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1-ЧАСТЬ

On the modeling of the standard options pricing process

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Abstract. The problem of the pricing of European and American type standard options is investigated for the Cox-Ross-Rubinstein discrete model of financial market. The form of a fair price and minimal hedge is found for one class of nonselffinanced strategies. The obtained results make it possible to construct a complex of programs for numerical examples.

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1. We consider the financial (B,S) – market consisting only of two assets: a bonds – $B=(B_n)$ and stocks – $S=(S_n)$, $n=0,1,...,N$. According to the Cox-Ross-Rubinstein model we have

$$B_n=(1+r)B_{n-1}, \quad B_0>0, \tag{1}$$

$$S_n=(1+\rho_n)S_{n-1}, \quad S_0>0, \tag{2}$$

where $r>0$ is an interest rate, ρ_n is a sequence of independent, identicaly distributed random variables taking only two values a and b, $a<r<b$ (see [1]).

Let us now assume that there is some investor who has the initial capital $X_0=x>0$ and wants to raise this capital in the future by using the capability of the (B,S) – market. In that case, we deal with the so-called investment problem. Suppose that at the moment $n=0$ the investor purchased β_0 quantity of bonds and γ_0 quantity of stocks. Then we have

$$X_0=X_0^\pi=\beta_0B_0+\gamma_0S_0, \tag{3}$$

where $\pi=\pi_0=(\beta_0,\gamma_0)$ is the investor's portfolio or strategy .

Assume that there is a sequence of functions $g=(g_n)$, $n=0,1,...,N$, $g_0=0$, and the investor transformed his portfolio π_0 to the new portfolio $\pi_1=(\beta_1,\gamma_1)$ in a maner such that the equality

$$X_0^\pi=\beta_1B_0+\gamma_1S_0+g_1 \tag{4}$$

is satisfied. At the moment $n=1$ we have

$$X_1^\pi=\beta_1B_1+\gamma_1S_1. \tag{5}$$

Analogously, for any moments $n-1$ and n we have

$$X_{n-1}^\pi=\beta_nB_{n-1}+\gamma_nS_{n-1}+g_n, \tag{6}$$

$$X_n^\pi=\beta_nB_n+\gamma_nS_n. \tag{7}$$

The strategy $\pi_n=(\beta_n,\gamma_n)$ is called nonselffinanced. If $g_n\equiv0$ then π_n is called selffinanced. If $X_0^\pi=X_0=x$ and $X_N^\pi\geq f_N$ then π_n is called a hedge, where $f=f_N$ is some payoff function, if $X_N^\pi=f_N$ then π_n is called a minimal hedge. Denote by Π the set of all hedges.

2. Now let us define a standard European call option. This is a derivative security with the payoff function

$$f = f_N = \max(S_N - K, 0). \tag{8}$$

The owner of this option enjoys the right to buy a stock at a price K at a certain moment N. If $S_N > K$, then the owner will buy a stock at a price K, sell it at a price S_N and have a gain

$$f_N = S_N - K - C_N, \tag{9}$$

where $C_N = \min\{x > 0 : \Pi = \varnothing\}$ is the so called fair price of option.

The problem of the investor (option seller) consist in the following: using the fair price C_N it is required to construct a minimal hedge.

Assume that the sequence of functions $g = (g_n)$ defined by the equality

$$g_n = c_1\beta_n B_{n-1} + c_2\gamma_n S_{n-1}, \quad 0 < c_1 < 1, \quad 0 < c_2 < 1. \tag{10}$$

Lemma 1. At each moment n, $n = 0,1,\dots,N-1$, a minimal hedge $\pi^*_{n+1} = \left(\beta^*_{n+1}, \gamma^*_{n+1}\right)$ is defined by the following equalities

$$\beta^*_{n+1} = \frac{(1+b)f((1+a)S_n) - (1+a)f((1+b)S_n)}{(1+r)(b-a)B_n}, \tag{11}$$

$$\gamma^*_{n+1} = \frac{f((1+a)S_n) - f((1+b)S_n)}{(b-a)B_n}, \tag{12}$$

where f is some payoff functuion.

Lemma 2. The capital of the minimal hedge π^*_{n+1} is defined by the equality

$$X^{\pi^*}_n = \frac{1+c_1}{1+r}\cdot[pf((1+b)S_n) + (1-p)f((1+a)S_n)], \tag{13}$$

where

$$p = \frac{r - c_1(1+a) + c_2(1+r) - a}{(b-a)(1+c_1)}. \tag{14}$$

Theorem 1. The fair price of an European standard call option is defined by the following equality

$$C_N = S_0\sum_{k=k_0}^{N} C_N^k p^k(1-p)^{N-k}\left(\frac{(1+c_1)(1+a)}{1+r}\right)^N\left(\frac{1+b}{1+a}\right)^k -$$

$$-K\cdot\left(\frac{1+c_1}{1+r}\right)^N \sum_{k=k_0}^{N} C_N^k\cdot p^k\cdot(1-p)^{N-k}, \tag{15}$$

where k_0 is the smallest integer number for which the inequality

$$S_0\cdot((1+c_1)(1+a))^N\cdot\left(\frac{1+b}{1+a}\right)^{k_0} > K.$$

Suppose that

$$B_0 = 20, \quad r = \frac{1}{5}, \quad S_0 = 100, \quad K = 100,$$

$$\rho_n = b = \frac{3}{5} \quad or \quad \rho_n = a = -\frac{2}{5}, \quad n = 0,1,\dots,N. \tag{16}$$

Example 1. Let we have the function (8), $N = 2$ $(n = 0,1,2)$, $c_1 = \frac{1}{40}$, $c_2 = \frac{1}{50}$. Then we have

$$C_2 = \frac{609\cdot 203\cdot 13}{40000}, \quad \pi^*_1 = \left(-\frac{609\cdot 13}{4000}, \frac{609\cdot 13}{10000}\right),$$

$$g_1 = \frac{609\cdot 13\cdot 3}{40000}, \quad X^{\pi^8}_0 = C_2.$$

Case I. $S_1 = 160$, $\pi_2^* = \left(\frac{13}{40}, \frac{39}{40}\right)$, $g_2 = \frac{117}{100}$.

1) If $S_2 = 256$, then $X_2^{\pi^*} = f(S_2) = 156$,

2) If $S_2 = 96$, then $X_2^{\pi^8} = f(S_2) = 0$.

Case II. $S_1 = 60$, $\pi_2^* = (0, 0)$, $g_2 = 0$.

1) If $S_2 = 96$, then $X_2^{\pi^*} = f(S_2) = 0$,

2) If $S_2 = 36$, then $X_2^{\pi^*} = f(S_2) = 0$.

3. Consider now the American standard put option with the payoff function

$$f = f_n = \max(K - S_n, 0). \tag{17}$$

Let we have the sequence of functions (17) and introduce the operator

$$Tf(x) = \frac{1+c_1}{1+r} \cdot \left[pf((1+b)x) + (1-p) f((1+a)x) \right], \tag{18}$$

where p is defined by (14).

Lemma 3. The fair price of the American standard put option $P_n^A(x)$ satisfies the following recurrent equations

$$P_n^A(x) = \max\left(f_n(x), TP_{n+1}^A(x)\right), \quad n = 0,1,\dots,N-1. \tag{19}$$

lemma 4. The rational (optimal stopping) moment τ^* is defined by the inequality

$$f(S_n) \geq TP_{n+1}^A(S_n). \tag{20}$$

Let's introduce the following notations

$$S_{1,j} = S_0 (1+b)^j (1+a)^{1-j},$$

$$f_{1,j} = f(S_{1,j}), \quad j = 0,1; \quad N = 1, \quad n = 0,1;$$

$$S_{2,j} = S_0 (1+b)^j (1+a)^{2-j},$$

$$f_{2,j} = f(S_{2,j}), \quad j = 0,1,2; \quad N = 2, \quad n = 0,1,2.$$

Theorem 2. The fair (rational) price of the American standard put option can be calculated by the following recurrent equality

$$P_{N-k,j}^A = \max\left\{ f_{N-k,j}, \frac{1+c_1}{1+r} \cdot \left[pP_{N-k+1,j+1}^A + (1+p) P_{N-k+1,j}^A \right] \right\}, \tag{21}$$

where p is defined by (14), $k = 0,1,\dots,N$; $j = 0,1,\dots,N-k$.

Example 2. Let we have the values (16) and the function (17). Suppose that $c_1 = c_2 = 0$, $N = 2$ $(n = 0,1,2)$. Then we have

$$P_2^A = P_{0,0}^A = \frac{42}{3}, \quad \pi_1^* = \left(\frac{79}{30}, -\frac{29}{25}\right), \quad X_0^{\pi^*} = P_2^A.$$

1) If $S_{1,1} = 160$, then $\pi_2^* = \left(\frac{2}{9}, -\frac{1}{40}\right)$, $\tau^* = 2$,

2) If $S_{1,0} = 60$, then $\pi_2^* = \left(\frac{125}{36}, -1\right)$, $\tau^* = 1$.

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Some results for the first-order autoregressive model

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We shall consider the first- order autoregressive process ($Y_1, Y_2, \ldots,$) defined by

$$Y_t = \beta Y_{t-1} + u_t, \quad t = 2,3,\ldots, \tag{1}$$

where $|\beta| < 1$, $\{u_1, u_2, \ldots,\}$ is a sequence of independent and identically distributed (i.i.d.) $N(0,\sigma^2)$ random variables.If $\overline{\beta}$ denotes the LS estimator of β based on n observations $(Y_1, Y_2, \ldots, Y_n)$, then the s-periods-ahead LS forecast is $\overline{Y}_{n+s} = \overline{\beta}^s Y_n$ .The least-squares estimator of β is:

$$\overline{\beta} = \sum_{t=2}^{n} y_t y_{t-1} / \sum_{t=2}^{n} y_{t-1}^2, \tag{2}$$

In the non –stationary case the least-squares estimator is the maximum likelihood estimator. The forecast error is therefore

$$\overline{Y}_{n+s} - Y_{n+s} = (\overline{\beta}^s - \beta^s)Y_n - \sum_{j=0}^{s-1} \beta^j u_{n+s-j} \tag{3}$$

From (3)we obtain the forecast bias,

$$M(\overline{Y}_{n+s} - Y_{n+s}) = M(\overline{\beta}^s Y_n), \tag{4}$$

and the mean-square forecast error(MSFE) ,

$$M(\overline{Y}_{n+s} - Y_{n+s})^2 = \beta^{2s}\,\mathrm{var}(Y_n) + M(\overline{\beta}^{2s} Y_n^2) - 2\beta^s M(\overline{\beta}^s Y_n^2) + \sigma^2 \sum_{j=0}^{s-1} \beta^{2j}. \tag{5}$$

Some proporties estimator (2) was considered in [1].From (4) we see that the expectation of the forecast error exist if and only if, $M(\overline{\beta}^s Y_n)$ exists, that is, if and only if $0 \le s < n-1$. Similarly, from(5), we see that the MSFE exists if and only if $M(\overline{\beta}^{2s} Y_n^2)$,$M(\overline{\beta}^s Y_n^2)$ exist, that is, if and onli if, $0 \le s < (n-1)/2$

Theorem. The expectation of the forecast error of $\overline{Y}_{n+s}$ exists if and only if $1 \le s \le n-2$, in which case

$$M(\overline{Y}_{n+s} - Y_{n+s}) = 0,$$

since $\overline{\beta}$ is a consistent estimator of β

$$\lim_{n\to\infty} MSFE = \sigma^2 (1-\beta^{2s}) / (1-\beta^2)$$

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