Mathematics

Stochastic Integral Representation of Multidimensional Polynomial Poisson Functionals

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ABSTRACT. The paper suggests a method which allows one to construct explicit expressions for integrands which take part in the stochastic integral representation for multidimensional polynomial functionals of Poisson processes and for these functionals the formulas for calculation of the predictable projection of their stochastic derivatives are given. *© 2008 Bull. Georg. Natl. Acad. Sci.*

Key words: *Poisson process, stochastic derivative, predictable projection.*

According to the well-known result obtained by Ma, Protter, Martin [1], if $F \in D_{2,1}^M$, then the Ocone-Haussmann-Clark's representation $F = E(F) + \int_{(0,T]} {}^p(D_t^M F) dM_t$ is valid. Here M is the so-called normal martingales (a martingale is said to be normal, if $\langle M, M \rangle_t = t$), $D_{2,1}^M$ denotes the space of square integrable functionals having the first order stochastic derivative, and ${}^p(D_t^M F)$ is a predictable projection of the stochastic derivative $D_t^M F$ of the functional F. But, in this case (exactly, when the quadratic variation $[M, M]$ is not deterministic), unlike Wiener's one, it is impossible to define in a generally adopted manner an operator of stochastic differentiation to obtain the structure of Sobolev spaces, which allows one to construct explicitly the stochastic derivative operator in many cases. Consequently, the Ocone-Haussmann-Clark's formula makes it impossible to construct explicitly the operator of the stochastic derivative of the functionals of the Compensated Poisson process (which, obviously, belongs to a class of normal martingales $\langle M, M \rangle_t = t$, but its quadratic variation is not deterministic, $[M, M]_t = N(t) = M(t) + t$), saying nothing of the construction of its predictable projection.

Our approach (in the one-dimensional case) within the framework of nonanticipative stochastic calculus of semimartingales allows one to construct explicitly the expression for the integrand of the stochastic integral in the theorem of martingale representation for square integrable functionals of the Compensated Poisson process, and to derive a formula allowing one to construct explicitly predictable projections of their stochastic derivatives [2, 3]. In this paper we consider the multidimensional case.

Let $(\Omega; \Im; \Im_t, t \in [0, T]; P)$ be a filtered probability space satisfying the usual conditions. Assume that the standard Poisson process $(N_t, \Im_t)$ is given on it and that $\Im_t$ is generated by N ($\Im_t = \Im_t^N$), $\Im = \Im_T$. Denote $M_t := N_t - t$, $\nu_n(t) := E[(M_t)^n]$.

Let us denote

$$\nabla^n f(x_1,x_2,...,x_n):=f(x_1+1,x_2+1,...,x_n+1)-f(x_1,x_2,...,x_n),$$

$$\nabla_{x_i} f(x_1,...,x_i,...,x_n):=f(x_1,...,x_i+1,...,x_n)-f(x_1,...,x_i,...,x_n),$$

$$\nabla_{x_{i_1}}\nabla_{x_{i_2}}\cdots\nabla_{x_{i_k}} f(x_1,...,x_n):=\nabla_{x_{i_1}}[\nabla_{x_{i_2}}[\cdots[\nabla_{x_{i_k}} f(x_1,...,x_n)]\cdots]].$$

It is not difficult to see that for any permutation $\sigma(1),\sigma(2),...,\sigma(k)$ of $i_1,i_2,...,i_k$ we have:

$$\nabla_{x_{\sigma(1)}}\nabla_{x_{\sigma(2)}}\cdots\nabla_{x_{\sigma(k)}} f(x_1,...,x_n)=\nabla_{x_{i_1}}\nabla_{x_{i_2}}\cdots\nabla_{x_{i_k}} f(x_1,...,x_n)$$

And, moreover,

$$\nabla^n f(x_1,x_2,...,x_n)=\sum_{k=1}^{n}\sum_{\substack{i_1,i_2,...,i_k=1\\ i_1<i_2<\cdots<i_k}}^{n}\nabla_{x_{i_1}}\nabla_{x_{i_2}}\cdots\nabla_{x_{i_k}} f(x_1,x_2,...,x_n).$$

Denote

$$\nabla_t^n f(M_{t_1},M_{t_2},...,M_{t_n})=\sum_{k=1}^{n}\{\sum_{\substack{i_1,i_2,...,i_k=1\\ i_1<i_2<\cdots<i_k}}^{n}[\nabla_{x_{i_1}}\nabla_{x_{i_2}}\cdots\nabla_{x_{i_k}} f(x_1,x_2,...,x_n)|_{x_1=M_{t_1},x_2=M_{t_2},...,x_n=M_{t_n}}]\times$$

$$\times I_{[0,t_{i_1}]}(t)\cdot I_{[0,t_{i_2}]}(t)\cdots I_{[0,t_{i_k}]}(t)\}.$$

Theorem 1. *For any polynomial function $P_n(x_1,x_2,...,x_n)$ the functional $P_n(M_{t_1},M_{t_2},...,M_{t_n})$ admits (P-a.s.) the following stochastic integral representation:*

$$P_n(M_{t_1},M_{t_2},...,M_{t_n})=EP_n(M_{t_1},M_{t_2},...,M_{t_n})+\int_{(0,t_1\vee t_2\vee\cdots\vee t_n]}E[\nabla_t^n P_n(M_{t_1},M_{t_2},...,M_{t_n})\,|\,\Im_{t-}]dM_t\,.$$

Sketch of the proof. For the sake of simplicity, in order to avoid cumbersome expressions and calculations, we consider the two-dimensional case ($n=2$, $x_1=x$, $x_2=y$, $t_1=S$, $t_2=T$). Thus, we must verify the following representation:

$$P_2(M_S,M_T)=E[P_2(M_S,M_T)]+\int_{(0,S\vee T]}E[\nabla_t^2 P_2(M_S,M_T)\big|\Im_{t-}]dM_t\ (P\text{-a.s.}), \tag{1}$$

Where $\nabla_t^2 P_2(M_S,M_T)=\nabla_x[\nabla_y P_2(M_S,M_T)]I_{[0,S]}(t)I_{[0,T]}(t)+\nabla_x P_2(M_S,M_T)I_{[0,S]}(t)+\nabla_y P_2(M_S,M_T)I_{[0,T]}(t)$

Fix $u\le S\le T$ and consider the power functional $M_S^n\cdot M_T^m$ ($m,n\in N$). According to the well-known properties of the Compensated Poisson process, we can easily obtain that: $E[M_T^m\,|\,\Im_S^M]=\sum_{i=0}^{m}C_m^i\nu_{m-i}(T-S)M_S^i$ and

$$X_u:=E\Big[M_S^n\cdot W_T^m\Big|\Im_u^M\Big]=E\Big[W_S^n E\{W_T^m\Big|\Im_S^M\}\Big|\Im_u^M\Big]=\sum_{i=0}^{m}\sum_{j=0}^{n+i}C_m^iC_{n+i}^j\nu_{m-i}(T-S)\nu_{n+i-j}(S-u)M_u^j\,.$$

Therefore, due to the Ito's formula, using the Propositions 1.5 and 2.1 [3], it is not difficult to see that:

$$X_u=\sum_{i=0}^{m}\sum_{j=0}^{n+i}C_m^iC_{n+i}^j\nu_{m-i}(T-S)\times$$

$$\times\{\int_{(0.u]}j\nu_{n+i-j}(S-t)M_{t-}^{j-1}dM_t+\int_{(0,u]}\nu_{n+i-j}(S-t)\sum_{k=2}^{j}C_j^kM_{t-}^{j-k}dN_t-\int_{(0,u]}M_{t-}^j\sum_{k=0}^{n+i-j-2}C_{n+i-j-2}^k\nu_k(S-t)dt\}.$$

Comparing now the terms with the same powers of M_{t-} in the two last integrals, due to the equality $dN_t-dt=dM_t$, we conclude that:

$$X_u=\sum_{i=0}^{m}\sum_{j=0}^{n+i}C_m^iC_{n+i}^j\nu_{m-i}(T-S)[\int_{(0.u]}j\nu_{n+i-j}(S-t)M_{t-}^{j-1}dM_t+\int_{(0,u]}\nu_{n+i-j}(S-t)\sum_{k=2}^{j}C_j^kM_{t-}^{j-k}dM_t].$$

Analogously, if $S < u \le T$, one can obtain:

$$X_u = E\left[M_S^n \cdot M_T^m \middle| \Im_u^M\right] = M_S^n E\left[M_T^m \middle| \Im_u^M\right] = M_S^n \sum_{i=0}^{m} C_m^i \nu_{m-i}(T-u) M_u^i =$$

$$= M_S^n \sum_{i=0}^{m} C_m^i [\int_{(0,u]} i\nu_{m-i}(T-t) M_{t-}^{i-1} dM_t + \int_{(0,u]} \nu_{m-i}(T-t) \sum_{j=2}^{i} C_i^j M_{t-}^{i-j} dN_t + \int_{(0,u]} -M_{t-}^i \sum_{k=0}^{m-i-2} C_{m-i}^k \nu_k (T-t) dt] =$$

$$= \sum_{i=0}^{m} C_m^i M_S^n \int_{(0,u]} [\nu_{m-i}(T-t) \sum_{j=1}^{i} C_i^j M_{t-}^{i-j}] dM_t \ \ (P\text{-a.s.}).$$

Summing up the above results, due to the relation $X_T = M_S^n \cdot M_T^m$, we ascertain that the following representation is valid:

$$M_S^n M_T^m = E[M_S^n M_T^m] + \sum_{i=0}^{m} \sum_{j=0}^{n+i} C_m^i C_{n+i}^j \nu_{m-i}(T-S) \int_{(0,S]} \nu_{n+i-j}(S-t)[jM_{t-}^{j-1} + \sum_{k=2}^{j} C_j^k M_{t-}^{j-k}] dM_t +$$

$$+ \sum_{i=0}^{m} C_m^i M_S^n \int_{(S,T]} [\nu_{n-i}(T-t) \sum_{j=1}^{i} C_i^j M_{t-}^{i-j}] dM_t \ \ (P\text{-a.s.}).$$

Furthermore, it is not difficult to see that:

$$\nabla_t^2 (M_S^m \cdot M_T^n) = \{[(M_S+1)^n (M_T+1)^m - M_S^n (M_T+1)^m] - [(M_S+1)^n M_T^m - M_S^n \cdot M_T^m]\} I_{(0,S]}(t) +$$

$$+ [(M_S+1)^n M_T^m - M_S^n \cdot M_T^m] I_{(0,S]}(t) + [M_S^n (M_T+1)^m - M_S^n \cdot M_T^m] I_{(0,T]}(t) =$$

$$= \{[(M_S+1)^n - M_S^n] \cdot [(M_T+1)^m - M_T^m]\} I_{(0,S]}(t) +$$

$$+ [(M_S+1)^n - M_S^n] \cdot M_T^m I_{(0,S]}(t) + [(M_T+1)^m - M_T^m] \cdot M_S^n I_{(0,T]}(t) =$$

$$= \sum_{i=0}^{n-1} C_n^i M_S^i \sum_{j=0}^{m} C_m^j M_T^j I_{(0,S]}(t) + \sum_{j=0}^{m-1} C_m^j M_T^j M_S^n I_{(0,T]}(t) =$$

$$= [\sum_{i=0}^{n-1} C_n^i M_S^i \sum_{j=0}^{m} C_m^j M_T^j + \sum_{j=0}^{m-1} C_m^j M_T^j M_S^n] I_{(0,S]}(t) + \sum_{j=0}^{m-1} C_m^j M_T^j M_S^n I_{(S,T]}(t) =$$

$$= [\sum_{i=0}^{n-1} \sum_{j=0}^{m} C_n^i C_m^j M_T^j M_S^i + \sum_{j=0}^{m-1} C_m^j M_T^j M_S^n] I_{(0,S]}(t) + \sum_{j=0}^{m-1} C_m^j M_T^j M_S^n I_{(S,T]}(t).$$

On the other hand, using arguments similar to those presented above, we have:

$$\int_{(0,T]} E[\nabla_t^2 (M_S^n \cdot M_T^m) | \Im_{t-}^M] dM_t = \sum_{i=0}^{m} \sum_{j=0}^{n+i} C_m^i C_{n+i}^j \nu_{m-i}(T-S) \{ \int_{(0,S]} \nu_{n+i-j}(S-t)[jM_{t-}^{j-1} +$$

$$+ \sum_{k=2}^{j} C_j^k M_{t-}^{j-k}] dM_t + \sum_{i=0}^{m} C_m^i M_S^n \int_{(S,T]} [\nu_{n-i}(T-t) \sum_{j=1}^{i} C_i^j M_{t-}^{i-j}] dM_t \ \ (P\text{-a.s.}).$$

Thus, we see that the representation is true for the power functional $M_S^n \cdot M_T^m$. Hence, taking into account the linearity of the operator ∇_t^2, of the mathematical expectation, and of the stochastic integral, we can see that the representation (1) is valid. □

Theorem 2. *For the functional* $P_n(M_{t_1}, M_{t_2}, ..., M_{t_n})$ *the following relation is valid:*

$$^p[D_t^M P_n(M_{t_1}, M_{t_2}, ..., M_{t_n})] = E[\nabla_t^2 P_n(M_{t_1}, M_{t_2}, ..., M_{t_n}) | \Im_{t-}^M] \ (dP \otimes d\lambda\text{-a.s.}), \quad (2)$$

Where $^p[D_t^M P_n(M_{t_1}, M_{t_2}, ..., M_{t_n})]$ denotes the predictable projection of the stochastic derivative with respect to the Compensated Poisson process (as a normal martingale) of functional $P_n(M_{t_1}, M_{t_2}, ..., M_{t_n})$.

Proof. According to the Ocone-Haussmann-Clark representation [1], we have:

$$P_n(M_{t_1}, M_{t_2}, ..., M_{t_n}) = EP_n(M_{t_1}, M_{t_2}, ..., M_{t_n}) + \int_{(0, t_1 \vee t_2 \vee \cdots \vee t_n]} \{^p[D_t^M P_n(M_{t_1}, M_{t_2}, ..., M_{t_n})]\} dM_t \ \ (P\text{-a.s.}).$$

Let us denote by η_t the difference between the left and right sides of relation (2) and denote by ξ_t the following stochastic process:

$$\xi_t := \int_{[0,t]} \{^p[D_u^M P_n(M_{t_1}, M_{t_2}, ..., M_{t_n})] - E[\nabla_u^2 P_n(M_{t_1}, M_{t_2}, ..., M_{t_n}) \mid \Im_{u-}^M]\} dM_u = \int_{[0,t]} \eta_u dM_u .$$

Then, due to Theorem 1, it is clear that $\xi_T = 0$ (P-a.s.). On the other hand, according to the Ito's formula, we obtain that:

$$\xi_T^2 = 2 \int_{[0,T]} \xi_{t-} \eta_t dM_t + \int_{[0,T]} \eta_t^2 d[M, M]_t \ \ (P\text{-a.s.}).$$

Taking now the mathematical expectation from both sides of the last relation, using the well-known properties of the square and predictable characteristics of the martingale, we conclude that:

$$0 = E \int_{[0,T]} \eta_t^2 d[M, M]_t = E \int_{[0,T]} \eta_t^2 d\langle M, M\rangle_t = E \int_{[0,T]} \eta_t^2 dt .$$

Therefore it is clear that $\eta_t = 0$ ($dP \otimes d\lambda$ -a.s.). Hence, the relation (2) is true. □

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