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MARTINGAL REPRESENTATION OF WIENER FUNCTIONAL

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Abstract. We have developed some methods of obtaining the martingal integral representation of nonsmooth (in Malliavin sense) Wiener functionals and have found explicit form of integrands in this representations.

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We have developed one method of obtaining the stochastic integral representation of nonsmooth (in Malliavin sense) Wiener functionals and have found explicit form of integrands in this representations. This method demands smoothness only for conditional mathematical expectation of the considered functional, instead of the usual requirement of smoothness of the functional (as it was in the well-known Clark-Ocone formula). The offered method allows us to obtain the integral representations for the indicator $I_{\{a\leq h(w_T)\leq b\}}$ (which as it is known is not differentiable in the Malliavin sense), for the functional of integral type $\int_0^T I_{\{a\leq f(w_t)\leq b\}}h(w_t)dt$ (which as it is proved also is not differentiable in the Malliavin sense) and other nonsmooth functionals.

We introduce below a method for finding the integrand of Clark's integral representation formula for square integrable functionals F of the Wiener process w with Malliavin differentiable conditional expectations $E[F|\Im_t^w]$, $t<T$. This method allows to obtain the explicit form of the integrand in case when the functional F has no Malliavin derivative. Some applications of the main result are also presented.

On the probability space $(\Omega,\Im,P)$ is given the standard Wiener process $w=(w_t)$, $t\in[0,T]$ and $(\Im_t^w)$, $t\in[0,T]$ is the natural filtration generated by the Wiener process w. We consider the functionals of the Wiener process, i.e. the random variables that are $\Im_T^w$-measurable.

We denote by $L_2([0,T]\times\Omega)=L_2([0,T]\times\Omega,\mathcal{B}([0,T])\otimes\Im,\lambda\times P)$ (where $\mathcal{B}([0,T])$ is the Borel $\sigma-$ algebra on $[0,T]$ and λ is the Lebesgue measure) the set of square integrable processes, and $L_a^2([0,T]\times\Omega)$ represents the subspace of adapted (to the filtration $(\Im_t^w)$, $t\in[0,T]$) processes; $L_2([0,T])=L_2([0,T],\mathcal{B}([0,T]),\lambda)$. Let $L_{2,T}$ denote the set of measurable functions $u:R\to R$, such that $u(\cdot)\rho(\cdot,T)\in L_2:=L_2(R,\mathcal{B}(R),\lambda)$, where $\rho(x,T)=\exp\{-\frac{x^2}{2T}\}$.

Theorem A. *(Clark, 1970, [1]). Let F be a square integrable $\Im_T^w$-measurable random variable. Then there exists a unique adapted stochastic process $v\in L_a^2([0,T]\times\Omega)$ such that*

$$F=E[F]+\int_0^T v_t\mathrm{d}w_t. \tag{1}$$

As it is familiar in Malliavin calculus, we introduce the norm

$$||F||_{1,2}=\{E[F^2]+E[||D.F||^2_{L_2([0,T])}]\}^{1/2},$$

where D is the Malliavin derivative operator and $D_{1,2}$ denotes the Hilbert space which is the closure of the class of smooth Brownian functionals S[1] with respect to the norm $||\cdot||_{1,2}$.

When the random variable F belongs to the space $D_{1,2}$, it turns out that the integrand in the Clark representation (1) can be identified as the optional projection of the derivative of F.

Theorem B. *(Clark-Ocone's representation formula, 1984, [2]). If F is differentiable in the Malliavin sense, $F \in D_{1,2}$, then the stochastic integral representation is fulfilled*

$$F = E[F] + \int_0^T E[D_t F|\Im_t^w]\mathrm{d}w_t. \tag{2}$$

Theorem C. *(Jaoshvili, Purtukhia, 2005, [3]). Let the function $f \in L_{2,T/\alpha}, 0 < \alpha < 1$, and it has the generalized derivative of the first order $\partial f/\partial x$, such that $\partial f/\partial x \in L_{2,T/\beta}, 0 < \beta < 1/2$, then the following integral representation holds*

$$f(w_T) = Ef(w_T) + \int_0^T E[\frac{\partial f}{\partial x}(B_T)|\Im_t^w]\mathrm{d}w_t. \tag{3}$$

Let F be a square integrable random variable. Denote

$$g_t = E[F|\Im_t^w], \quad t \in [0,T]. \tag{4}$$

Below we use the well-known statement:

Proposition 1. *For all bounded or positive measurable function f we have the relation*

$$E[f(w_t)|\Im_u^w] = \int_R f(y)p(u,t,w_u,dy) \quad (P-a.s.),$$

where $p(u,t,w_u,A)$ is the transition probability of the Brownian motion.

Theorem 1. *(Glonti, Purtukhia, 2014 [5]). If u_t has the Wiener-Chaos decomposition $u_t = \sum_{n=0}^{\infty} I_n(f_n^t(\cdot))$ with kernels, measurable in all their variables, then average process $\int_0^T u_t dt$ has the development*

$$\int_0^T u_t dt = \sum_{n=0}^{\infty} I_n(\int_0^T f_n^t(\cdot)dt).$$

Corollary of Theorem 1. *For any real numbers $a < b$ the integral*

$$\int_0^T I_{\{a\le w_t \le b\}} h(w_t)dt$$

is not in the space $D_{2,1}$.

[1]Here S denotes the class of a random variables which has the form

$$F = f(w_{t_1}, ..., w_{t_n}),\ f \in C_p^{\infty}(R^n),\ t_i \in [0,T],\ n \ge 1,$$

where $C_p^{\infty}(R^n)$ is the set of all infinitely continuously differentiable functions $f : R^n \to R$ such that f and all of its partial derivatives have polynomial growth.

Theorem 2. *Suppose that there exists a sequence* $(t_n)_{n\geq 1}$ *in* $[0,T)$, $t_n \uparrow T$, *such that* $g_{t_n} \in D_{1,2}$, $n \geq 1$. *Then we have the integral representation*

$$g_T = F = E[F] + \int_0^T v_s \mathrm{d}w_s, \tag{5}$$

where

$$v_s := \lim_{t_n \to T} E[D_s g_{t_n} | \Im_s^w] \quad \text{in the} \quad L_2([0,T] \times \Omega).$$

Corollary of Theorem 2. *Let* $F \in D_{1,2}$, *then the Clark-Ocone's representation formula (2) follows from the Theorem 2 and the following relation is fulfilled*

$$\lim_{t_n \to T} E[D_s g_{t_n} | \Im_s^w] = E[D_s \lim_{t_n \to T} g_{t_n} | \Im_s^w] \quad \text{in} \quad L_2([0,T] \times \Omega).$$

Remark 1. Despite the fact that the stochastic derivative operator is not a continuous operator, in our case we have "continuity" in some weak sense.

Example 1. Consider the square integrable $\Im_T^w$-measurable random variable

$$F = F(x) = I_{\{w_T \leq x\}}, \quad x \in R.$$

It is well-known that indicator of event $A \in \Im$ is Malliavin differentiable if and only if probability $P(A)$ is equal to zero or one (see [4], Chapter I, Proposition 1.2.6, page 30). Therefore, in general, $F(x) = I_{\{w_T \leq x\}}$, $(x \in R)$ are not Malliavin differentiable.

According to the result of Theorem 2, using Proposition 1 for the computation of g_t, due to the chain rule of stochastic differentiation of a composite function (see [4], Chapter I, Proposition 1.2.3, page 28) the Malliavin derivative of g_t, we can obtain the following integral representation:

$$I_{\{w_T \leq x\}} = \Phi(\frac{x}{\sqrt{T}}) - \int_0^T \varphi(\frac{x - w_s}{\sqrt{T-s}}) \mathrm{d}w_s, \quad x \in R, \tag{6}$$

where Φ is the standard normal distribution function and φ is its density.

Remark 2. Formula (6) can be also deduced from Theorem C.

Note that, the integral representation (6) is obtained in [6] as an example for illustration of Theorem C.

As an illustration of Corollary of Theorem 2 we give the following example.

Example 2. Let $F = w_T^+ = max(0, w_T)$. In this case F has the Malliavin derivative

$$D_t F = D_T w_T^+ = I_{\{w_T > 0\}} I_{[0,T]}(t)$$

and using Clark-Ocone's representation formula (2), according to Proposition 1, we have

$$\begin{aligned} w_T^+ &= E w_T^+ + \int_0^T E[I_{\{w_T > 0\}} | \Im_t^w] \mathrm{d}w_t \\ &= \sqrt{\frac{T}{2\pi}} + \int_0^T \Phi(\frac{w_t}{\sqrt{T-t}}) \mathrm{d}w_t. \end{aligned}$$

On the other hand, using again Proposition 1, we can compute

$$g_t = E[w_T^+|\Im_t^w] = E[I_{\{w_T>0\}}w_T|\Im_t^w]$$
$$= \frac{1}{\sqrt{2\pi(T-t)}}\int_0^\infty x\exp\{-\frac{(x-w_t)^2}{2(T-t)}\}\mathrm{d}x.$$

Therefore, due to the chain rule of stochastic differentiation of an ordinary integral and a composite function, using the standard technique of integration, we can directly show that

$$\lim_{t\to T} E[D_s g_t|\Im_s^w] = E[D_s \lim_{t\to T} g_t|\Im_s^w] \quad \text{in} \quad L_2([0,T]\times\Omega).$$

But according to the Corollary of Theorem 2, it is a general fact, which follows from the Malliavin differentiable of F.

Proposition 2. *Let $h = h(x)$, $x \in R$, be a nondecreasing function. Then the indicator $I_{\{h(B_T)\le x\}}$ allows the representation*

$$I_{\{h(w_T)\le x\}} = P(w_T \le h^{-1}(x)) - \int_0^T \varphi(\frac{h^{-1}(x)-w_t}{\sqrt{T-t}})\mathrm{d}w_t, \;\; x \in R. \tag{7}$$

Theorem 3. *(Glonti, Purtukhia, 2014 [5]). Let $F = (B_T - K)^+ I_{\{B_T^*\le L\}}$ (where $w_T^* = max_{0\le t\le T} B_t$). Then the following stochastic integral representation is fulfilled*

$$F = EF + \int_0^T [\Phi(\frac{w_t - K}{\sqrt{T-t}}) - \Phi(\frac{w_t - 2L + K}{\sqrt{T-t}}) - \frac{2(L-K)}{\sqrt{T-t}}\varphi(\frac{L-w_t}{\sqrt{T-t}})]dw_t$$

(where K and $L > 0$ are constants).

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