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STOCHASTIC INTEGRAL REPRESENTATION OF POISSON FUNCTIONALS WITH AN EXPLICIT FORM OF THE INTEGRAND

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Abstract. A more convenient and practical form for finding the explicit expression of the integrand expression in the Clark-Haussmann-Ocone representation of functionals of the Poisson process is found.

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1 Introduction and preliminaries. In the 80th of the past century, it turned out ([1]) that the martingale representation theorems (along with the Girsanov's measure change theorem) play an important role in the modern financial mathematics. In particular, using the integrand of the stochastic integral appearing in the integral representation, one can construct hedging strategies in the European options of different types. After Clark ([2]) had obtained the formula for the stochastic integral representation for Wiener functionals, many authors tried to find the integrand explicitly, and the corresponding results were obtained when the functionals were smooth in some sense.

The first proof of the martingale representation theorem was implicitly provided by Ito (1951). Many years later, Dellacherie (1974) gave a simple new proof of Ito's theorem using Hilbert space techniques. Many other articles were written afterward on this problem and its applications but one of the pioneer works on explicit descriptions of the integrand is certainly the one by Clark (1970). Those of Haussmann (1979), Ocone (1984), Ocone and Karatzas (1991) and Karatzas, Ocone and Li (1991) were also particularly significant. A nice survey article on the problem of martingale representation was written by Davis (2005).

The constructive integral representation is based on the Malliavin (stochastic) derivative and in the Wiener case it is known as the Clark-Ocone formula ([3]): if F is differentiable in the Malliavin sense, $F \in D_{2,1}^W$, then the integrand in the Clark representation is $E[D_t^W F|\Im_t^W]$, where D_t^W is the so called Malliavin stochastic derivative of F.

It has turned out that the requirement of smoothness of functional can be weakened by the requirement of smoothness only of its conditional mathematical expectation. We (together with prof. O. Glonti, 2014) considered Wiener functionals which are not stochastically differentiable and established the method of finding the integrand (see [8]).

The stochastic integral representation in the case of so called normal martingales[1] M for functionals from the class $D_{2,1}^M$ is known as the Clark-Haussmann-Ocone formula ([4]):

[1]A martingale M is called normal if its predictable square characteristic $\langle M, M\rangle_t$ is deterministic.

let M be a normal martingale with the CRP (chaos representation property), and let $F \in L_2(\Omega)$. If $F \in D^M_{2,1}$, then

$$F = EF + \int_0^T {}^p(D^M_t F) dM_t,$$

where ${}^p(D^M_t F)$ is the predictable projection of the stochastic derivative $(D^M_t F)$ of the functional F.

As we see, this representation analogously to the Wiener case requires existence of a stochastic derivative. On the other hand, in this case, unlike the Wiener's one, it is impossible to define in a generally adopted manner an operator of stochastic differentiation to obtain the structure of the Sobolev space $D^M_{2,1}$. Here, the determination of the stochastic derivative is based on the expansion in series of multiple stochastic integrals of the functional, whereas the Wiener case involves, besides the above-mentioned approach, the structure of Sobolev spaces.

For a class of normal martingales one fails to define the space $D^M_{p,1}$ $(1 \leq p < 2)$ in a commonly adopted manner (i.e., by closing a class of smooth functionals with respect to the corresponding norm). In our work ([5]) we defined the space $D^M_{p,1}$ $(1 < p < 2)$ for the normal martingales (the Banach space $D^M_{p,1}$ which is closure of $D^M_{2,1}$ under the norm $||F||_{p,1} := ||F||_{L_p(\Omega)} + E||D^M_{\cdot} F||_{L_2([0,T])}$) and generalized the Clark-Haussmann-Ocone formula for the functionals of this space.

2 Main result. Ma, Protter, Martin gave an example showing that two possible ways of determination of a stochastic derivative coincide if and only if the quadratic martingale characteristic $[M, M]$ is the deterministic function (as, for example, in the Wiener case $[W, W]_t = t$). Consequently, the Clark-Haussmann-Ocone formula makes it impossible to construct explicitly the operator of the stochastic derivative of the functionals of the Compensated Poisson process, saying nothing on the construction of its predictable projection.

Our approach (see [6], [7]) within the framework of nonanticipative stochastic calculus of semimartingales allows one to construct explicitly the expression for the integrand of the stochastic integral in the theorem of martingale representation for square integrable Poisson functionals, in particular, we will generalize the Clark-Haussmann-Ocone formula for Poisson functionals. We proposed a new approach to the definition of the stochastic derivative of the operator of Poisson functionals and obtained the explicit form of the integrand in the integral representation. Here a more convenient and practical form for finding the explicit expression of the integrand expression in the Clark-Haussmann-Ocone representation of functionals of the Poisson process N will be found. In particular, in the conditional mathematical expectation of the above-mentioned integrand the σ-algebra

For example, among the known martingales, the normal martingale is the Wiener process W_t (because $\langle W, W\rangle_t = t$) and the Compensated Poisson process $M_t = N_t - t$ (because $\langle M, M\rangle_t = t$, but unlike to the Wiener process, here the square martingale characteristic is not deterministic: $\langle M, M\rangle_t = N_t$), where N_t is the Poisson process.

$\Im_{t-}^N = \sigma(\cup_{s<t}\Im_s^N)$ can be replaced by a more natural σ-algebra $\Im_t^N = \sigma(N_s : 0 \le s \le t)$, which, in turn, allows us to more effectively use the well-known properties of the Poisson process.

Let $(\Omega, \Im, (\Im_t)_{0\le t\le T}, P)$ be a filtered probability space, satisfying the usual conditions. Assume that the standard Poisson process N_t is given on it ($P_k := P(N_t = k) = e^{-t}t^k/k!$, $k = 0, 1, 2, \ldots$) and that $\Im_t$ is generated by N ($\Im_t = \Im_t^N$), $\Im = \Im_T$. Let $M_t := N_t - t$ (the compensated Poisson process) and $\Delta M_t = M_t - M_{t-}$.

Let $Z^+ = \{0, 1, 2, \ldots\}$; $\Delta_- f(k) = f(k) - f(k-1)$ ($f(k) = 0,\ k < 0$); $\Delta_-^n := (\Delta_-)^n$ and define the Poisson-Sharle's polynomials: $\Pi_n(k) = (-1)^n \Delta_-^n P_k/P_k,\ n \ge 1$; $\Pi_0 = 1$. It is known that the system of normalized Poisson-Charlier polynomials is a basis in $L_2(Z^+) := \{f : \sum_{k=0}^{\infty} f^2(k) < \infty\}$.

Let

$$L_2^T := \{f : e^{-T}\sum_{k=0}^{\infty} f^2(k)T^k/k! < \infty\}.$$

This is a Banach space with the basis $\{k^n e^{-T}T^k/k!\}$. Denote $\Delta_+^x f(x) = f(x+1) - f(x)$ ($\Delta_+^x f(M_T) := \Delta_+^x f(x)|_{x=M_T}$). The following theorem was proved in [7].

Theorem 1. *Let $f \in L_2^T$ and for some $\epsilon > 0 : \Delta_+^x f(\cdot - T) \in L_2^{(1+\epsilon)T}$, then the stochastic integral below is well defined and the following representation is valid:*

$$f(M_T) = E[f(M_T)] + \int_{(0;T]} E[\Delta_+^x f(M_T)|\Im_{t-}]dM_t \quad (P\text{-}a.s.).$$

Although we have established here explicit form of an integrand, which does not require calculation of a stochastic derivative of functional and its predictable projection, we still encounter practical difficulties. For example, calculation of conditional mathematical expectation with respect to the σ-algebra $\Im_{t-}^N$ is quite a difficult task. It is much more convenient to calculate the conditional mathematical expectation with respect to the σ-algebra $\Im_t^N$, since in this case the Markov property of the Poisson process can be used.

The class of smooth Poisson functionals S^M is the class of random variables which has the form

$$F = f(M_{t_1}, ..., M_{t_n}),\ f \in C_p^\infty(R^n),\ t_i \in [0, T],\ n \ge 1,$$

where $C_p^\infty(R^n)$ is the set of all infinitely continuously differentiable functions $f : R^n \to R$ such that f and all of its partial derivatives have polynomial growth.

Definition. The stochastic (Malliavin) derivative of a smooth random variable $F \in S^M$ is the stochastic process $D_t^M F$, given by

$$D_t^M F = \sum_{k=1}^{n} \sum_{i_1<\cdots<i_k} \Delta_+^{i_1}(\cdots(\Delta_+^{i_k} f(M_{t_1}, ..., M_{t_n})))I_{[0,t_{i_1}]}(t)\cdots I_{[0,t_{i_k}]}(t),$$

where $\Delta_+^i f(x_1, ..., x_i, ..., x_n) = f(x_1, ..., x_i + 1, ..., x_n) - f(x_1, ..., x_i, ..., x_n)$.

Theorem 2. *Let $f \in L_2^T$ and for some $\epsilon > 0 : \Delta_+^x f(\cdot - T) \in L_2^{(1+\epsilon)T}$, then the stochastic integral below is well defined and the following representation is valid:*

$$f(M_T) = E[f(M_T)] + \int_{(0;T]} E[\Delta_+^x f(M_T - \Delta M_t)|\Im_t]dM_t \quad (P\text{-}a.s.).$$

REFERENCES

1. Harrison, M., Pliska, S.R. Martingales and stochastic integrals in the theory of continuous trading. *Stochastic Processes and Applications*, **11** (1981), 215-260.
2. Clark, M.C. The representation of functionals of Brownian motion by stochastic integrals. *J. Ann. Math. Stat.,* **41** (1970), 1282-1295.
3. Ocone, D. Malliavin calculus and stochastic integral representation formulas of diffusion processes. *J. Stochastics,* **12** (1984), 161-185.
4. Ma, J., Protter, P., Martin, J.S. Anticipating integrals for a class of martingales, *Bernoulli* **4, 1** (1998), 81-114.
5. Purtukhia, O. An extension of Ocone-Haussmann-Clark's formula for a class of normal martingales. *Proceedings of A. Razmadze Mathematical Institute*, **132** (2003), 127-136.
6. Jaoshvili, V., Purtukhia, O. Stochastic integral representation of functionals of Poisson processes. *Proceedings of A. Razmadze Mathematical Institute*, **143** (2007), 37-60.
7. Jaoshvili, V., Purtukhia, O. An extension of the Ocone-Haussmann-Clark formula for the compensated Poisson processes. *SIAM J. Theory of Probability and Its Applications,* **53, 2** (2009), 316-321.
8. Glonti, O., Purtukhia, O. On one integral representation of functionals of Brownian motion. *SIAM J. Theory of Probability and Its Applications*, **61, 1** (2017), 133-139.

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