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BANACH SPACE VALUED FUNCTIONALS OF THE ONE DIMENSIONAL WIENER PROCESS

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Abstract. In this article is considered the problem of representation of the Banach space-valued functionals of the one dimensional Wiener process by the Ito stochastic integral. Earlier in [1] we developed this problem in case, when the joint distribution of the Wiener process and its functional is Gaussian. Here is developed the general case: firstly for the weak second order Banach space valued functional is found the generalized random process as an integrand. Further, for the one dimensional functional of the Wiener process the sequence of adapted step functions converging to the integrand function is found, which is generalization of the corresponding one dimensional result for the Gaussian case (see [2]). In addition, a sequence of adapted step functions of generalized random elements that converge to an integrand of generalized random process is constructed.

Keywords and phrases: Functional of the Wiener process, Ito stochastic integral, generalized random element, covariance operator of the (generalized) random element in a Banach space.

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1 Introduction and preliminaries. In developing of the Ito stochastic analysis in a Banach space the main problem is construction of the stochastic integral in an arbitrary separable Banach space. This problem is considered in the following cases: in the first case the integrand adapted (to the σ-algebra generated by the Wiener process) process is a Banach space valued and the stochastic integral is constructed by the one dimensional Wiener process; in the second case the integrand adaptive process is an operator valued (from the Banach space to the Banach space) and the stochastic integral is constructed by the Wiener process in a Banach space; in the third case the integrand adapted process is operator-valued (from the Hilbert space to the Banach space) and the stochastic integral is constructed by the cylindrical Wiener process in a Hilbert space. In all of these cases the main difficulties are same. Therefore, for simplicity, we consider the first case (the Wiener process is one dimensional).

Using traditional methods, to find the suitable conditions that guarantee the construction of the Ito stochastic integral in a Banach space is possible only in a very narrow class of Banach spaces. This class is so called UMD Banach spaces class (see survey in [3]). In our approach is constructed the generalized stochastic integral for a wide class of adapted Banach space valued random processes and the problem of existence of the stochastic integral is reduced to the problem of decomposability of the generalized random element (cylindrical random element or random linear function) (see [4]).

In this article we consider the problem of representation of the functional of the Wiener process by the stochastic integral in an arbitrary separable Banach space. This problem is, in some sense, opposite to the problem of existence of the stochastic integral: in this case we have the Banach space valued random element and the problem is to find the integrand Banach space valued adapted process the stochastic integral of which is this random element. In [1] we considered this problem in case, when the joint distribution of the Wiener process and its functional is Gaussian. In [5] is considered this problem for UMD Banach space case, where, under special conditions of the Wiener functional, it is represented by the stochastic integral and is generalized the Clark-Ocone formula of representation of the functional of the Wiener process by the Malliavin derivative.

Let X be a real separable Banach space. X^* – its conjugate, (Ω, B, P) – a probability space. Remember, that the continuous linear operator $T : X^* \to L_2(\Omega, B, P)$ is called the generalized random element (GRE) [1]. Denote by $\mathcal{M}_1 := L(X^*, L_2(\Omega, B, P)$ the Banach space of GRE with the norm $\|T\|^2 = \sup_{\|x^*\|\le 1} E(Tx^*)^2$.

We can realize the weak second order random element ξ as an element of $\mathcal{M}_1$, $T_\xi x^* = \langle \xi, x^* \rangle$ (boundedness of this operator follows by closed graph theorem), but not conversely: in an infinite dimensional Banach space for any $T : X^* \to L_2(\Omega, B, P)$, there does not always exists the random element $\xi : \Omega \to X$ such that $Tx^* = \langle \xi, x^* \rangle$ for all $x^* \in X^*$. The problem of existence of such random element is the well known problem of decomposability (radonizability) of the GRE. Denote by $\mathcal{M}_2$ the linear normed space of all random elements of the weak second order with the norm $\|\xi\|^2 = \sup_{\|x^*\|\le 1} E\langle \xi, x^* \rangle^2$.

Thus, we have $\mathcal{M}_2 \subset \mathcal{M}_1$. The family of linear operators $(T_t)_{t\in[0,1]}$ is called the weak second order generalized random process (GRP) if $T_t x^*$ is $B([0, 1]) \times B(\Omega)$ measurable and

$$\|T_t\|^2 \equiv \sup_{\|x^*\|\le 1} \int_0^1 E(T_t x^*)^2 dt < \infty.$$

Denote by $\mathcal{M}_1^{(\lambda,P)}$ the Banach space of such GRP. The Banach space valued stochastic process $f(t, \omega)$, $t \in [0, 1]$ is called a weak second order random process, if for all $x^* \in X^*$,

$$\int_0^1 E\langle f(t, \omega), x^* \rangle^2 dt < \infty.$$

Weak second order random process realizes the GRP $T_f : X^* \to L_2([0, 1] \times \Omega)$: $T_f x^* = \langle f(t, \omega), x^* \rangle$. Denote by $\mathcal{M}_2^{(\lambda,P)}$ the normed linear spaces of $f(t, \omega)$, $t \in [0, 1]$, with norm

$$\sup_{\|x^*\|\le 1} (\int_0^1 E\langle f(t, \omega), x^* \rangle^2 dt)^{\frac{1}{2}}.$$

We have $\mathcal{M}_2^{(\lambda,P)} \subset \mathcal{M}_1^{(\lambda,P)}$. Let $(W_t)_{t\in[0,1]}$ – be a real valued Wiener process. Denote by F_t^W the σ-algebra generated by the random variables $(W_s)_{s\le t}$ $(F_t^W = \sigma(W_s, s \le t))$, which is completed by P-null sets. Suppose that ξ is F_1^W- measurable weak second order random

[1]sometimes it is used the terms: random linear function or cylindrical random element

element i.e., ξ is the functional of the Wiener process. Our main aim is to represent the random element ξ by the Ito stochastic integral $\xi = E\xi + \int_0^1 f(t,\omega)dW_t$, where $f(t,\omega)$ is a Banach space valued F_t^W-adapted random process, but, in general, it is impossible. We have the following result: For all weak second order Wiener functional we always have integrand as a GRP, that is, an element of the Banach space $\mathcal{M}_1^{(\lambda,P)}$. In developing of this problem firstly, in [1] we considered the case when ξ is a Gaussian random element which with the Wiener process generates the mutually Gaussian system. Even in this case we constructed the example (see example 1 in [1]), where a) the integrand function $f(t)$ (in this case the integrand is nonrandom) is X-valued; b) the integrand function is not X-valued, but it is X^{**}-valued and c) the integrand function is not X^{**}-valued, but it is a GRE $T : X^* \to L_2[0,1]$.

Proposition 1. *Let ξ be a Banach space valued weak second order functional of the one dimensional Wiener process. There exists the unique F_t^W-adapted GRP $T : X^* \to L_2([0,1]\times\Omega)$ such that for all $x^* \in X^*$*

$$\langle \xi, x^* \rangle = E\langle \xi, x^* \rangle + \int_0^1 Tx^*(t,\omega)dW_t. \tag{1}$$

For any GRP $T : X^* \to L_2([0,1]\times\Omega)$ from $\mathcal{M}_1^{(\lambda,P)}$, the correlation operator of T is called the linear, bounded operator from X^* to X^{**}, $R_T = T^*T$.

Proposition 2. *If for any functional of the Wiener process ξ, $\langle \xi, x^* \rangle = \int_0^1 Tx^*(t,\omega)dW_t$, then $R_T = T^*T$ maps X^* to X.*

As it is known (see [2] Theorem 5.6), for the one dimensional case, if the joint distribution of the Wiener process and its one dimensional functional is Gaussian, then the sequence of step functions $f_n(t) = \sum_{i=0}^{2^n-1} 2^n E(\xi - E\xi)(W_{\frac{i+1}{2^n}} - W_{\frac{i}{2^n}})I_{(\frac{i}{2^n},\frac{i+1}{2^n}]}(t)$ converges in $L_2[0,1]$ to the integrand function $f(t)$, $\int_0^1 f^2(t)dt < \infty$ and $\xi_n = E\xi + \int_0^1 f_n(t)dW_t$ converges in $L_2(\Omega, B, P)$ to $\xi = E\xi + \int_0^1 f(t)dW_t$.

Theorem 1. *Let the square integrable random variable ξ be a functional of the Wiener process. The sequence of step functions*

$$f_n(t,\omega) = \sum_{i=0}^{2^n-1} 2^n E((\xi_{\frac{i+1}{2^n}} - \xi_{\frac{i}{2^n}})(W_{\frac{i+1}{2^n}} - W_{\frac{i}{2^n}})|F^W_{\frac{i}{2^n}})I_{(\frac{i}{2^n},\frac{i+1}{2^n}]}(t) \tag{2}$$

converges in $L_2([0,1],\Omega)$ to the F_t^W-adapted random process $f(t,\omega)$ and

$$\xi = \lim_{n\to\infty} \int_0^1 f_n(t,\omega)dW_t = \int_0^1 f(t,\omega)dW_t.$$

Lemma. *Let $\xi = \int_0^1 f(t,\omega)dW(t)$ be a real valued functional of the Wiener process. Then for any $0 \le a \le b$*

$$E((\xi_b - \xi_a)(W_b - W_a)|F_a^W) = E(\int_a^b f(t,\omega)dt|F_a^W),$$

where $\xi_t = E(\xi|F_t^W) = \int_0^t f(s,\omega)dW(s)$.

Theorem 2. *Let* ξ *be a Banach space valued* F_1^W *measurable weak second order random element, such that in representation (0.1) the GRP* $T \in \mathcal{M}_1^{\lambda,P}$ $T:[0,1]\to\mathcal{M}_1$ *is separable valued and*

$$\int_0^1 \|T\|^2_{\mathcal{M}_1} < \infty.$$

There exists the sequence of F_t^W*-adapted step functions* $T_n(t,\omega)$, $n \in N$ *converging in* $\mathcal{M}_1^{\lambda,P}$ *to the* F_t^W*-adapted GRP* $T : X^* \to L_2([0,1],\Omega)$ *such that the sequence of the stochastic integrals*

$$\int_0^1 T_n x^*(t,\omega)dW_t$$

converges to

$$\int_0^1 Tx^*(t,\omega)dW_t = \langle \xi - E\xi, x^*\rangle \quad in \ \ \mathcal{M}_1.$$

REFERENCES

1. Mamporia, B., Purtukhia, O. On functionals of the Wiener process in a Banach space. *Transactions of A. Razmadze Mathematical Institute, Elsevier.*, **172**, 3 (2018), 420-428.
2. Liptser, R.S. Shiriaev, A.N. Statistics of Random Processes (Russian). *Nauka, Moscow*, 1974, *Springer*, (2001), pp. 427.
3. Van Neerven, J.M.A.M., Veraar, M., Weis, L. Stochastic integration in UMD Banach spaces a survey. *Stochastic Analysis: A series of Lectures*, (2015), 297-332.
4. Mamporia, B. Stochastic differential equation for generalized stochastic processes in a Banach space.*Theory of Probability and its Applications, SIAM*, **56**, 4 (2012), 602-620.
5. Maas, J., Van Neerven, J.M.A.M. A Clark-Ocone formula in UMD spaces. *Electron commun. Probab.*, **13** (2008), 151-164.

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