Proceedings of A. Razmadze
Mathematical Institute
Vol. **168** (2015), 25–40

HEDGING OF EUROPEAN OPTION OF EXOTIC TYPE

O. GLONTI†, V. JAOSHVILI AND O. PURTUKHIA

Abstract. We consider Exotic type European Option in the case of Bachelier financial market model whose payoff function is a certain combination of the Binary and Asian options payoff functions and investigate the hedging problem. The Clark stochastic integral representation formula for corresponding Wiener functionals with the explicit form of integrand is given.

რეზიუმე. ფინანსური ბაზრის ბაშელიეს მოდელის შემთხვევაში ჩვენ ვიხილავთ ეგზოტიკური ტიპის ევროპულ ოფციონს, რომლის გადასახადის ფუნქცია წარმოადგენს ბინარული და აზიური ოფციონის გარკვეულ კომბინაციას და ვიკვლევთ ჰეჯირების პრობლემას. შესაბამისი ვინერის ფუნქციონალისათვის დადგენილია კლარკის სტოქასტური ინტეგრალური წარმოდგენის ინტეგრანდი ცხადი სახით.

1. Introduction and Preliminaries

We have developed some methods of obtaining the stochastic integral representation of nonsmooth (in the Malliavin sense) Wiener functionals and their applications in the problems of hedging of European Options. In turn, for receiving obvious integral expressions, we use the result of stochastic integral representation proven by us earlier, which demands the smoothness only of a conditional mathematical expectation of the considered functional, instead of the usual requirement of smoothness of the functional (as it was in the well-known Clark-Ocone formula). The suggested method allows to remove integral representation for the indicator $I_{\{K_1 \leq S_T \leq K_2\}}$ (which is, as is known, not differentiable in Malliavin sense) for the functional of integral type $\int_0^T I_{\{K_1 \leq S_t \leq K_2\}} dt$ (which is, as it has been proved, also not differentiable in Malliavin sense), etc.

The payoff functions of derivative securities with forms, more complicated than standard European or American call and put options are known as Exotic Options. One of such kind Exotic Options is the so-called Binary

2010 *Mathematics Subject Classification.* 60H07; 60H30; 62P05.
Key words and phrases. Exotic option, Bachelier model, Clark-Ocone representation, hedging problem.

Option. It is an option with discontinuous payoff function. The simplest examples of Binary Options are call and put options “cash or nothing”. The payoff function of the call option has the form $BC_T = QI_{\{S_T>K\}}$, and for the put option – $BP_T = QI_{\{S_T<K\}}$, where K is the strike price at the time of execution T. This is also common the Binarny Option “an asset or nothing”. There are same conditions as in “cash or nothing” option, but the difference is that an owner of the call option receives price of the asset S_T instead of amount Q. The Standard European Call Option (i.e. the option with the payoff function $(S_T - K)^+$) is equivalent to a long position (the bought asset) in the “an asset or nothing” option and short position (the sold asset) in the “cash or nothing” option, when $Q = K$.

Moreover, the so-called Asian Options are also of the type of Exotic Option. The payoff function of this option depends on average value of the price of an asset during the certain period of option life time. The payoff function of the Asian Option has by definition the following representation: $C_T^A = (A_S(T_0, T) - K)^+$, where

$$A_S(T_0, T) = \frac{1}{T - T_0} \int_{T_0}^{T} S_t dt$$

is arithmetic mean of the prices of asset at time interval $[T_0, T]$, K is a strike and $S = (S_t)$ $(0 \le t \le T\ )$ is geometrical Brownian motion. The main difficulty in pricing and hedging of the Asian Option is that the random variable $A_S(T_0, T)$ is no lognormal distributed and, therefore it is rather difficult to obtain explicit formulas of pricing of this option.

We consider an Exotic Option which is a certain combination of the Binary and Asian Options and investigate the hedging problem. In particular, we study the European Option with payoff function $Q \int_0^T I_{\{K_1 \le S_t \le K_2\}} dt$, and for this purpose we give the Clark stochastic integral representation of such kind payoff function with the explicit form of integrand.

Let on the probability space $(\Omega, \Im, P)$ be given the Wiener process $w = (w_t)$, $t \in [0, T]$ and $(\Im_t^w)$, $t \in [0, T]$ be the natural filtration generated by the Wiener process w. Consider the Bachelier market model with a risk-free asset price evolution described by

$$dB_t = rB_t dt, \quad B_0 = 1, \tag{1.1}$$

where $r \ge 0$ is an interest rate and risky asset price evolution

$$dS_t = \mu dt + \sigma dw_t, \quad S_0 = 1, \tag{1.2}$$

where $\mu \in R$ is an appreciation rate and $\sigma > 0$ is a volatility coefficient.

Let

$$Z_T = exp\Big\{ -\frac{\mu - r}{\sigma} w_T - \frac{1}{2}\Big(\frac{\mu - r}{\sigma}\Big)^2 T \Big\}$$

and $\widetilde{P}_T$ be the measure on $(\Omega, \Im_T^w)$ such that

$$d\widetilde{P}_T = Z_T dP.$$

From Girsanov's Theorem it follows (see Shiryaev [1]) that under this measure (martingale measure),

$$\widetilde{w}_t = w_t + \frac{\mu - r}{\sigma} t$$

is the standard Wiener process and

$$dS_t = rdt + \sigma d\widetilde{w}_t, \quad S_0 = 1,$$

or

$$S_t = 1 + rt + \sigma \widetilde{w}_t. \tag{1.3}$$

Consider the problem of "replication" the European Option of Exotic Type with the payoff of integral type

$$G = \int_0^T I_{\{K_1 \le S_t \le K_2\}} dt \tag{1.4}$$

(where K_1 and K_2 are some positive constants, $K_1 < K_2$), i.e. one needs to find a trading strategy (β_t, γ_t), $t \in [0, T]$ such that the capital process

$$X_t = \beta_t B_t + \gamma_t S_t, \quad X_T = G \tag{1.5}$$

under the self-financing condition

$$dX_t = \beta_t dB_t + \gamma_t dS_t. \tag{1.6}$$

From the relations (1.3), (1.5) and (1.6), we have

$$G = X_T = X_0 + \int_0^T r(\beta_t B_t + \gamma_t) dt + \int_0^T \sigma \gamma_t d\widetilde{w}_t. \tag{1.7}$$

Our problem is to find the trading strategy $(\gamma, \beta) = (\gamma_t, \beta_t)$, $t \in [0, T]$. It is well-known that this problem is equivalent to finding a martingale representation of the payoff G with explicit form of integrand. Note that G is square integrable, but not differentiable in Malliavin sense, functional of the Wiener process $\widetilde{w} = (\widetilde{w}_t)$, $t \in [0, T]$, and therefore we try to obtain the Clark integral representation with the known integrand applying a nonconventional method (because the Clark-Ocone's well-known method here is not applicable).

2. Main Notions and Results

The anticipating stochastic calculus for the Wiener process is based on a variational (Malliavin) derivative operator D (gradient) and its adjoint (divergence or Skorohod integral). There are two basically different methods to define these operators (Nualart and Pardoux 1988, Nualart and Zakai 1986, Nualart 1995). The first approach uses the Wiener chaos decomposition and the seconed one consists in defining the operator D as a directional derivative on a space of test functions.

Starting from the 70th of the past century, many attempts were made to weak the requirement for the integrand to be adapted for the integrand of the Ito's stochastic integral as well as in the theory of "the extension of filtration". Skorokhod (1975) suggested absolutely different method, symmetric with respect to the time inversion and not requiring for the integrand to be independent of the future Wiener process. Towards this end, he required for the integrand to be smooth in a certain sense, i.e., its stochastic differentiability. This idea was later on developed in the works of Gaveau-Trauber (1982), Nualart, Zakai (1986), Pardoux (1982), Protter, Malliavin (1979), etc. It turned out (as it was shown by Gaveau and Trauber in 1982) that the operator of Skorokhod stochastic integration coincides with the conjugate operator of stochastic differentiation in the sense of Malliavin. As is known, the original aim of Malliavin's infinite-dimensional stochastic investigation was to study the density smoothness of a solution of a stochastic differential equation. The situation changed in 1991 when Karatsas and Ocone showed how one can apply in financial mathematics the Ocone's theorem of stochastic integral representation (the martingale representation theorem) for the functional of diffusion processes. This theorem was subsequently called the Ocone-Haussmann-Clark formula. It has been used in constructing hedging strategies at full financial markets driven by Brownian motion. Due to this result, the interest in Malliavin calculus on the part of mathematicians and financial researchers grew essentially. Since that time Malliavin's theory has been actively developing. Also, an active search for new areas of its application is being carried out. Malliavin's methods for jump processes (in particular, for Levy's processes) were developed by many authors. Despite the fact that in the general case, financial markets driven by Levy's processes are not full, the Ocone-Haussmann-Clark formula nevertheless plays an important role in financial applications.

In the 80th of the past century, it turned out (Harison, Pliska (1981)) that the martingale representation theorems (along with the Girsanov's measure change theorem) play an important role in the modern financial mathematics. According to the well-known Clark's formula (see Clark [2]), if F is an

$\Im_T^w$-measurable square integrable random variable, then

$$F = EF + \int_0^T \varphi_t(\omega) dw_t$$

for some adapted (to the filtration $\Im_t^w$, $t \in [0,T]$) and square integrable random process $\varphi_t(\omega)$ ($\varphi_\cdot(\cdot) \in L_2([0,T] \times \Omega)$). Due to the so-called Ocone-Clark's formula (see Ocone [3]): $\varphi_t(\omega) = E[D_t F | \Im_t^w]$, where $D_t F$ is the stochastic derivative (the so-called Malliavin's derivative) of the functional F. But in the cases if the functional F has no stochastic derivative, its application is impossible.

The derivative of a smooth random variable F of the form

$$F = f(w(h_1), \ldots, w(h_n)), \quad f \in C_p^\infty(R^n), \quad h_i \in L_2([0,T])$$

is the stochastic process $D_t F$ given by

$$D_t F = \sum_{i=1}^n \frac{\partial f}{\partial x_i}((w(h_1), \ldots, w(h_n)) h_i(t).$$

For example, $D_t w(h) = h(t)$. We will consider DF as an element of $L_2([0,T] \times \Omega)$, that means DF is a square integrable process indexed by the parameter space $[0,T]$. In order to interpret DF as a directional derivative, for any element $h \in L_2([0,T])$ we can write

$$\langle DF, h \rangle_{L_2} = \lim_{\epsilon \to 0} \frac{1}{\epsilon}[f(w(h_1) + \epsilon \langle h_1, h \rangle_{L_2}, \ldots, w(h_n) + \epsilon \langle h_n, h \rangle_{L_2}) - f].$$

This means, the scalar product $\langle DF, h \rangle_{L_2}$ is the derivative at $\epsilon = 0$ of the random variable F composed with shifted white noise $w(A) + \varepsilon \int_A h dt$.

D is closable as an operator from $L_2(\Omega)$ to $L_2(\Omega; L_2([0,T]))$. We denote its domain by $D_{2,1}$. This means that $D_{2,1}$ is equal to the adherence of the class of smooth random variables with respect to the norm

$$||F||_{2,1} := ||F||_{L_2(\Omega)} + |||DF|||_{L_2(\Omega; L_2([0,T]))}.$$

For the completeness of the statement we present some results from the Nualart's book [4] below:

Proposition 2.1. *Let F be a square integrable random variable with development*

$$F = \sum_{n=0}^\infty I_n(f_n(\cdot))$$

(*where $I_n(f_n(\cdot))$ is the multiple Wiener-Ito stochastic integral of the function $f_n \in L_2([0,T]^n)$*).

Then F belongs to the space $D_{2,1}$ if and only if

$$\sum_{n=1}^{\infty} nn! ||f_n(\cdot)||^2_{L_2([0,T]^n)} < \infty,$$

and in this case we have

$$D_t F = \sum_{n=1}^{\infty} n I_{n-1}(f_n(\cdot, t))$$

(see [4, Proposition 1.2.7]).

Proposition 2.2. *Let φ be a continuously differentiable function with bounded derivative. Suppose that $F \in D_{2,1}$. Then $\varphi(F) \in D_{2,1}$ and*

$$D(\phi(F)) = \frac{\partial \varphi}{\partial x} DF$$

(see [4, Proposition 1.2.3]).

Proposition 2.3. *Let $A \in \Im$. Then the indicator function of A belongs to $D_{2,1}$ if and only if $P(A)$ is equal to zero, or to unity*

(see [4, Proposition 1.2.6]).

Let $p(u, t, w_u, A)$ be the transition probability of the Wiener process w, i.e. $P[w_t \in A | \Im^w_u] = p(u, t, w_u, A)$, where $0 \le u \le t$, A is a Borel subset of R and

$$p(u, t, x, A) = \frac{1}{\sqrt{2\pi(t-u)}} \int\limits_A \exp\Big\{ - \frac{(x-y)^2}{2(t-u)} \Big\} dy.$$

For the computation of conditional mathematical expectation below we use the well-known statement:

Proposition 2.4. *For all bounded or positive measurable functions f we have the relation*

$$E[f(w_t) | \Im^w_u] = \int\limits_R f(y) p(u, t, w_u, dy) \quad (P - a.s.). \tag{2.1}$$

Theorem 2.5. *Suppose that $g_t = E[F|\Im^w_t]$ is Malliavin differentiable ($g_t(\cdot) \in D_{2,1}$) for almost all $t \in [0, T)$. Then we have the stochastic integral representation*

$$g_T = F = EF + \int\limits_0^T \nu_u dw_u, \quad (P - a.s.),$$

where

$$\nu_u := \lim_{t \to T} E[D_u g_t | \Im^w_u] \quad \text{in the} \quad L_2([0,T] \times \Omega)$$

(see, [5, Theorem 1]).

Theorem 2.6. *If u_t has the Wiener-Chaos decomposition* $u_t = \sum_{n=0}^{\infty} I_n(f_n^t(\cdot))$ *with kernels, measurable in all their variables, then the average process* $\int_0^T u_t dt$ *has the development*

$$\int_0^T u_t dt = \sum_{n=0}^{\infty} I_n\Big(\int_0^T f_n^t(\cdot)dt\Big).$$

Proof. It is well-known that if the square integrable random variable

$$F = \sum_{n=0}^{\infty} I_n(f_n(\cdot))$$

belongs to the space $D_{2,\infty}$, then $f_n = \frac{1}{n!}E(D^n F)$ for every n.

Therefore, if for almost every $t \in [0,T] : u_t \in D_{2,\infty}$, the kernels g_n of the average process $\int_0^T u_t dt$ we can calculate as

$$g_n = \frac{1}{n!}E\Big(D^n\Big(\int_0^T u_t dt\Big)\Big) = \int_0^T \frac{1}{n!}E(D^n u_t)dt = \int_0^T f_n^t(\cdot)dt.$$

Hence, the proof can be completed by using the standard technique of approximation. □

Theorem 2.7. *For any real number $K_1 < K_2$, the integral*

$$\int_0^T I_{\{K_1 \le w_t \le K_2\}} dt$$

is not in the space $D_{2,1}$.

Proof. Since for any $t \in [0,T]$ the probability of the event $\{K_1 \le w_t \le K_2\}$ is not zero or unity $1 \ne P\{K_1 \le w_t \le K_2\} > 0$, due to Proposition 2.3, the indicator function $I_{\{a \le w_t \le b\}}$ is not in space $D_{2,1}$. For all $t \in [0,T]$, $I_{\{K_1 \le w_t \le K_2\}}$ is a square integrable random variable, and hence it has the Wiener-Chaos decomposition

$$I_{\{K_1 \le w_t \le K_2\}} = \sum_{n=0}^{\infty} I_n(f_n^t(\cdot)),$$

where the deterministic kernels $f_n^t(\cdot)$ are symmetric and depend on the parameter t. Using the standard approximation technique to the process $I_{\{K_1 \le w_t \le K_2\}}$ in $L_2([0,T] \times \Omega)$ by a sequence of simple processes, these kernels can be chosen to be measurable in all their variables. Hence, due to

Theorem 2.6, we have

$$\int_0^T I_{\{K_1\le w_t\le K_2\}}dt=\sum_{n=0}^{\infty} I_n\Big(\int_0^T f_n^t(\cdot)dt\Big).$$

Further, according to Proposition 2.2, the series

$$\sum_{n=1}^{\infty} nn!||f_n^t(\cdot)||_{L_2([0,T]^n)}$$

is nonconvergent for all $t\in[0,T]$.

On the other hand, according to the Fubini Theorem, we can conclude that the integral

$$\int_0^T\sum_{n=1}^{\infty} nn!||f_n^t(\cdot)||^2_{L_2([0,T]^n)}dt=\sum_{n=1}^{\infty}\int_0^T nn!||f_n^t(\cdot)||^2_{L_2([0,T]^n)}dt$$

is also nonconvergent, because otherwise we obtain

$$\sum_{n=1}^{\infty} nn!||f_n^t(\cdot)||^2_{L_2([0,T]^n)}<\infty$$

for almost all $t\in[0,T]$.

Therefore, using Proposition 2.1, we easily ascertain that the theorem is true. □

Theorem 2.8. *In scheme* (1.2), *for any real number* K *and* $\theta\in[0,T]$, *the functional* $I_{\{S_\theta\le K\}}$ *admits the following stochastic integral representation:*

$$I_{\{S_\theta\le K\}}=\Phi\Big(\frac{K-1-r\theta}{\sigma\sqrt{\theta}}\Big)-$$

$$-\int_0^\theta\frac{1}{\sqrt{2\pi(\theta-u)}}\exp\Big\{-\frac{(K-1-r\theta-\sigma\widetilde{w}_u)^2}{2\sigma^2(\theta-u)}\Big\}d\widetilde{w}_u.^1 \tag{2.2}$$

Proof. First, we will check that the conditions of Theorem 2.5 are satisfied. Fix $t<\theta$. Using the well-known properties of Wiener's process and conditional mathematical expectation, it is not difficult to see that

$$g_t^\theta:=\widetilde{E}[I_{\{S_\theta\le K\}}|\Im_t^{\widetilde{w}}]=$$

$$=\widetilde{E}[I_{\{\widetilde{w}_\theta\le\frac{K-1-r\theta}{\sigma}\}}|\Im_t^{\widetilde{w}}]=\widetilde{E}[I_{\{\widetilde{w}_\theta-\widetilde{w}_t\le\frac{K-1-r\theta-\sigma\widetilde{w}_t}{\sigma}\}}|\Im_t^{\widetilde{w}}]=$$

$$=\widetilde{E}[I_{\{\widetilde{w}_\theta-\widetilde{w}_t\le\frac{K-1-r\theta-\sigma\widetilde{w}_t}{\sigma}\}}|\widetilde{w}_t]=\widetilde{E}[I_{\{\widetilde{w}_\theta-\widetilde{w}_t\le\frac{K-1-r\theta-\sigma y}{\sigma}\}}]|_{y=\widetilde{w}_t}=$$

[1]Here and below, Φ is the standard normal distribution function.

$$= \widetilde{P}\Big\{\frac{\widetilde{w}_\theta - \widetilde{w}_t}{\sqrt{\theta - t}} \le \frac{K - 1 - r\theta - \sigma y}{\sigma\sqrt{\theta - t}}\Big\}\Big|_{y=\widetilde{w}_t} =$$

$$= \Phi\Big(\frac{K - 1 - r\theta - \sigma y}{\sigma\sqrt{\theta - t}}\Big)\Big|_{y=\widetilde{w}_t} = \Phi\Big(\frac{K - 1 - r\theta - \sigma\widetilde{w}_t}{\sigma\sqrt{\theta - t}}\Big).$$

Hence, due to Proposition 2.2 (see also [8]), the random variable $g_t^\theta = \widetilde{E}[I_{\{S_\theta \le K\}}|\Im_t^{\widetilde{w}}]$ is Malliavin differentiable ($g_t(\cdot) \in D_{2,1}$) for all $t \in [0, \theta)$.

According to Theorem 2.5, we have the following stochastic integral representation:

$$I_{\{S_\theta \le K\}} = \widetilde{E}[I_{\{S_\theta \le K\}}] + \int_0^T \nu_u^\theta d\widetilde{w}_u, \quad (P - a.s.), \tag{2.3}$$

where

$$\nu_u^\theta := \lim_{t \to T} \widetilde{E}\{D_u([I_{\{S_\theta \le K\}}|\Im_t^{\widetilde{w}}])\} \quad \text{in the} \quad L_2([0, T] \times \Omega).$$

Clearly,

$$\widetilde{E}[I_{\{S_\theta \le K\}}] = \widetilde{P}\{S_\theta \le K\} =$$

$$= \widetilde{P}\Big\{\widetilde{w}_\theta \le \frac{K - 1 - r\theta}{\sigma}\Big\} = \Phi\Big(\frac{K - 1 - r\theta}{\sigma\sqrt{\theta}}\Big). \tag{2.4}$$

Further, due to Proposition 2.2, we have

$$D_u\Phi\Big(\frac{K - 1 - r\theta - \sigma\widetilde{w}_t}{\sigma\sqrt{\theta - t}}\Big) =$$

$$= -\frac{\sigma}{\sigma\sqrt{\theta - t}}\varphi\Big(\frac{K - 1 - r\theta - \sigma\widetilde{w}_t}{\sigma\sqrt{\theta - t}}\Big) I_{[0,t]}(u).^2$$

Hence, according to Theorem 2.5, to find the integrand in the relation (2.2) we have to calculate the following limit

$$\nu_u^\theta := \lim_{t \to \theta} \widetilde{E}[D_u g_t^\theta|\Im_u^{\widetilde{w}}] = \lim_{t \to \theta} \widetilde{E}\Big[D_u\Phi\Big(\frac{K - 1 - r\theta - \sigma\widetilde{w}_t}{\sigma\sqrt{\theta - t}}\Big)|\Im_u^{\widetilde{w}}\Big] =$$

$$= \lim_{t \to \theta}\Big\{ -\frac{1}{\sqrt{\theta - t}}\widetilde{E}\Big[\varphi\Big(\frac{K - 1 - r\theta - \sigma\widetilde{w}_t}{\sigma\sqrt{\theta - t}}\Big)|\Im_u^{\widetilde{w}}\Big] I_{[0,t]}(u)\Big\}. \tag{2.5}$$

Using Proposition 2.5, we can write

$$\widetilde{E}\Big[\varphi\Big(\frac{K - 1 - r\theta - \sigma\widetilde{w}_t}{\sigma\sqrt{\theta - t}}\Big)|\Im_u^{\widetilde{w}}\Big] =$$

$$= \frac{1}{\sqrt{2\pi(t - u)}} \int_{-\infty}^{\infty} \varphi\Big(\frac{K - 1 - r\theta - \sigma x}{\sigma\sqrt{\theta - t}}\Big) \exp\Big\{ -\frac{(x - \widetilde{w}_u)^2}{2(t - u)}\Big\} dx =$$

---

[2]Here and below φ is the standard normal distribution density function.

$$=\frac{1}{2\pi\sqrt{(t-u)}}\int\limits_{-\infty}^{\infty}\exp\Big\{-\frac{(K-1-r\theta-\sigma x)^2}{2\sigma^2(\theta-t)}-\frac{(x-\widetilde{w}_u)^2}{2(t-u)}\Big\}dx. \tag{2.6}$$

To calculate the last integral, we transform its subintegral expression, in particular, we allocate a full square from argument of an exponential function. We have

$$-\frac{(K-1-r\theta-\sigma x)^2}{2\sigma^2(\theta-t)}-\frac{(x-\widetilde{w}_u)^2}{2(t-u)}=$$
$$=-\frac{(K-1-r\theta-\sigma x)^2(t-u)+\sigma^2(x-\widetilde{w}_u)^2(\theta-t)}{2\sigma^2(\theta-t)(t-u)}=$$
$$=-\frac{x^2-2x[\frac{(K-1-r\theta)(t-u)+\sigma(\theta-t)\widetilde{w}_u}{\sigma(\theta-u)}]+\frac{(K-1-r\theta)^2(t-u)+\sigma^2(\theta-t)\widetilde{w}_u^2}{\sigma^2(\theta-u)}}{2(\theta-t)(t-u)/(\theta-u)}=$$
$$=-\frac{[x-\frac{(K-1-r\theta)(t-u)+\sigma(\theta-t)\widetilde{w}_u}{\sigma(\theta-u)}]^2}{2(\theta-t)(t-u)/(\theta-u)}+$$
$$+\frac{\frac{-(K-1-r\theta)^2(t-u)(\theta-t)+2\sigma(K-1-r\theta)(t-u)(\theta-t)\widetilde{w}_u-\sigma^2(\theta-t)(t-u)\widetilde{w}_u^2}{\sigma^2(\theta-)}}{2\sigma^2(\theta-t)(t-u)(\theta-u)}=$$
$$=-\frac{[x-\frac{(K-1-r\theta)(t-u)+\sigma(\theta-t)\widetilde{w}_u}{\sigma(\theta-u)}]^2}{2(\theta-t)(t-u)/(\theta-u)}-$$
$$-\frac{-(K-1-r\theta-\sigma\widetilde{w}_u)^2}{2\sigma^2(\theta-u)}. \tag{2.7}$$

From the relations (2.6) and (2.7), using the well-known property of the distribution density function, we easily find that

$$-\frac{1}{\sqrt{\theta-t}}\widetilde{E}\Big[\varphi\Big(\frac{K-1-r\theta-\sigma\widetilde{w}_t}{\sigma\sqrt{\theta-t}}\Big)|\Im_u^{\widetilde{w}}\Big]=$$
$$=-\frac{1}{2\pi\sqrt{(t-u)(\theta-t)}}\exp\Big\{-\frac{-(K-1-r\theta-\sigma\widetilde{w}_u)^2}{2\sigma^2(\theta-u)}\Big\}\times$$
$$\times\int\limits_{-\infty}^{\infty}\exp\Big\{-\frac{[x-\frac{(K-1-r\theta)(t-u)+\sigma(\theta-t)\widetilde{w}_u}{\sigma(\theta-u)}]^2}{2(\theta-t)(t-u)/(\theta-u)}\Big\}dx=$$
$$=-\frac{\sqrt{(\theta-t)(t-u)/(\theta-u)}}{\sqrt{\theta-t}}\frac{1}{\sqrt{2\pi(t-u)}}\exp\Big\{-\frac{(K-1-r\theta-\sigma\widetilde{w}_u)^2}{2\sigma^2(\theta-u)}\Big\}=$$
$$=-\frac{1}{\sqrt{2\pi(\theta-u)}}\exp\Big\{-\frac{(K-1-r\theta-\sigma\widetilde{w}_u)^2}{2\sigma^2(\theta-u)}\Big\}. \tag{2.8}$$

Hence, according to the relation (2.8), we have

$$\nu_u^\theta := \lim_{t\to\theta} \widetilde{E}[D_u g_t^\theta | \Im_u^{\widetilde{w}}] =$$

$$= \lim_{t\to\theta} \Big\{ -\frac{1}{\sqrt{2\pi(\theta-u)}} \exp\Big\{ -\frac{(K-1-r\theta-\sigma\widetilde{w}_u)^2}{2\sigma^2(\theta-u)} \Big\} I_{[0,t]}(u) \Big\} =$$

$$= -\frac{1}{\sqrt{2\pi(\theta-u)}} \exp\Big\{ -\frac{(K-1-r\theta-\sigma\widetilde{w}_u)^2}{2\sigma^2(\theta-u)} \Big\} I_{[0,\theta]}(u). \tag{2.9}$$

Combining now the relations (2.3), (2.4) and (2.9), we easily ascertain that the representation (2.2) is fulfilled. □

3. Hedging of Option

Theorem 3.1. *In scheme* (1.2)*, for any real numbers $K_1 < K_2$, the functional G from* (1.4) *admits the following stochastic integral representation:*

$$\int_0^T I_{\{K_1 \le S_t \le K_2\}} dt = \int_0^T \Big[\Phi\Big(\frac{K-1-rt}{\sigma\sqrt{t}}\Big)\Big]\Big|_{K=K_1}^{K_2} dt -$$

$$-\int_0^T \int_u^T \frac{1}{\sqrt{t-u}} \Big[\varphi\Big(\frac{K-1-rt-\sigma\widetilde{w}_u}{\sigma\sqrt{t-u}}\Big)\Big]\Big|_{K=K_1}^{K_2} dt d\widetilde{w}_u. \tag{3.1}$$

Proof. Taking from the both parts of expression (2.2) the integral with respect to $d\theta$, using the stochastic type Fubini's theorem (see [6, Lemma III.4.1] or [7, Corollary of the Lemma IV.2.4]), it is not difficult to see that the following stochastic integral representation is fulfilled

$$\int_0^T I_{\{S_\theta \le K\}} d\theta = \int_0^T \Phi\Big(\frac{K-1-r\theta}{\sigma\sqrt{\theta}}\Big) d\theta -$$

$$-\int_0^T \int_0^\theta \frac{1}{\sqrt{2\pi(\theta-u)}} \exp\Big\{ -\frac{(K-1-r\theta-\sigma\widetilde{w}_u)^2}{2\sigma^2(\theta-u)} \Big\} d\widetilde{w}_u d\theta =$$

$$= \int_0^T \Phi\Big(\frac{K-1-r\theta}{\sigma\sqrt{\theta}}\Big) d\theta -$$

$$-\int_0^T \int_u^T \frac{1}{\sqrt{\theta-u}} \varphi\Big(\frac{K-1-r\theta-\sigma\widetilde{w}_u}{\sigma\sqrt{\theta-u}}\Big) d\theta d\widetilde{w}_u. \tag{3.2}$$

It is clear that

$$\int\limits_0^T I_{\{K_1\le S_t\le K_2\}}dt=\int\limits_0^T I_{\{S_t\le K_2\}}dt-\int\limits_0^T I_{\{S_t<K_1\}}dt. \tag{3.3}$$

From the relations (3.2) and (3.3), we easily obtain the representation (3.1). □

Corollary 3.2. *In the case $r=0$, for any real numbers $1\le K_1<K_2$,* [3] *the functional G from* (1.4) *admits the following stochastic integral representation:*

$$G=\int\limits_0^T I_{\{K_1\le S_t\le K_2\}}dt=\int\limits_0^T\Big[\Phi\Big(\frac{K-1}{\sigma\sqrt{t}}\Big)\Big]\Big|_{K=K_1}^{K_2}dt-$$
$$-\int\limits_0^T\int\limits_u^T\frac{1}{\sqrt{t-u}}\Big[\varphi\Big(\frac{K-1-\sigma\widetilde{w}_u}{\sigma\sqrt{t-u}}\Big)\Big]\Big|_{K=K_1}^{K_2}dtd\widetilde{w}_u, \tag{3.4}$$

where

$$\int\limits_0^T\Big[\Phi\Big(\frac{K-1}{\sigma\sqrt{t}}\Big)\Big]\Big|_{K=K_1}^{K_2}dt=T\Big[\Phi\Big(\frac{K_2-1}{\sigma\sqrt{T}}\Big)-\Phi\Big(\frac{K_1-1}{\sigma\sqrt{T}}\Big)\Big]+$$
$$+\Big(\frac{K_1-1}{\sigma}\Big)^2\Big[\frac{1}{2}erf\Big(\frac{v}{\sqrt{2}}\Big)+\frac{\varphi(v)}{v}\Big]\Big|_{v=\frac{K_1-1}{\sigma\sqrt{T}}}^{\frac{K_2-1}{\sigma\sqrt{T}}}+$$
$$+\Big[\Big(\frac{K_2-1}{\sigma}\Big)^2-\Big(\frac{K_1-1}{\sigma}\Big)^2\Big]\times$$
$$\times\Big\{-\frac{1}{2}\Big[1-erf\Big(\frac{K_2-1}{\sigma\sqrt{2T}}\Big)\Big]+\frac{\sigma\sqrt{T}}{K_2-1}\varphi\Big(\frac{K_2-1}{\sigma\sqrt{T}}\Big)\Big\}. \tag{3.5}$$

and

$$\int\limits_u^T\frac{1}{\sqrt{t-u}}\Big[\varphi\Big(\frac{K-1-\sigma\widetilde{w}_u}{\sigma\sqrt{t-u}}\Big)\Big]\Big|_{K=K_1}^{K_2}dt=$$
$$=\Big[\sqrt{\frac{2}{\pi}}\sqrt{T-u}\exp\Big\{-\frac{K^2}{2(T-u)}\Big\}+$$
$$+Kerf\Big(\frac{K}{\sqrt{2(T-u)}}\Big)\Big]\Big|_{K=\frac{K_1-1-\sigma\widetilde{w}_u}{\sigma}}^{\frac{K_2-1-\sigma\widetilde{w}_u}{\sigma}}. \tag{3.6}$$

[3]The case of other possible values of constants K_1 and K_2 can be considered similarly.

Proof. It is clear that in our case, the inequality $\frac{K_1-1}{\sigma\sqrt{t}} \leq v \leq \frac{K_2-1}{\sigma\sqrt{t}}$ is equivalent to

$$\Big(\frac{K_1-1}{\sigma v}\Big)^2 \leq t \leq \Big(\frac{K_2-1}{\sigma v}\Big)^2.$$

Therefore, due to the Fubini's Theorem, we have

$$\int\limits_0^T \Big[\Phi\Big(\frac{K-1}{\sigma\sqrt{t}}\Big)\Big]\Big|_{K=K_1}^{K_2} dt = \frac{1}{\sqrt{2\pi}}\int\limits_0^T \int\limits_{\frac{K_1-1}{\sigma\sqrt{t}}}^{\frac{K_2-1}{\sigma\sqrt{t}}} \exp\Big\{-\frac{v^2}{2}\Big\}dvdt =$$

$$= \frac{1}{\sqrt{2\pi}}\int\limits_{\frac{K_1-1}{\sigma\sqrt{T}}}^{\frac{K_2-1}{\sigma\sqrt{T}}} \int\limits_{(\frac{K_1-1}{\sigma v})^2}^{T} \exp\Big\{-\frac{v^2}{2}\Big\}dtdv+$$

$$+\frac{1}{\sqrt{2\pi}}\int\limits_{\frac{K_2-1}{\sigma\sqrt{T}}}^{+\infty} \int\limits_{(\frac{K_1-1}{\sigma v})^2}^{(\frac{K_2-1}{\sigma v})^2} \exp\Big\{-\frac{v^2}{2}\Big\}dtdv =$$

$$= \frac{1}{\sqrt{2\pi}}T\int\limits_{\frac{K_1-1}{\sigma\sqrt{T}}}^{\frac{K_2-1}{\sigma\sqrt{T}}} \exp\Big\{-\frac{v^2}{2}\Big\}dv-$$

$$-\frac{1}{\sqrt{2\pi}}\Big(\frac{K_1-1}{\sigma}\Big)^2\int\limits_{\frac{K_1-1}{\sigma\sqrt{T}}}^{\frac{K_2-1}{\sigma\sqrt{T}}} v^{-2}\exp\Big\{-\frac{v^2}{2}\Big\}dv+$$

$$+\frac{1}{\sqrt{2\pi}}\Big[\Big(\frac{K_2-1}{\sigma}\Big)^2-\Big(\frac{K_1-1}{\sigma}\Big)^2\Big]\int\limits_{\frac{K_2-1}{\sigma\sqrt{T}}}^{+\infty} v^{-2}\exp\Big\{-\frac{v^2}{2}\Big\}dv :=$$

$$= -I_1 - I_2 + I_3. \tag{3.7}$$

Clearly,

$$I_1 := \frac{1}{\sqrt{2\pi}}T\int\limits_{\frac{K_1-1}{\sigma\sqrt{T}}}^{\frac{K_2-1}{\sigma\sqrt{T}}} \exp\Big\{-\frac{v^2}{2}\Big\}dv = T\Big[\Phi\Big(\frac{K-1}{\sigma\sqrt{T}}\Big)\Big]\Big|_{K=K_1}^{K_2}. \tag{3.8}$$

Further, according to the standard technique of integration, it is not difficult to see that for any constants $0 < a < b$ we easily obtain

$$\int_a^b v^{-2} \exp\Big\{-\frac{v^2}{2}\Big\}dv = \Big[-\sqrt{\frac{\pi}{2}}erf\Big(\frac{v}{\sqrt{2}}\Big) - \frac{\exp\{-v^2/2\}}{v}\Big]\Big|_{v=a}^{b}, \tag{3.9}$$

where $erf(\cdot)$ is the so-called error function, i.e.,

$$erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x \exp\{-v^2\}dv.$$

Therefore, on the one hand, we have

$$I_2 := \frac{1}{\sqrt{2\pi}}\Big(\frac{K_1-1}{\sigma}\Big)^2 \int_{\frac{K_1-1}{\sigma\sqrt{T}}}^{\frac{K_2-1}{\sigma\sqrt{T}}} v^{-2}\exp\Big\{-\frac{v^2}{2}\Big\}dv =$$

$$= \Big(\frac{K_1-1}{\sigma}\Big)^2\Big[-\frac{1}{2}erf\Big(\frac{v}{\sqrt{2}}\Big) - \frac{\varphi(v)}{v}\Big]\Big|_{v=\frac{K_1-1}{\sigma\sqrt{T}}}^{\frac{K_2-1}{\sigma\sqrt{T}}}. \tag{3.10}$$

On the other hand, due to the relations $erf(+\infty) = 1$ and

$$\lim_{v\to+\infty}\frac{\varphi(v)}{v} = 0,$$

we conclude that

$$I_3 := \frac{1}{\sqrt{2\pi}}\Big[\Big(\frac{K_2-1}{\sigma}\Big)^2 - \Big(\frac{K_1-1}{\sigma}\Big)^2\Big]\int_{\frac{K_2-1}{\sigma\sqrt{T}}}^{+\infty} v^{-2}\exp\Big\{-\frac{v^2}{2}\Big\}dv =$$

$$= \Big[\Big(\frac{K_2-1}{\sigma}\Big)^2 - \Big(\frac{K_1-1}{\sigma}\Big)^2\Big]\times$$

$$\times\Big\{-\frac{1}{2}\Big[1-erf\Big(\frac{K_2-1}{\sigma\sqrt{2T}}\Big)\Big] + \frac{\sigma\sqrt{T}}{K_2-1}\varphi\Big(\frac{K_2-1}{\sigma\sqrt{T}}\Big)\Big\}. \tag{3.11}$$

Combining now the relations (3.8)-(3.12), we ascertain that the (3.6) is fulfilled.

Due to the standard technique of integration, it is easy to see that for any constants $0 < a < b$, we have

$$\int_a^b \frac{1}{\sqrt{v}}\exp\Big\{-\frac{K^2}{2v}\Big\}dv =$$

$$= \Big[2\sqrt{v}\exp\Big\{-\frac{K^2}{2v}\Big\} + \sqrt{2\pi}K erf\Big(\frac{K}{\sqrt{2v}}\Big)\Big]\Big|_{v=a}^{b}. \tag{3.12}$$

According to the relation (3.13), it is not difficult to verify that the relation (3.7) and therefore the proof is complete. □

In the case $r = 0$, the result of Corollary 3.2 gives us the possibility to find the component γ_t of the hedging strategy $\pi = (\beta_t, \gamma_t)$, $t \in [0, T]$ which is defined by the integrand of representation (3.4) and is equal to

$$\gamma_t = -\frac{1}{\sigma} \int_t^T \frac{1}{\sqrt{u-t}} \Big[\varphi\Big(\frac{K - 1 - \sigma \widetilde{w}_t}{\sigma\sqrt{u-t}}\Big)\Big]\Big|_{K=K_1}^{K_2} du. \tag{3.13}$$

Further, using the result of Corollary 3.2, we can find the capital process

$$X_t = \widetilde{E}[G|\Im_t^{\widetilde{w}}] = \widetilde{E}G +$$
$$+ \int_0^t \int_u^T \frac{1}{\sqrt{v-u}} \Big[\varphi\Big(\frac{K - 1 - \sigma \widetilde{w}_u}{\sigma\sqrt{v-u}}\Big)\Big]\Big|_{K=K_1}^{K_2} dv d\widetilde{w}_u. \tag{3.14}$$

It is well-known (see Shiryaev [1], or (1.5) in this paper) that the second component β_t of hedging strategy π :

$$\beta_t = X_t - \gamma_t S_t. \tag{3.15}$$

Therefore, the hedging strategy $\pi = (\beta_t, \gamma_t)$, $t \in [0, T]$ in the problem of "replication" of Exotic type European Option with payoff G given by (1.4) in the case of Bachelier financial market model, is defined by the relations (3.13), (3.14) and (3.5) and the price C of this option

$$C = \widetilde{E}G = \int_0^T \Big[\Phi\Big(\frac{K-1}{\sigma\sqrt{t}}\Big)\Big]\Big|_{K=K_1}^{K_2} dt.$$

Acknowledgement

The research was partially supported by Shota Rustaveli National Scientific Grants No FR/308/5-104/12, FR/69/5-104/12.

References

1. A. N. Shiryaev, Essentials of Stochastic Finance, Facts, models, theory, (Translated from the Russian by N.Kruzhilin) Advanced Series on Statistical Science and Applied Probability, 3. *World Scientific Publishing, New Jersey, London, Singapore, Hong Kong* (2003).
2. M. C. Clark, The representation of functionals of Brownian motion by stochastic integrals. *Ann. Math. Stat.*, **41** (1970), 1282–1295 and **42** (1971), 1778.
3. D. Ocone, Malliavin's calculus and stochastic integral representations of functionals of diffusion processes. *Stochastics* **12** (1984), No. 3-4, 161–185.
4. D. Nualart, The Malliavin calculus and related topics. Second edition. Probability and its Applications (New York). *Springer-Verlag, Berlin*, 2006
5. O. Glonti and O. Purtukhia, Clark's representation of Wiener functionals and hedging of the barrier option. *Bull. Georgian Natl. Acad. Sci. (N.S.)* **8** (2014), No. 1, 32–39.

6. O. A. Glonti, Investigations in the theory of conditionally Gaussian processes. (Russian) *Metsniereba, Tbilisi*, 1985.
7. N. Ikeda and Sh. Watanabe, Stochastic differential equations and diffusion processes. North-Holland Mathematical Library, 24. *North-Holland Publishing Co., Amsterdam-New York; Kodansha, Ltd., Tokyo*, 1981.
8. V. Jaoshvili and O. Purtukhia, Stochastic integral representation of functionals of Wiener processes. *Bull. Georgian Acad. Sci.* **171** (2005), No. 1, 17–20.

(Received 28.03.2015)

Authors' addresses:

O. Purtukhia

A. Razmadze Mathematical Institute
I. Javakhishvili Tbilisi State University
6, Tamarashvili st., Tbilisi 0177, Georgia
E-mail: o.purtukhia@gmail.com

V. Jaoshvili

13, University st., Tbilisi, 0177, Georgia
E-mail: vakhtangi.jaoshvili@gmail.com