nu_m(t) = \sum_{k=0}^{m/2} a_k^m t^k$, therefore according to the Proposition 1.3, using the relation $[(m+1)/2] = [(m-1/2)] + 1$, we obtain that:

$$\nu_{m+1}(t) = t\sum_{i=0}^{m-1}C_m^i\sum_{k=0}^{i/2}a_k^i t^k = t\sum_{i=0}^{[(m-1)/2]} b_i t^i = \sum_{k=0}^{[(m+1)/2]} b_{k-1}t^k.$$

Hence, by virtue of the Mathematical Induction Method, the relation (1.3) is justified for any $n \geq 1$.

On the other hand, one can write:

$$\nu_n(t) = \sum_{k=0}^{\infty}(k-t)^n\frac{t^k}{k!}e^{-t} = \frac{\sum_{k=0}^{\infty}(k-t)^n\frac{t^k}{k!}}{\sum_{i=0}^{\infty}\frac{t^i}{i!}} = \sum_{k=0}^{[n/2]} a_k^n t^k,$$

i.e.

$$\sum_{k=0}^{\infty}(k-t)^n\frac{t^k}{k!} = \sum_{i=0}^{\infty}\frac{t^i}{i!}\cdot\sum_{k=0}^{[n/2]}a_k^n t^k.$$

Therefore, using the Newton's binomial formula and the rule of multiplication of series, equating with each other the coefficients at the same degrees of t in the both side of equality, it is not difficult to see that the coefficients a_k^n in (1.3) satisfy the relations (1.4). □

Proof of Proposition 1.5. We have:

$$\frac{d\nu_n(t)}{dt} = \frac{d}{dt}\Big[\sum_{k=0}^{\infty}(k-t)^n\frac{t^k}{k!}e^{-t}\Big] = -\sum_{k=0}^{\infty}n(k-t)^{n-1}\frac{t^k}{k!}e^{-t}+$$

$$+\sum_{k=1}^{\infty}(k-t)^n k\frac{t^{k-1}}{k!}e^{-t} - \sum_{k=0}^{\infty}(k-t)^n\frac{t^k}{k!}e^{-t} := I_1 + I_2 + I_3,$$

where

$$I_1 = -n\sum_{k=0}^{\infty}(k-t)^{n-1}\frac{t^k}{k!}e^{-t} = -n\nu_{n-1}(t),$$

$$I_2 = \sum_{k=1}^{\infty}(k-t)^n\frac{t^{k-1}}{(k-1)!}e^{-t} = \sum_{k=1}^{\infty}(k-1-t+1)^n\frac{t^{k-1}}{(k-1)!}e^{-t} =$$

$$=\sum_{k=1}^{\infty}\sum_{i=0}^{n}C_n^i(k-1-t)^i\frac{t^k}{(k-1)!}e^{-t}=\sum_{i=0}^{n}C_n^i\sum_{k=1}^{\infty}(k-1-t)^i\frac{t^{k-1}}{(k-1)!}e^{-t}=$$

$$=\sum_{i=0}^{n}C_n^i\nu_i(t) = \sum_{i=0}^{n-2}C_n^i\nu_i(t) + n\nu_{n-1}(t) + \nu_n(t)$$

and

$$I_3 = -\sum_{k=0}^{\infty}(k-t)^n\frac{t^k}{k!}e^{-t} = -\nu_n(t).$$

Combining the relations obtained above, we conclude that the proposition is valid. □

Acknowledgement

The work has been financed by the Georgian National Foundation grant No. 337/07, 06_223_3-104.

References

1. J. M. C. Clark, The representation of functionals of Brownian motion by stochastic integrals. *Ann. Math. Statist.* **41**(1970) 1282–1295.
2. A. V. Skorohod, On generalization of a stochastic integral. *Theor. Prob. Appl.* **20**(1975), 219-233.
3. Paul Malliavin, Stochastic calculus of variation and hypoelliptic operators. *Proceedings of the International Symposium on Stochastic Differential Equations* (*Res. Inst.*

Math. Sci., Kyoto Univ., Kyoto, 1976), pp. 195–263, *Wiley, New York-Chichester-Brisbane*, 1978.

4. U. G. Haussmann, On the integral representation of functionals of Ito processes. *Stochastics* **3**(1979), No. 1, 17–27.
5. Daniel Ocone, Malliavin's calculus and stochastic integral representations of functionals of diffusion processes. *Stochastics* **12**(1984), No. 3-4, 161–185.
6. J. Michael Harrison and Stanley R. Pliska, Martingales and stochastic integrals in the theory of continuous trading. *Stochastic Process. Appl.* **11**(1981), No. 3, 215–260.
7. Bernard Gaveau and Philip Trauber, L'intégrale stochastique comme opérateur de divergence dans l'espace fonctionnel. (French) *J. Funct. Anal.* **46**(1982), No. 2, 230–238.
8. David Nualart and Moshe Zakai, Generalized stochastic integrals and the Malliavin calculus. *Probab. Theory Relat. Fields* **73** (1986), No. 2, 255–280.
9. D. Nualart and É Pardoux, Stochastic calculus with anticipating integrands. *Probab. Theory Related Fields* **78** (1988), No. 4, 535–581.
10. Jin Ma, Philip Protter and Jaime San Martin, Anticipating integrals for a class of martingales. *Bernoulli* **4**(1998), No. 1, 81–114.
11. A. N. Shiryaev and Ĭor, M. On stochastic integral representations of functionals of Brownian motion. I. (Russian) Teor. Veroyatnost. i Primenen. 48 (2003), No. 2, 375–385; English transl.: Theory Probab. Appl. 48 (2004), No. 2, 304–313.
12. O. Purtukhia, An extension of the Ocone-Haussmann-Clark formula for a class of normal martingales. *Proc. A. Razmadze Math. Inst.* **132**(2003), 127–136.
13. V. Jaoshvili and O. Purtukhia, Stochastic Integral Representation of Functionals of Wiener Processes. *Bull. Georg. Acad. Sci.* **171**(2005), No. 1, 17–20.
14. V. Jaoshvili and O. Purtukhia, Stochastic Integral Representation of Functionals of Poisson Pro-cesses I. *Bull. Georg. Acad. Sci.* **172**(2005), No. 2, 189–192.
15. A. N. Kolmogorov and C. V. Fomin, The elements of theory of functions and functional analysis. (Russian) *Nauka, Moscow*, 1989.
16. R. Sh. Liptser and A. N. Shiryaev, Martingale theory. (Russian) *Nauka, Moscow*, 1986.

(Received 20.06.2006)

Authors' addresses:

V. Jaoshvili and O. Purtukhia

I. Javakhishvili Thilisi State University
2, University St., Tbilisi 380143
Georgia

O. Purtukhia

A. Razmadze Mathematical Institute,
Georgian Academy of Sciences,
1, M. Aleksidze St, 0193 Tbilisi
Georgia