Available online at www.sciencedirect.com
ScienceDirect
Transactions of A. Razmadze Mathematical Institute 172 (2018) 420–428
Transactions of A. Razmadze Mathematical Institute
www.elsevier.com/locate/trmi

Original article

On functionals of the Wiener process in a Banach space

Badri Mamporia[a,*], Omar Purtukhia[b]

[a] *Georgian Technical University N. Muskhelishvili Institute of Computational Mathematics, 4 Grigol Peradze st., Tbilisi 0131, Georgia*
[b] *Iv. Javakhishvili Tbilisi State University, A.Razmadze Mathematical Institute, Faculty of Exact and Natural Sciences, Department of Mathematics, Tbilisi, Georgia*

Received 1 June 2018; received in revised form 27 July 2018; accepted 29 July 2018
Available online 14 August 2018

Abstract

In development of stochastic analysis in a Banach space one of the main problem is to establish the existence of the stochastic integral from predictable Banach space valued (operator valued) random process. In the problem of representation of the Wiener functional as a stochastic integral we are faced with an inverse problem: we have the stochastic integral as a Banach space valued random element and we are looking for a suitable predictable integrand process. There are positive results only for a narrow class of Banach spaces with special geometry (UMD Banach spaces). We consider this problem in a general Banach space for a Gaussian functional.

Keywords: Wiener process; Functional of the Wiener process; Ito stochastic integral; Covariance operator in Banach space

1. Introduction and preliminaries

The problem of construction of the Ito stochastic integral in a Banach space is developing in three directions. In the first (relatively) direction the integrand is Banach space valued predictable random process and the stochastic integral is taken by the one dimensional Wiener process. In the second direction the integrand is operator valued (from Banach space to Banach space) predictable random process and stochastic integral is taken from Wiener process in a Banach space. In the third direction the integrand is operator-valued (from Hilbert space to Banach space) predictable process and stochastic integral is taken from cylindrical Wiener process in a Hilbert space. In all of these cases difficulties are the same. Therefore, for simplicity, in this article we consider the first case (Wiener process is one dimensional).

Using traditional methods, to find the suitable conditions that guarantee the construction of the stochastic integral is possible only in a very narrow class of Banach spaces. This class is so called UMD Banach spaces class (see survey in [1]). We consider the generalized stochastic integral for a wide class of predictable random processes and

* Corresponding author.
E-mail addresses: badrimamporia@yahoo.com (B. Mamporia), o.purtukhia@gmail.com (O. Purtukhia).
Peer review under responsibility of Journal Transactions of A. Razmadze Mathematical Institute.

https://doi.org/10.1016/j.trmi.2018.07.007

the problem of existence of the stochastic integral we reduced to the problem of decomposability of the generalized random element (see [2]).

In this article we consider the problem of representation of the Wiener functional by the stochastic integral in an arbitrary separable Banach space. This problem is, in some sense, opposite to the problem of existence of the stochastic integral: here we have the stochastic integral as a random element and the problem is to find the integrand as a Banach space valued predictable process. In this direction there exists the following result in UMD Banach space case: under special condition every Wiener functional is represented by the stochastic integral and is generalized the Clark–Ocone formula of representation of the functional of the Wiener process by Malliavin derivative (see [3]).

Let X be a real separable Banach space. X^* – its conjugate, (Ω, B, P) – a probability space. Let $(W_t)_{t\in[0,1]}$ -be a real valued Wiener process. Denote by F_t^W the minimal σ-algebra generated by the random variables $(W_s)_{s\leq t}$ $(F_t^W = \sigma(W_s, s \leq t))$. The random element ξ is a weak second order if for all x^*, $E\langle\xi, x^*\rangle^2 < \infty$. Suppose that ξ is F_1^W- measurable i.e., ξ is the functional of the Wiener process. Our main aim is to represent the random element ξ by the Ito stochastic integral

$$\xi = E\xi + \int_0^1 f(t, \omega) dW_t,$$

where $f(t, \omega)$ is Banach space valued predictable random process. In the development of this difficult problem firstly, in this article, we consider the case when ξ is a Gaussian random element which with the Wiener process generates mutually Gaussian system. In this case the integrand (if it exists) will be nonrandom function. Remember, that the continuous linear operator $T : X^* \to L_2(\Omega, B, P)$ is called the generalized random element (GRE).[1]

Denote by $\mathcal{M}_1 := L(X^*, L_2(\Omega, B, P))$ the Banach space of GRE with the norm

$$\|T\|^2 = \sup_{\|x^*\|\leq 1} E(Tx^*)^2.$$

We can realize the weak second order random element ξ as an element of $\mathcal{M}_1$, $T_\xi x^* = \langle\xi, x^*\rangle$, but not conversely: in infinite dimensional Banach space for all $T : X^* \to L_2(\Omega, B, P)$, there does not always exist the random element $\xi : \Omega \to X$ such that $Tx^* = \langle\xi, x^*\rangle$ for all $x^* \in X^*$. The problem of existence of such random element is the well known problem of decomposability of the GRE. Denote by $\mathcal{M}_2$ the linear normed space of all random elements of the weak second order with the norm

$$\|\xi\|^2 = \sup_{\|x^*\|\leq 1} E\langle\xi, x^*\rangle^2.$$

Thus, we have $M_2 \subset M_1$. The family of random processes $(T_t)_{t\in[0,1]}$ is called the generalized random processes (GRP). In this paper we will consider the linear bounded operators $T : X^* \to L_2[0, 1]$. In this special case instead of $L_2(\Omega, B, P)$, we have $L_2([0, 1], B([0, 1]), \lambda)$, Nevertheless we use the term GRE in this special case too. The decomposability problem is: for the GRE $T : X^* \to L_2[0, 1]$ existence of the weak second order function $f : [0, 1] \to X$ such that for all $x^* \in X^*$, $Tx^* = \langle f, x^*\rangle$ λ-a.e. Denote by $\mathcal{M}_1^\lambda$ the linear space of GRE $T : X^* \to L_2[0, 1]$. $\mathcal{M}_1^\lambda$ is a Banach space with the norm

$$\|T\|^2 = \sup_{\|x^*\|\leq 1} \int_0^1 [Tx^*(t)]^2 dt = \sup_{\|x^*\|\leq 1} \|Tx^*\|_{L_2}^2.$$

Denote by $\mathcal{M}_2^\lambda$ the linear space of functions $f : [0, 1] \to X$, such that $\int_0^1 \langle f(t), x^*\rangle^2 < \infty$. $\mathcal{M}_2^\lambda \subset \mathcal{M}_1^\lambda$.

2. Integral representation of functionals

For simplicity assume that $E\xi = 0$.

Proposition 2.1. *Let ξ be a F_1^W-measurable Gaussian random element. There exists a GRE $T : X^* \to L_2[0, 1]$, such that for all $x^* \in X^*$,*

$$\langle\xi, x^*\rangle = \int_0^1 Tx^*(t) dW_t. \tag{2.1}$$

[1] sometimes it is used the terms: random linear function or cylindrical random element.

Proof. We use the technique developed in one dimensional case (see [4]). Denote

$$F_n^W = \sigma\{W_{\frac{1}{2^n}}, W_{\frac{2}{2^n}}, \ldots, W_1\} = \sigma\{W_{\frac{1}{2^n}}, (W_{\frac{2}{2^n}} - W_{\frac{1}{2^n}}), \ldots, (W_1 - W_{\frac{2^n-1}{2^n}})\}$$
$$= \sigma\{2^{\frac{n}{2}} g_1, 2^{\frac{n}{2}} g_2, \ldots, 2^{\frac{n}{2}} g_{2^n}\} = \sigma\{g_1, g_2, \ldots, g_{2^n}\},$$

where

$$g_1, g_2, \ldots, g_{2^n}, \quad g_i = 2^{\frac{n}{2}}(W_{\frac{i+1}{2^n}} - W_{\frac{i}{2^n}})$$

are independent, standard Gaussian random variables. Denote $\xi_n \equiv E(\xi|F_n^W)$ — the conditional mathematical expectation.

It is obvious that

$$\xi_n = E(\xi|F_n^W) = \sum_{i=0}^{2^n-1} E(\xi g_i) g_i$$
$$= \sum_{i=0}^{2^n-1} 2^n E(\xi(W_{\frac{i+1}{2^n}} - W_{\frac{i}{2^n}})(W_{\frac{i+1}{2^n}} - W_{\frac{i}{2^n}})) = \int_0^1 f_n(t) dW_t,$$

where

$$f_n(t) = \sum_{i=0}^{2^n-1} 2^n E\xi(W_{\frac{i+1}{2^n}} - W_{\frac{i}{2^n}}) I_{(\frac{i}{2^n}, \frac{i+1}{2^n}]}(t).$$

As $(\xi_n)_{n\in N}$ is Gaussian martingale, we have

$$E\|\xi_n - \xi_m\|^2 = \left\| \int_0^1 f_n(t) dW_t - \int_0^1 f_m(t) dW_t \right\|^2 \to 0.$$

That is

$$\xi = \lim_{n\to\infty} \int_0^1 f_n(t) dW_t.$$

For all $x^* \in X^*$ denote $Tx^*(t) = \lim_{n\to\infty}\langle f_n(t), x^*\rangle$. We have

$$\sup_{\|x^*\|\le 1} \int_0^1 (Tx^*(t))^2 dt = \sup_{\|x^*\|\le 1} E\langle \xi, x^*\rangle^2 \le E\|\xi\|^2 < \infty.$$

Therefore, $T : X^* \to L_2[0, 1]$ is GRE and

$$\langle \xi, x^*\rangle = \int_0^1 Tx^*(t) dW_t. \quad \square$$

Remark 2.1. Note that if we have two representations of the F_1^W measurable random element ξ, by the stochastic integral

$$\langle \xi, x^*\rangle = \int_0^1 T_1 x^*(t) dW_t = \int_0^1 T_2 x^*(t) dW_t,$$

then

$$0 = \sup_{\|x^*\|\le 1} E\left(\int_0^1 (T_1x^*(t) - T_2x^*(t)) dW(t)\right)^2 = \sup_{\|x^*\|\le 1} \int_0^1 ((T_1 - T_2)x^*)^2 dt.$$

Hence, the representation (2.1) is unique.

The main problem is to find the function $f : [0, 1] \to X$ such that $Tx^*(t) = \langle f(t), x^*\rangle$ a. e. for all $x^* \in X^*$. In this case we will have $\xi = \int_0^1 f(t) dW_t$. The following example shows that, in general, such function (even $f : [0, 1] \to X^{**}$) does not exist.

Example. Let $X = c_0$ and denote $e_{k,n}(t) = a_n I_{(\frac{k-1}{n},\frac{k}{n}]}(t)$, $n \in N, k = 1, 2, \ldots, n$, the sequence $(a_n)_{n\in N}$ we will choose later. Suppose $f(t) \equiv (e_{kn}(t))_{n\in N,k\leq n}$, $t \in [0, 1]$. $f : [0, 1] \to R^N$. Consider the map $T(t) : l_1 \to R^1$, $t \in [0, 1]$.

$$T(t)\vec{\lambda} = \sum_{n=1}^{\infty}\sum_{k=1}^{n} \lambda_{nk} e_{nk}(t),$$

where $\vec{\lambda} = (\lambda_{nk})_{n\in N,k\leq n} \in l_1$ $(\sum_{n=1}^{\infty}\sum_{k=1}^{n} \|\lambda_{nk}\| < \infty)$.

We will show that T is the linear bounded operator from l_1 to $L_2[0, 1]$. We have

$$\int_0^1 (T(t)(\vec{\lambda}))^2 dt = \int_0^1 \sum_{n=1}^{\infty}\sum_{k=1}^{n}\sum_{m=1}^{\infty}\sum_{l=1}^{m} \lambda_{nk}\lambda_{ml} a_n a_m I_{(\frac{k-1}{n},\frac{k}{n}]}(t) I_{(\frac{l-1}{m},\frac{l}{m}]}(t).$$

If we demand that $|a_n| \leq n^{\frac{1}{2}}$, we receive

$$\int_0^1 (T(t)(\vec{\lambda}))^2 dt \leq \int_0^1 \sum_{n=1}^{\infty}\sum_{k=1}^{n}\sum_{m=1}^{\infty}\sum_{l=1}^{m} \lambda_{nk}\lambda_{ml}(mn)^{\frac{1}{2}} \min\Big(\frac{1}{n}, \frac{1}{m}\Big)$$
$$\leq \sum_{n=1}^{\infty}\sum_{k=1}^{n}\sum_{m=1}^{\infty}\sum_{l=1}^{m} |\lambda_{nk}||\lambda_{ml}| = \|\vec{\lambda}\|_{l_1}^2.$$

Therefore, $T : c_0^* \to L_2[0, 1]$ is bounded linear operator, $\|T\| \leq 1$.

On the other hand,

$$\int_0^1 T(t)\vec{\lambda} dw_t = \sum_{n=1}^{\infty}\sum_{k=1}^{n} \lambda_{nk} a_n (W_{\frac{k}{n}} - W_{\frac{k-1}{n}}) = \langle\vec{\lambda}, \xi\rangle,$$

where $\xi := ((W_{\frac{k}{n}} - W_{\frac{k-1}{n}})a_n)_{n\in N,k\leq n}$.

Let us check the sufficient condition of N. Vakhania (see [5], prop. 5.5.8) on the belonging of the random element ξ to the Banach space c_0. If we denote $\sigma_{k,n} = E[a_n(W_{\frac{k}{n}} - W_{\frac{k-1}{n}})]^2 = \frac{a_n^2}{n}$, then

$$\sum_{n=1}^{\infty}\sum_{k=1}^{n} \exp\Big(-\frac{\mu}{\sigma_{kn}}\Big) = \sum_{n=1}^{\infty}\sum_{k=1}^{n} \exp\Big(-\frac{n\mu}{a_n^2}\Big).$$

For example, if $a_n = n^{\frac{1}{3}}$, then the series $\sum_{n=1}^{\infty} nexp(-n^{\frac{1}{3}}\mu)$ converges for all fixed μ. Therefore, by the theorem N. Vakhania $\xi \in c_0$.

In this example we have $\langle\xi, x^*\rangle = \int_0^1 T(t)x^* dW_t$, where $T(t) : X^* \to L_2[0, 1]$. There does not exist $f : [0, 1] \to X$ or $f : [0, 1] \to X^{**}$ such that $T(t)x^* = \langle f(t), x^*\rangle$ (if $a_n \to 0$, then $f \in X$; if $a_n = 1, n = 1, 2, \ldots,$ then $f \in X^{**}$; if $a_n \to \infty$, $a_n \leq n^{\frac{1}{3}}$, then $f \bar{\in} X^{**}$).

Proposition 2.2. *Let $\xi : \Omega \to X$ be Gaussian F_1^W measurable random element. $T_\xi : X^* \to L_2[0, 1]$ be such that $\langle\xi, x^*\rangle = \int_0^1 Tx^*(t)dW(t)$, then $T_\xi \in \bar{\mathcal{M}}_2^\lambda \subseteq \mathcal{M}_1^\lambda$, $T^*T : X^* \to X \subset X^{**}$ is Gaussian covariance, there exists $a_\xi \in X$ such that*

$$\int_0^1 T_\xi x^*(t)dt = \langle a, x^*\rangle, \quad for \;\; all \;\; x^* \in X^*.$$

Proof. From the proof of Proposition 2.1, we have

$$f_n(t) = \sum_{i=0}^{2^n-1} 2^n E\xi(W_{\frac{i+1}{2^n}} - W_{\frac{i}{2^n}}) I_{(\frac{i}{2^n},\frac{i+1}{2^n}]}(t), \quad f_n \in \mathcal{M}_2^\lambda$$

and

$$\|f_n - T\|^2_{\mathcal{M}_1^\lambda} = \sup_{\|x^*\|\le 1} \int_0^1 T_\xi x^*(t) - \langle f_n(t), x^*\rangle^2 dt \to 0.$$

Therefore, $T_\xi \in \bar{\mathcal{M}}_2^\lambda$.

As $T_\xi \in \bar{\mathcal{M}}_2^\lambda$, by the proposition 1 of [2] $T_\xi^* T_\xi : X^* \to X$. This statement follows also from the equality

$$\langle T_\xi^* T_\xi x^*, y^*\rangle = \int_0^1 T_\xi x^*(t) T_\xi y^*(t)dt = E\langle \xi, x^*\rangle\langle \xi, y^*\rangle = \langle R_\xi x^*, y^*\rangle$$

and as $R_\xi : X^* \to X$, then $T^*T : X^* \to X \subset X^{**}$ too.

Further, we have

$$\int_0^1 f_n(t)dt = \sum_{i=0}^{2^n-1} 2^n E\xi(W_{\frac{i+1}{2^n}} - W_{\frac{i}{2^n}})2^{-n} = \sum_{i=0}^{2^n-1} E\xi(W_{\frac{i+1}{2^n}} - W_{\frac{i}{2^n}})$$
$$= E\xi\left(\sum_{i=0}^{2^n-1}(W_{\frac{i+1}{2^n}} - W_{\frac{i}{2^n}})\right) = E\xi W_1.$$

Denote $a := E(\xi W_1)$, then we have

$$\int_0^1 T_\xi x^*(t)dt = \langle a, x^*\rangle, \quad for \ \ all \ \ x^* \in X^*.$$

Whereas that T_ξ is not X-valued function, the integral $\int_0^1 T_\xi(.)(t)dt$ is X-valued, that is, the Pettis integral from T_ξ exists. More clearly, if there exists $f : [0, 1] \to X$ such that $T_\xi x^* = \langle f(t), x^*\rangle$ for all $x^* \in X^*$, then a is the Pettis integral from f. □

Proposition 2.3. *For any X-valued F_1^W-measurable Gaussian random element ξ and $g \in L_2[0, 1]$, there exists $a_g \in X$ such that*

$$\int_0^1 T_\xi x^*(t)g(t)dt = \langle a_g, x^*\rangle, for \ \ all \ \ x^* \in X^*.$$

Proof. Consider the family of σ-algebras

$$F_n = \sigma\left\{\left(\frac{i}{2^n}, \frac{i+1}{2^n}\right], \ \ i = 0, 1, \ldots, 2^n\right\}.$$

It is evident that $F_n \subset F_{n+1}$, $n = 1, 2, \ldots$; $E(g|F_n) \to g \ \ in \ \ L_2[0, 1]$;

$$E(g|F_n)(t) = \sum_{i=0}^{2^n-1} 2^n \int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} g(s)ds I_{(\frac{i}{2^n}, \frac{i+1}{2^n}]}(t);$$

$$\int_0^1 T_\xi x^*(t)g(t)dt = \lim_{n\to\infty} \int_0^1 \sum_{i=0}^{2^n-1} 2^n E(\xi(W_{\frac{i+1}{2^n}} - W_{\frac{i}{2^n}}))I_{(\frac{i}{2^n}, \frac{i+1}{2^n}]}(t) \sum_{i=0}^{2^n-1} g_i I_{(\frac{i}{2^n}, \frac{i+1}{2^n}]}(t)dt$$
$$= \lim_{n\to\infty} \int_0^1 \sum_{i=0}^{2^n-1} [2^n E(\xi(W_{\frac{i+1}{2^n}} - W_{\frac{i}{2^n}})g_i)]I_{(\frac{i}{2^n}, \frac{i+1}{2^n}]}(t)dt$$
$$= \lim_{n\to\infty} \sum_{i=0}^{2^n-1} [E[\xi(W_{\frac{i+1}{2^n}} - W_{\frac{i}{2^n}})g_i]]$$
$$= \lim_{n\to\infty} E\xi(\sum_{i=0}^{2^n-1}(W_{\frac{i+1}{2^n}} - W_{\frac{i}{2^n}})g_i) = E\xi \int_0^1 g(t)dW_t,$$

where

$$g_i \equiv 2^n \int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} g(t)dt.$$

As

$$E\|\xi \int_0^1 g(t)dW_t\| \leq (E\|\xi\|^2)^{\frac{1}{2}} + (\int_0^1 g(t)^2 dt)^{\frac{1}{2}} < \infty,$$

the Bochner integral $\int_0^1 T_\xi x^*(t)g(t)dt \equiv a_g$ exists. □

3. Representation of functionals in the form of series

Consider now the minimal subspace of $L_2(\Omega, B, P)$, consisting only random variables $(W_t, t \in [0, 1])$. Denote $G_0 = L(W_t, t \in [0, 1])$. G_0 consists only Gaussian Random variables. Denote $G := \bar{G_0}$. $G \subset L_2(\Omega, B, P)$ is a Hilbert space. As limit of Gaussian random variables is also Gaussian, G-contains only Gaussian random variables. Denote $\xi_t = E(\xi|F_t^W)$, $t \in [0, 1]$. $\xi_t : \Omega \to X$. $(\xi_t)_{t\in[0,1]}$ is a Gaussian process of independent increments. Firstly we consider one dimensional case and give the representation of the functional of the Wiener process by the sum of independent Gaussian random variable.

Theorem 3.1. *Let $\xi_t : \Omega \to R^1$ be F_t^W-measurable Gaussian random process, $f : [0, 1] \to R^1$ be such, that*

$$\xi(t) = \int_0^t f(\tau)dW(\tau).$$

For any orthonormal Basis $(e_n)_{n\in N}$ of $L_2[0, 1]$, there exists the sequence of independent, identically distributed, standard Gaussian random variables $(g_n)_{n\in N}$, such that

$$\xi_t = \sum_{k=1}^{\infty} \int_0^t f(\tau)e_k(\tau)d\tau g_k.$$

The convergence of the sum is a.s. uniformly in t.

Proof. As $(W_t)_{t\in[0,1]}$ has a.s. continuous sample paths, we can consider the corresponding $C[0, 1]$ valued random element $W : \Omega \to C[0, 1]$. The covariance operator $R_W : C[0, 1]^* \to C[0, 1]$,

$$R_W\varphi(t) = \int_0^1 \min(t, s)d\varphi(s)$$

admits the factorization (see [4] factorization lemma 3.1.1) through the Hilbert space $L_2[0, 1]$, $R_W = AA^*$:

$$A : L_2[0, 1] \to C[0, 1], \quad Ah(t) = \int_0^t h(\tau)d\tau, \quad A^*(t) : C[0, 1]^* \to L_2[0, 1],$$
$$A^*\delta_t = \chi_{[0,t]}(\tau), \quad \delta_t \in C[0, 1]^*, \quad \langle\delta_t, \psi\rangle = \psi(t), \quad \psi \in C[0, 1], \quad t \in [0, 1].$$

We have also another factorization of R_W: $R_W = T^*T$, $T : C[0, 1]^* \to G$, $T\delta_t = \langle W, \delta_t\rangle = W_t$. By the factorization lemma, there exists the isometric operator $I : G \to L_2[0, 1]$, such that $IT = A^*$. Therefore, $I(T\delta_t) = I(W_t) = A^*\delta_t = \chi_{[0,t]}(\tau)$. Thus,

$$EW_t g_k = \langle T\delta_t, g_k\rangle = \langle IT\delta_t, e_k\rangle = \langle\chi_{[0,t]}, e_k\rangle = \int_0^t e_k(\tau)d\tau.$$

Accordingly, we have $W_t = \sum_{k=1}^{\infty} E(W_t g_k)g_k$. Therefore

$$W_t = \sum_{k=1}^{\infty} \int_0^t e_k(\tau)d\tau g_k. \tag{3.1}$$

We have convergence of sum (3.1) in $C[0, 1]$. Therefore, this formula gives representation of the Wiener process by the a.s. uniformly in t convergent sum of independent Gaussian random variables.

Return now to the functional of the Wiener process $\xi = \int_0^1 f(t)dW_t$. $\xi_t = E(\xi|F_t^W) = \int_0^t f(\tau)dW_\tau$. $(\xi_t)_{t\in[0,1]}$ is a Gaussian process of independent increments. Dispersion of the random variable ξ_t is $\int_0^t f^2(\tau)d\tau$. As $(\xi_t)_{t\in[0,1]}$ is the process with continuous sample paths, we can consider corresponding random element $\xi : \Omega \to C[0,1]$. The covariance operator of this random element is $R_\xi : C[0,1]^* \to C[0,1]$,

$$R_\xi\varphi(t) = \int_0^1 \min(t,s)f(s)d\varphi(s), \quad R_\xi = AA^*, \quad A : L_2[0,1] \to C[0,1],$$
$$Ah(t) = \int_0^t h(\tau)f(\tau)d(\tau), \quad A^* : C[0,1]^* \to L_2[0,1], \quad A^*\delta_t = \chi_{[0,t]}(\tau)f(\tau).$$

It is clear that

$$\langle R\delta_t, \delta_s\rangle = \langle AA^*\delta_t, \delta_s\rangle = \langle T\delta_t, T\delta_s\rangle$$
$$= \int_0^1 \chi_{[0,t]}(\tau)f(\tau)\chi_{[0,s]}(\tau)f(\tau)d\tau = \int_0^{\min(t,s)} f^2(\tau)d\tau.$$

Further, we have:

$$\xi = \int_0^1 f(t)dW_t = \lim_{n\to\infty}\int_0^1 f_n(t)dW_t,$$

where

$$f_n(t) = \sum_{i=0}^{2^n-1} 2^n E\xi(W_{\frac{i+1}{2^n}} - W_{\frac{i}{2^n}})I_{(\frac{i}{2^n},\frac{i+1}{2^n}]}(t).$$

Hence,

$$\xi = \lim_{n\to\infty}\sum_{i=0}^{2^n-1} 2^n E\xi(W_{\frac{i+1}{2^n}} - W_{\frac{i}{2^n}})(W_{\frac{i+1}{2^n}} - W_{\frac{i}{2^n}})$$
$$= \lim_{n\to\infty}\sum_{i=0}^{2^n-1} 2^n E\xi(W_{\frac{i+1}{2^n}} - W_{\frac{i}{2^n}})\sum_{k=1}^{\infty}\int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} e_k(\tau)d(\tau)g_k$$
$$= \lim_{n\to\infty}\sum_{k=1}^{\infty}\int_0^1\sum_{i=0}^{2^n-1} 2^n E\xi(W_{\frac{i+1}{2^n}} - W_{\frac{i}{2^n}})I_{(\frac{i}{2^n},\frac{i+1}{2^n}]}(\tau)e_k(\tau)d(\tau)g_k$$
$$= \lim_{n\to\infty}\sum_{k=1}^{\infty}\int_0^1 f_n(\tau)e_k(\tau)d(\tau)g_k = \sum_{k=1}^{\infty}\int_0^1 f(\tau)e_k(\tau)d(\tau)g_k,$$

as

$$\lim_{n\to\infty} E\left(\sum_{k=1}^{\infty}\int_0^1 f(\tau)e_k(\tau)d(\tau)g_k - \sum_{k=1}^{\infty}\int_0^1 f_n(\tau)e_k(\tau)d(\tau)g_k\right)^2$$
$$= \lim_{n\to\infty} E\left(\sum_{k=1}^{\infty}\int_0^1 (f_n(\tau) - f(\tau))e_k(\tau)d(\tau)g_k\right)^2$$
$$= \lim_{n\to\infty}\sum_{k=1}^{\infty}\left(\int_0^1 (f_n(\tau) - f(\tau))e_k(\tau)d(\tau)\right)^2$$
$$= \lim_{n\to\infty}\sum_{k=1}^{\infty}\langle (f(\tau) - f_n(\tau)), e_k\rangle^2 = \lim_{n\to\infty}\| f(\tau) - f_n(\tau)\|^2_{L_{[0,1]}} \to 0.$$

We received that

$$\xi = \int_0^1 f(t)dW_t = \sum_{k=1}^{\infty}\int_0^1 f(\tau)e_k(\tau)d(\tau)g_k.$$

Therefore

$$\xi_t = \sum_{k=1}^{\infty} \int_0^t f(\tau) e_k(\tau) d(\tau) g_k. \tag{3.2}$$

Consider the partial sum

$$(\xi_t)_n = \sum_{k=1}^{n} \int_0^t f(\tau) e_k(\tau) d(\tau) g_k$$

of independent, $C[0, 1]$-valued random elements. By the Ito–Nisio theorem [6], we have convergence of the last sum in $C[0, 1]$. Therefore, the last sum converges a.s. uniformly in t. □

Remark 3.1. N.Wiener [7] shows that the series

$$W_t \equiv g_0 t + \sum_{k=1}^{\infty} \sum_{n=2^{k-1}}^{2^k-1} n^{-1} g_n \sqrt{2} \sin \pi n t$$

converges (along an appropriate subsequence) uniformly in t to the Wiener process. Paul Levy [8] simplified Wiener's construction using Haar functions. Z.Ciesielsky [9] proved a.s. uniformly in t convergence of the series (3.1) in case, when $(e_k)_{k\in N}$ are Haar functions. K.Ito and M.Nisio [6] proved convergence of the sum (3.1) uniformly in t for an arbitrary orthonormal Basis $(e_k)_{k\in N}$ of $L_2[0, 1]$. Representation any fixed Wiener process by the sum (3.1) requires an additional effort, for example, to use the factorization lemma.

Consider now the corresponding problem for Banach space valued Wiener functional. Let $\xi : \Omega \to X$ be F_1^W measurable Gaussian random element. By Proposition 2.1, there exists GRE $T : X^* \to L_2[0, 1]$ such that

$$\langle \xi, x^* \rangle = \int_0^1 T x^*(t) dW_t,$$

for all $x^* \in X^*$. Denote $\xi_t = E(\xi | F_t^W)$. $(\xi_t)_{t\in[0,1]}$ is the Gaussian process with independent increments in a Banach space X. By continuity of the family $(F_t^W)_{t\in[0,1]}$ follows stochastically continuity of the process $(\xi_t)_{t\in[0,1]}$ and, as it is a stochastically continuous Gaussian process with independent increments, it has continuous sample paths (see [10]). Therefore, we can consider the corresponding random element in a Banach space $C([0, 1], X)$. The following theorem is similar to Theorem 3.1 for the case of Banach space X.

Theorem 3.2. *Let $\xi_t : \Omega \to X$ be F_t^W-measurable Gaussian random process, $T_\xi : X^* \to L_2[0, 1]$ be corresponding GRE, $\xi_t = \int_0^t T_\xi x^*(\tau) dW_\tau$. For any orthonormal basis $(e_n)_{n\in N}$ of $L_2[0, 1]$ there exists the sequence of independent, identically distributed, standard Gaussian random variables $(g_n)_{n\in N}$ such that*

$$\xi_t = \sum_{k=1}^{\infty} \int_0^t T_\xi(\tau) e_k(\tau) d\tau g_k. \tag{3.3}$$

The elements of the sum are X-valued and convergence of the sum is a.s. uniformly in t in X.

Proof. For any fixed orthonormal basis $(e_n)_{n\in N}$ of $L_2[0, 1]$, there exists the sequence of independent, identically distributed, standard Gaussian random variables $(g_n)_{n\in N}$ such that $W_t = \int_0^t e_k(\tau) d\tau g_k$. By Theorem 3.1, for any $x^* \in X^*$, we have

$$\langle \xi_t, x^* \rangle = \sum_{k=1}^{\infty} \int_0^t T_\xi x^*(\tau) e_k(\tau) d\tau g_k.$$

By Proposition 2.3 for an arbitrary $k \in N$, and $t \in [0, 1]$ $\int_0^t T_\xi(\tau) e_k(\tau) d\tau \equiv a_k(t)$ belongs to X. As $(\xi_t)_{t\in[0,1]}$ is X-valued Gaussian process of independent increments with continuous sample paths, we can consider corresponding random element with values in Banach space valued continuous functions: $\bar{\xi} : \Omega \to C([0, 1], X)$. In the process of proof of Proposition 2.1 we have considered the sequence

$$f_n(t) = \sum_{i=0}^{2^n-1} 2^n E\xi(W_{\frac{i+1}{2^n}} - W_{\frac{i}{2^n}}) I_{(\frac{i}{2^n}, \frac{i+1}{2^n}]}(t),$$

for which we have $\| f_n - T_\xi \|_{M_1^\lambda} \to 0$ and

$$\xi = \int_0^1 T_\xi x^*(t) dW_t = \lim_{n\to\infty} \int_0^1 f_n(t) dW_t.$$

Analogously to the proof of Theorem 3.1 we obtain

$$\xi = \lim_{n\to\infty} \int_0^1 f_n(t) dW_t = \lim_{n\to\infty} \sum_{i=0}^{2^n-1} 2^n E\xi(W_{\frac{i+1}{2^n}} - W_{\frac{i}{2^n}})(W_{\frac{i+1}{2^n}} - W_{\frac{i}{2^n}})$$

$$= \lim_{n\to\infty} \sum_{k=1}^{\infty} \int_0^1 f_n(\tau) e_k(\tau) d(\tau) g_k = \sum_{k=1}^{\infty} \int_0^1 T_\xi(\tau) e_k(\tau) d(\tau) g_k$$

According to Proposition 2.3 we have

$$\int_0^1 T_\xi(\tau) e_k(\tau) d(\tau) = \int_0^1 a_k(\tau) d(\tau) \in X.$$

Therefore, the last sum converges in X to ξ.

Consider the partial sum

$$\sum_{k=1}^{n} \int_0^t T_\xi(\tau) e_k(\tau) d(\tau) g_k \tag{3.4}$$

of independent $C([0, 1], X)$ valued random elements. By the Ito–Nisio Theorem this sum converges in $C([0, 1], X)$ to $(\xi_t)_{t\in[0,1]}$, therefore, we have a.s. uniformly in t convergence of the partial sum (3.4) to the sum (3.3). □

References

[1] J.M.A.M. van Neerven, M. Veraar, L. Weis, Stochastic Integration in UMD Banach Spaces –A Survey, in: Stochastic Analysis: A Series of Lectures, 2015, pp. 297–332.
[2] B. Mamporia, Stochastic differential equation for generalized stochastic processes in a Banach space, Teor. Veroyatn. Primen. 56 (4) (2011) 704—725; Theory of Probability and its Applications, SIAM, 2012, 602–620 (4).
[3] J. Maas, Jan, J. van Neerven, A Clark-Ocone formula in UMD banach spaces, Electron. Comm. Probab. 13 (2008) 151–164.
[4] R.S. Liptser, A.N. Shiriaev, Statistics of Random Processes, Nauka, Springer, Moscow, 1974 (in Russian), (2001).
[5] N.N. Vakhania, V.I. Tarieladze, S.A. Chobanyan, Probability Distributions on Banach Spaces, Nauka, Moscow, 1985. English translation: Reidel, Dordrecht, the Netherlands, 1987.
[6] K. Ito, M. Nisio, On the convergence of sums of independent Banach space valued random variables, Osaka J. Math. 5 (1968) 35–48.
[7] N. Wiener, Differential space, J. Math. Phys. 58 (1923) 131–174.
[8] P. Levy, Processus Stochastiques et Mouvement Brownien. Suivi d'une Note de M. Loéve, Gauthier-Villars, Paris, 1948 (French).
[9] Z. Ciesielski, Hölder conditions for realizations of Gaussian processes, Trans. Amer. Math. Soc. 99 (1961) 403–413.
[10] I.I. Gikhman, A.V. Skorokhod, The Theory of Stochastic Processes. II, Nauka, Moscow, 1973. Translated from the Russian by S Kotz. Reprint of the 1975 edition. Classics in Mathematics. Springer-Verlag, Berlin, 2004.