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STOCHASTIC INTEGRAL REPRESENTATION OF PATH-DEPENDENT NON-SMOOTH BROWNIAN FUNCTIONALS

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Abstract. Here we consider stochastically non-smooth path-dependent Brownian functionals and investigate questions of their stochastic integral representation. The class of functionals under consideration includes such non-smooth functionals to which not only the well-known Clark-Ocone formula, but also its generalization Glonti-Purtukhia, is not applicable. Moreover, for smooth functionals from the proved result it is easy to obtain the Clark-Ocone formula.

Keywords and phrases: Stochastic integral representation, Malliavin's derivative, Clark-Ocone formula.

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1 Introduction. Stochastic integral representation of Brownian functionals (also known as martingale representation theorem) states that any square-integrable Brownian functional is equal to a stochastic integral with respect to Brownian motion. The first proof of the martingale representation theorem was implicitly provided by Ito himself [1]. Indeed, Theorem 4.2 [1] states: any L_2-functional F of Brownian motion can be expressible as the form: $F=\sum_{n=0}^{\infty} I_n(f_n)$ (where $I_n(f_n)$ is a multiple stochastic integral with respect to Brownian motion B_t). Further, according to Theorem 5.1 [1], the multtple stochasticintegral can be expressible as iterated stochastic intgrals. Therefore, we can write

$$F=EF+\sum_{n=1}^{\infty} I_n(f_n):=EF+\sum_{n=1}^{\infty}\int_0^T \widetilde{I}_{n-1}(g_n(\cdot,t))dB_t$$

$$=EF+\int_0^T\sum_{n=1}^{\infty}\widetilde{I}_{n-1}(g_n(\cdot,t))dB_t:=EF+\int_0^T G(t)dB_t,$$

where

$$I_n(f_n):=\int_0^T\cdots\int_0^T f_n(t_1,...t_{n-1},t)dB_{t_1}\cdots dB_{t_{n-1}}dB_t$$

and

$$\widetilde{I}_{n-1}(g_n):$$

$$=n!\int_0^T\Big(\int_0^{t_{n-1}}\Big(\cdots\int_0^{t_3}\Big(\int_0^{t_2} f_n(t_1,...t_{n-1},t)dB_{t_1})dB_{t_2}\Big)\cdots\Big)dB_{t_{n-1}}\Big)dB_t.$$

Many years later, Dellacherie (1974) gave a simple new proof of Ito's theorem using Hilbert space techniques. Many other articles were written afterward on this problem and its applications but one of the pioneer works on explicit descriptions of the integrand is certainly the one by Clark ([2]). Those of Haussmann (1979), Ocone ([3]), Ocone and Karatzas (1991) and Karatzas, Ocone and Li (1991) were also particularly significant. A nice survey article on the problem of martingale representation was written by Davis (2005). In many papers using Malliavin calculus or some kind of differential calculus for stochastic processes, the results are quite general but unsatisfactory from the explicitness point of view. Shiryaev and Yor (2003) proposed a method based on Ito's formula to find explicit martingale representations for functionals of the running maximum of Brownian motion.

In all cases described above investigated functionals, were stochastically (in Malliavin sense) smooth. It has turned out that the requirement of smoothness of functional can be weakened by the requirement of smoothness only of its conditional mathematical expectation (see, [4]). Here we will consider functionals that do not satisfy even these weakened conditions. Such functionals include, for example, the Lebesgue integral (with respect to the time variable) of stochastically nonsmooth square-integrable processes.

2 Main results. Let on the probability space $(\Omega, \Im, P)$ the Brownian motion $B = (B_t)$, $t \in [0,T]$ be given and $\Im_t^B = \sigma\{B_s : 0 \le s \le t\}$.

Let $h(t,x)$ be a measurable function on $[0,T] \times R^1$. Denote $F(t,T) = \int_t^T h(u,B_u)du$ and $F := F(0,T) = \int_0^T h(u,B_u)du$.

Theorem 1. *If $h(t,x)$ is a bounded measurable function on $[0,T] \times R^1$, then the function $V(t,x) = E[F(t,T)|B_t = x]$ satisfies the requirements of the Ito formula and the following stochastic integral representation holds*

$$F = EF + \int_0^T V_x'(t,B_t)dB_t \quad (P-a.s.). \tag{1}$$

Proof. It is well known that for any bounded measurable function $h(s,x)$ and $0 \le s \le t$ we have

$$E[h(t,B_t)|\Im_s^B] = \int_{-\infty}^{\infty} h(t,y)p(s,t,B_s,dy),$$

where $p(s,t,B_s,A) = P(B_t \in A|\Im_s^B)$ is Brownian transition probability with

$$p(s,t,x,A) = \frac{1}{\sqrt{2\pi(t-s)}} \int_A \exp\Big\{-\frac{(x-y)^2}{2(t-s)}\Big\}dy.$$

Hence, due to the Markov property of the Brownian motion, it is easy to see that

$$V(t,x) = E[F(t,T)|B_t = x] = E[F(t,T)|B_t]|_{B_t=x}$$

$$= E[\int_t^T h(u,B_u)du|B_t]\big|_{B_t=x} = \int_t^T E[h(u,B_u)|B_t]du\big|_{B_t=x}$$

$$= \int_t^T \frac{1}{\sqrt{2\pi(u-t)}} \int_{-\infty}^{\infty} h(u,y) \exp\left\{-\frac{(B_t-y)^2}{2(u-t)}\right\} dydu\Big|_{B_t=x}$$

$$= \int_t^T \frac{1}{\sqrt{2\pi(u-t)}} \int_{-\infty}^{\infty} h(u,y) \exp\left\{-\frac{(x-y)^2}{2(u-t)}\right\} dydu.$$

From this it is not difficult to conclude that the function $V(t,x)$ satisfies the requirements of the Ito formula. Further, according to the Ito formula, we have

$$V(t,B_t) = V(0,B_0) + \int_0^t [V_s^{'}(s,B_s) + \frac{1}{2}V_{xx}^{''}(s,B_s)]ds$$

$$+ \int_0^t V_x^{'}(s,B_s)dB_s \quad (P-a.s.). \tag{2}$$

It is evident that the process

$$\int_0^t h(u,B_u)du + V(t,B_t) = \int_0^t h(u,B_u)du + E[F(t,T)|B_t]$$

$$= \int_0^t h(u,B_u)du + E[F(t,T)|\Im_t^B] = E\big[\int_0^t h(u,B_u)du|\Im_t^B\big]$$

$$+E\big[\int_t^T h(u,B_u)du|\Im_t^B\big] = E\big[\int_0^T h(u,B_u)du|\Im_t^B\big] := M_t$$

is a martingale.

On the other hand, according to Levy's theorem, M_t is a continuous martingale. Therefore, in equality (2), the term of bounded variation combined with an additional term ($\int_0^t h(u,B_u)du$) of the bounded variation of the martingale M is equal to zero and, taking into account the equalities $M_T = F$ and

$$M_0 = V(0,B_0) = E[F(0,T)|\Im_0^B] = EF \quad (P-a.s.)$$

we obtain representation (1). □

Remark. It should be noted that the result of Theorem 1 is especially interesting for stochastically non-smooth $h(u,B_u)$, although it is also useful for smooth $h(u,B_u)$. For example, if we consider the function $h(u,B_u) = g(u)I_{\{B_u \le C\}}$ for some constunt C, then on the one hand, this is not a Malliavin differentiable (see, Proposition 1.2.6 [5]), and, on the other hand, the path-dependent functional $F = \int_0^T g(u)I_{\{B_u\le C\}}du$ is also stochastically non-smooth, for which even the weakened Glonti-Purtukhia requirement fails (see, Theorem 2 [6]).

Theorem 2. *Let $h(u,B_u) \in D_{1,2}$[1] for almost all u. Then the Clark-Ocone representation for the functional $F = \int_0^T h(u,B_u)du$ follows from Theorem 1.*

[1] The class of smooth Brownian functionals S is the class of a random variables which has the form

$$F = f(B_{t_1},...,B_{t_n}),\ f \in C_p^\infty(R^n),\ t_i \in [0,T],\ n \ge 1,$$

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where $C_p^\infty(R^n)$ is the set of all infinitely continuously differentiable functions $f : R^n \to R$ such that f and all of its partial derivatives have polynomial growth.

The stochastic (Malliavin) derivative of a smooth random variable $F \in S$ is the stochastic process D_tF given by

$$D_tF = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(B_{t_1}, ..., B_{t_n}) I_{[0,t_i]}(t).$$

By $D_{2,1}$ we denote the Hilbert space ([5]), which is the closure of the class of smooth Brownian functionals with the following Sobolev-type norm:

$$||F||_{2,1} = ||F||_{L_2(\Omega)} + |||D.F|||_{L_2(\Omega;L_2([0,T]))}.$$