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Mathematics

Martingale Representation Theorems for Multidimensional Wiener Functionals

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ABSTRACT. The paper suggests a method which allows to construct explicit expressions for integrands taking part in the stochastic integral representation of functionals of Wiener processes in multidimensional case. *© 2008 Bull. Georg. Natl. Acad. Sci.*

Key words: *Wiener process, Malliavin derivative, Sobolev's average operator.*

0. In the 80s of the past century, it turned out (Harrison and Pliska (1981)) that the martingale representation theorems (along with Girsanov's measure change theorem) play an important role in modern financial mathematics. The well-known Clark formula [1] says nothing about finding explicitly the integrands which take part in the integral representation. In the case when the Wiener functional has a stochastic derivative, the Ocone-Clark's (Ocone (1984)) formula finds explicitly this integrand: if $F \in D_{2,1}$, then $F = EF + \int_0^T E[D_t F \mid \Im_t^w] dw_t$, where $D_t F$ is a Malliavin derivative. Our approach [2], within the classical Ito's calculus (unlike the Ocone's one) allows one to construct explicitly the integrand even so if the Wiener functional has no stochastic derivative. In this paper we consider the multidimensional case.

I. Let $(\Omega; \Im; \Im_t,\ t \geq 0; P)$ be a standard probability space with a given standard Wiener processes $(w_t, \Im_t)$, where $\Im_t = \Im_t^w$.

Proposition 1. For all $m, n, r \in N$ and $s, u, v \geq 0$ the following representation ($P-a.s.$) is valid:

$$w_s^m w_u^n w_v^r = E[w_s^m w_u^n w_v^r] +$$

$$+ \int_0^{s \vee u \vee v} E[\partial/\partial w_s (w_s^m w_u^n w_v^r) I_{\{t \leq s\}} + \partial/\partial w_u (w_s^m w_u^n w_v^r) I_{\{t \leq u\}} + \partial/\partial w_v (w_s^m w_u^n w_v^r) I_{\{t \leq v\}} \big| \Im_t^w] dw_t. \qquad (1)$$

Sketch of the proof. Denote $X_t := E[w_s^m w_u^n w_v^r \big| \Im_t^w]$. Let $t \leq s \leq u \leq v$. Due to the properties of Wiener processes we can easily calculate:

$$X_t := E[w_s^m w_u^n w_v^r \big| \Im_t^w] = \sum_{k=0}^{r}\sum_{j=0}^{n+k}\sum_{i=0}^{m+j}\{C_r^k C_{n+k}^j C_{m+j}^i \cdot$$

$$\cdot (r-k-1)!!(n+k-j-1)!!(m+j-i-1)!! \cdot (v-u)^{\frac{r-k}{2}}(u-s)^{\frac{n+k-j}{2}}(s-t)^{\frac{m+j-i}{2}} \cdot w_t^i\},$$

where $(2n-1)!! := 1\cdot 3 \cdot \dots \cdot (2n-1)$ and $(2n)!! := 0$.

According to Ito's formula, taking into account the relation $E[w_t^n \mid \Im_s^w] = \sum_{i=0}^{n}(t-s)^{(n-i)/2}\cdot(n-i-1)!! \cdot w_s^i \quad P-a.s.$ ($t>s$, it is not difficult to see that:

$$dX_t = \sum_{k=0}^{r}\sum_{j=0}^{n+k}\sum_{i=0}^{m+j}\{C_r^k C_{n+k}^j C_{m+j}^i \cdot (r-k-1)!!(n+k-j-1)!!(m+j-i-1)!! \cdot$$

$$\cdot (v-u)^{\frac{r-k}{2}}(u-s)^{\frac{n+k-j}{2}} \cdot [i(s-t)^{\frac{m+j-i}{2}} w_t^{i-1} dw_t +$$

$$+\frac{i(i-1)}{2}(s-t)^{\frac{m+j-i}{2}} w_t^{i-2}dt - \frac{m+j-i}{2}(s-t)^{\frac{m+j-i-2}{2}} w_t^i dt]\}.$$

Studying carefully the obtained relation, one can easily notice that the coefficients before any member with the same degree of w_t have the opposite sign and their absolute values coincide with each other. Hence, we have

$$dX_t = \sum_{k=0}^{r}\sum_{j=0}^{n+k}\sum_{i=0}^{m+j}\{C_r^k C_{n+k}^j C_{m+j}^i \cdot (r-k-1)!!(n+k-j-1)!!(m+j-i-1)!! \cdot$$

$$\cdot (v-u)^{\frac{r-k}{2}}(u-s)^{\frac{n+k-j}{2}} \cdot i(s-t)^{\frac{m+j-i}{2}} w_t^{i-1}\}dw_t \cdot$$

On the other hand, it is not difficult to see that

$$E[(E[w_s^m w_u^n w_v^r \big| \Im_s^w])'_{w_s} \big| \Im_t^w] = \sum_{k=0}^{r}\sum_{j=0}^{n+k}\sum_{i=0}^{m+j}\{C_r^k C_{n+k}^j C_{m+j}^i \cdot$$

$$\cdot (r-k-1)!!(n+k-j-1)!!(m+j-i-1)!! \cdot (v-u)^{\frac{r-k}{2}}(u-s)^{\frac{n+k-j}{2}} \cdot i(s-t)^{\frac{m+j-i}{2}} w_t^{i-1}\}.$$

From the last two relations we conclude that

$$dX_t = E[(E[w_s^m w_u^n w_v^r \big| \Im_s^w])'_{w_s} \big| \Im_t^w] dw_t.$$

Analogously, one can consider the cases: $s \le t \le u \le v$ and $s \le u \le t \le v$ and obtain that: $dX_t = E[(E[w_s^m w_u^n w_v^r \big| \Im_u^w])'_{w_u} \big| \Im_t^w] dw_t$ for $s \le t \le u \le v$, and $dX_t = E[(E[w_s^m w_u^n w_v^r \big| \Im_v^w])'_{w_v} \big| \Im_t^w] dw_t$ for $s \le u \le t \le v$.

Combining the above obtained relations, we conclude that the representation (1) is true. □

Proposition 2. For any polynomial function $P(x,y,z)$ $P-a.s.$ we have the representation:

$$P(w_s, w_u, w_v) = E[P(w_s, w_u, w_v)] +$$

$$+\int_0^{s\vee u\vee v} E[\partial/\partial x\, P(w_s,w_u,w_v)I_{\{t\le s\}} + \partial/\partial y\, P(w_s,w_u,w_v)I_{\{t\le u\}} + \partial/\partial z\, P(w_s,w_u,w_v)I_{\{t\le v\}} \big| \Im_t^w]dw_t . \quad (2)$$

II. Let $\rho(x,y,z,s,u,v) := \exp\left\{-x^2/2s - y^2/2u - z^2/2v\right\}$ and $\overline{\rho}(x,y,z,s,u,v) := \rho(x-y, y-z, z, s-u, u-v, v)$. Denote by $L_{2,s,u,v} := L_2[R^3, \rho(x,y,z,s,u,v)]$ (accordingly, $L_2^{s,u,v} := L_2[R^2, \overline{\rho}(x,y,z,s,u,v)]$) the space of measurable functions with finite norm: $\|f\|_{2,s,u,v} := \|f\cdot\rho\|_{L_2}$ (accordingly $\|f\|_2^{s,u,v} := \|f\cdot\overline{\rho}\|_{L_2}$) for fixed $s > u > v$.

Proposition 3. $L_2^{s,u,v}$ is a Banach space with the basis: $\{x^k y^p z^q \overline{\rho}(x,y,z,t_1,t_2,t_3)\}$.

Theorem 1. *If the function* $f(\cdot,\cdot,\cdot) \in C^1(R^3) \cap L_{2,s,u,v}$ *such that* $\partial/\partial x, \partial/\partial y, \partial/\partial z\, f(\cdot,\cdot,\cdot) \in L_{2,s/\alpha,u/\alpha,v/\alpha}$ (*for some number* $0 < \alpha < 1$), *then* (P-a.s.) *we have the representation*:

$$f(w_s,w_u,w_v) = E[f(w_s,w_u,w_v)] +$$

$$+\int_0^{s\vee u\vee v} E[\partial/\partial x\, f(w_s,w_u,w_v)I_{\{t\le s\}} + \partial/\partial x\, f(w_s,w_u,w_v)I_{\{t\le u\}} + \partial/\partial z\, f(w_s,w_u,w_v)I_{\{t\le v\}} \big| \Im_t^w]dw_t . \quad (3)$$

Sketch of the proof. At first we consider the case when $f(\cdot,\cdot,\cdot) \in C^3(R^3) \cap L_{2,s,u,v}$ and $\partial^3/\partial x\partial y\partial z\, f(\cdot,\cdot,\cdot) \in L_{2,s/\alpha,u/\alpha,v/\alpha}$, then there exists a sequence of polynomial functions $Q_n(x,y,z)$ such that as $n\to\infty$ in $L_{2,s/\alpha,u/\alpha,v/\alpha}$: $Q_n(x,y,z) \to \partial^3/\partial x\partial y\partial z\, f(x,y,z)$.

Let us construct the sequences of functions $P_n(x,y,z)$, $R_1^n(x,y,z)$, $R_2^n(x,y,z)$, $R_3^n(x,y,z)$ in the following way:

$$P_n(x,y,z) := f(0,0,0) - f(x,0,0) - f(0,y,0) - f(0,0,z) + ,$$

$$+ f(x,y,0) + f(x,0,z) + f(0,y,z) - \int_0^x\int_0^y\int_0^z Q_n(\theta,\sigma,\tau)d\theta d\sigma d\tau ,$$

$$R_1^n(x,y,z) := \partial/\partial x\, P_n(x,y,z),\ R_{21}^n(x,y,z) := \partial/\partial y\, P_n(x,y,z),\ R_3^n(x,y,z) := \partial/\partial z\, P_n(x,y,z).$$

The members of these sequences are respectively the sum of functions of one variable, of two variables and polynomial functions, therefore according to Proposition 2 and Theorem 4 [4] we obtain the representation (2).

On the other hand, as $n\to\infty$ in $L_2^{s,u,v}$ we have $P_n(x,y,z) \to f(x,y,z)$, $R_1^n(x,y,z) \to \partial/\partial x\, f(x,y,z)$, $R_2^n(x,y,z) \to \partial/\partial x\, f(x,y,z)$, $R_3^n(x,y,z) \to \partial/\partial x\, f(x,y,z)$.

Further, it is not difficult to show that as $n\to\infty$ in $L_2(\Omega)$:

$$P_n(w_s,w_u,w_v) \to f(w_s,w_u,w_v),\ R_1^n(w_s,w_u,w_v) \to \partial/\partial x\, f(w_s,w_u,w_v),$$

$$R_2^n(w_s,w_u,w_v) \to \partial/\partial y\, f(w_s,w_u,w_v) \text{ and } R_3^n(w_s,w_u,w_v) \to \partial/\partial z\, f(w_s,w_u,w_v).$$

Passing now to the limit as $n \to \infty$ in (2), written for $P_n(w_s, w_u, w_v)$, we obtain the relation (3) when $f(\cdot,\cdot,\cdot) \in C^3(R^3)$. Further, in the conditions of the theorem, one can prove the representation (3) for the function $T_\varepsilon f$ (where T_ε is the well-known Sobolev's average operator) and then passing to the limit as $\varepsilon \to 0$ we complete the proof of the theorem. □

Theorem 2. *If the function* $f(x,y,z) \in L_{2,s,u,v}$ *has generalized derivatives of first order such that for some* $0 < \alpha < 1$: $\delta/\delta x(f), \delta/\delta y(f), \delta/\delta z(f) \in L_{2,s/\alpha,u/\alpha,v/\alpha}$, *then* ($P$-a. s.) *the following representation is valid:*

$$f(w_s, w_u, w_v) = E[f(w_s, w_u, w_v)] +$$

$$+ \int_0^{s \vee u \vee v} E[\delta/_{\delta x} f(w_s, w_u, w_v) I_{\{t \le s\}} + \delta/_{\delta y} f(w_s, w_u, w_v) I_{\{t \le u\}} + \delta/_{\delta z} f(w_s, w_u, w_v) I_{\{t \le v\}} \big| \Im_t^w] dw_t . \qquad (4)$$

Sketch of the proof. In the conditions of the theorem there exists $f_\varepsilon := T_\varepsilon f \in C^\infty(R^3)$, such that $f_\varepsilon \to f$ in $L_{2,s,u,v}$ as $\varepsilon \to 0$. For f_ε we have the representation (3). Further, it is not difficult to show that:

$$\int_0^s E[\partial/_{\partial x} f_\varepsilon(w_s, w_u, w_v) \big| \Im_t^w] dw_t \xrightarrow{L_2(\Omega)} \int_0^s E[\delta/_{\delta x} f(w_s, w_u, w_v) \big| \Im_t^w] dw_t, \quad \text{as } \varepsilon \to 0,$$

$$\int_0^u E[\partial/_{\partial y} f_\varepsilon(w_s, w_u, w_v) \big| \Im_t^w] dw_t \xrightarrow{L_2(\Omega)} \int_0^u E[\delta/_{\delta y} f(w_s, w_u, w_v) \big| \Im_t^w] dw_t, \quad \text{as } \varepsilon \to 0$$

Vakhtang Jaoshvili, Omar Purtukhia

and

$$\int_0^v E[\partial/_{\partial z} f_\varepsilon(w_s, w_u, w_v) \big| \Im_t^w] dw_t \xrightarrow{L_2(\Omega)} \int_0^v E[\delta/_{\delta z} f(w_s, w_u, w_v) \big| \Im_t^w] dw_t, \quad \text{as } \varepsilon \to 0.$$

Below we justify only the first relation. Indeed, we have:

$$E\left(\int_0^s E[\delta/_{\delta x} \{f_\varepsilon(w_s, w_u, w_v) - f(w_s, w_u, w_v)\} \big| \Im_t^w] dw_t \right)^2 \le$$

$$\le E \int_0^s \left(E[\delta/_{\delta x} \{f_\varepsilon(w_s, w_u, w_v) - f(w_s, w_u, w_v)\} \big| \Im_t^w] \right)^2 dt \le$$

$$\le s \cdot \int_0^s E E[\delta/_{\delta x} \{f_\varepsilon(w_s, w_u, w_v) - f(w_s, w_u, w_v)\} \big| \Im_t^w]^2 dt \le$$

$$\le s \cdot \int_0^s E[\delta/_{\delta x} \{f_\varepsilon(w_s, w_u, w_v) - f(w_s, w_u, w_v)]^2 dt =$$

$$= s\cdot\int_0^s \frac{1}{\sqrt{2\pi s(u-s)(v-u)}}\iiint_{R^3}\frac{\delta}{\delta x}\{f_\varepsilon(x,y,z)-f(x,y,z)\}^2 e^{-\frac{x^2}{2s}}e^{-\frac{(y-x)^2}{2(u-s)}}e^{-\frac{(z-y)^2}{2(v-u)}}dxdydzdt =$$

$$= s\cdot\int_0^s \frac{1}{\sqrt{2\pi s(u-s)(v-u)}}dt\left\|\frac{\delta}{\delta x}\{f_\varepsilon(x,y,z)-f(x,y,z)\}\right\|_2^{s/\alpha, u/\alpha, v/\alpha} \xrightarrow{\varepsilon\to\infty} 0$$

Passing now to the limit in (3) we obtain (4). □

III. Let $\rho(x_1,x_2,...,x_n,t_1,t_2,...,t_n):=\exp\left\{-x_1^2/2t_1 - x_2^2/2t_2 - \cdots - x_n^2/2t_n\right\}$ and $\overline{\rho}(x_1,x_2,...,x_n,t_1,t_2,...,t_n):=$ $\rho(x_1-x_2,x_2-x_3,...,x_{n-1}-x_n,x_n,t_1-t_2,t_2-t_3,...,t_{n-1}-t_n,t_n)$. Denote by $L_{2,t_1,t_2,...,t_n}:=L_2[R^3,\rho(x_1,x_2,...,x_n,t_1,t_2,...,t_n)]$ (accordingly, $L_2^{t_1,t_2,...,t_n}:=L_2[R^2,\overline{\rho}(x_1,x_2,...,x_n,t_1,t_2,...,t_n)]$) the space of measurable functions with finite norm: $\|f\|_{2,t_1,t_2,...,t_n}:=\|f\cdot\rho\|_{L_2}$ (accordingly $\|f\|_2^{t_1,t_2,...,t_n}:=\|f\cdot\overline{\rho}\|_{L_2}$) for fixed $t_1>t_2>\cdots>t_n$.

Proposition 4. $L_2^{t_1,t_2,...,t_n}$ is a Banach space with the basis: $\{x_1^{k_1}\cdot x_2^{k_2}\cdot\cdots\cdot x_n^{k_n}\cdot \cdot\overline{\rho}(x_1,x_2,...,x_n,t_1,t_2,...,t_n)\}_{k_1,k_2,...,k_n\geq 0}$.

Theorem 3. *If the function* $f(x_1,x_2,...,x_n)\in C^1(R^n)\cap L_{2,t_1,t_2,...,t_n}$ *such that* $\partial/\partial x_1$, $\partial/\partial x_2$, …, $\partial/\partial x_n\, f(x_1,x_2,...,x_n)\in L_{2,t_1/\alpha,t_2/\alpha,...,t_n/\alpha}$ *(for some* $0<\alpha<1$*), then* $P-a.s.$:

$$f(w_{t_1},w_{t_2},...,w_{t_n}) = E[f(w_{t_1},w_{t_2},...,w_{t_n})] + \int_0^{t_1\vee t_2\vee\cdots\vee t_n} E[\partial/\partial x_1\, f(w_{t_1},w_{t_2},...,w_{t_n})I_{\{t\leq t_1\}} +$$

$$+\partial/\partial x_2\, f(w_{t_1},w_{t_2},...,w_{t_n})I_{\{t\leq t_2\}}+\cdots+\partial/\partial x_n\, f(w_{t_1},w_{t_2},...,w_{t_n})I_{\{t\leq t_n\}}\big|\Im_t^w]dw_t. \quad (5)$$

Theorem 4. *If the function* $f(x_1,x_2,...,x_n)\in L_{2,t_1,t_2,...,t_n}$ *has generalized derivatives of first order such that:* $\delta/\delta x_1(f),...,\delta/\delta x_n(f)\in L_{2,t_1/\alpha,t_2/\alpha,...,t_n/\alpha}$ $(0<\alpha<1)$, *then* $P-a.s.$:

$$f(w_{t_1},w_{t_2},...,w_{t_n}) = E[f(w_{t_1},w_{t_2},...,w_{t_n})] + \int_0^{t_1\vee t_2\vee\cdots\vee t_n} E[\delta/\delta x_1\, f(w_{t_1},w_{t_2},...,w_{t_n})I_{\{t\leq t_1\}} +$$

$$+\delta/\delta x_2\, f(w_{t_1},w_{t_2},...,w_{t_n})I_{\{t\leq t_2\}}+\cdots+\delta/\delta x_n\, f(w_{t_1},w_{t_2},...,w_{t_n})I_{\{t\leq t_n\}}\big|\Im_t^w]dw_t. \quad (6)$$

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მათემატიკა

მარტინგალური წარმოდგენის თეორემები მრავალგანზომილებიანი ვინერის ფუნქციონალებისათვის

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ნაშრომში გადმოცემულია ერთი მეთოდი, რომელიც საშუალებას იძლევა ცხადი სახით ავაგოთ ვინერის ფუნქციონალების ინტეგრალურ წარმოდგენაში მონაწილე სტოქასტური ინტეგრალის ინტეგრანდი მრავალგანზომილებიანი ვინერის ფუნქციონალებისათვის.

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