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Spaces Associated to Quadratic Endofunctors of the Category of Groups

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1. Introduction

Since the work of Kan [11], [9] it is well known that the homotopy category of simplicial groups is equivalent to the homotopy category of path connected pointed spaces. The equivalence carries a simplicial group S to its classifying space BS. For example if $S_{(n)}$ is the simplicial group freely generated by one element in degree n then $BS_{(n)} \simeq S^{n+1}$ is an $(n+1)$-sphere, $n \geq 0$. Now let Groups be the category of groups. We consider endofunctors T : Groups $\to$ Groups which preserve reflective coequalizers and filtered colimits. Then applying the functor T to $S_{(n-1)}$ yields a new simplicial group $TS_{(n-1)}$ and hence the *space*

$$K(T,n) = BTS_{(n-1)}, \quad n \geq 1,$$

associated to T. This space, in particular, is of importance if T is an additive functor. In this case the functor $T = (-) \otimes A$ is given by an abelian group A, in the sense that $T(G) = G^{ab} \otimes A$, and by the Dold-Kan theorem $K((-) \otimes A, n)$ coincides with the Eilenberg-MacLane space $K(A,n)$. Hence the spaces $K(T,n)$ are canonical generalizations of Eilenberg-MacLane spaces. As a next step we consider quadratic functors $T = (-) \otimes Q$, which by the classification in

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[6] are given by *square groups* Q. We apply a variety of concepts of quadratic algebra and quadratic homology theory in order to study the spaces $K((-)\otimes Q, n)$. Of course $K((-)\otimes Q, n)$ is $(n-1)$-connected and the first k-invariant of $X = K((-)\otimes Q, n)$ for $n \geq 2$ is given by a homomorphism

$$k : \Gamma(\pi_2) \to \pi_3 \quad \text{for} \quad n = 2,$$

$$k : \pi_n \otimes \mathbb{Z}/2\mathbb{Z} \to \pi_{n+1} \quad \text{for} \quad n \geq 3.$$

We say that k is *flat* if for $n = 2$ the homomorphism k is trivial if restricted to the kernel of $\Gamma(\pi_2) \to \pi_2 \otimes \pi_2$ and for $n \geq 3$ the group π_{n+1} satisfies $2\pi_{n+1} = 0$.

THEOREM 1. *For any square group Q the first k-invariant of $X = K((-)\otimes Q, n)$ is flat, $n \geq 2$. Moreover we have $\pi_m X = 0$ for $m \geq 2n+2$.*

We conjecture that conversely all flat k are realizable by a quadratic $K((-)\otimes Q, n)$. This is true for $n \geq 3$ and for many k for $n = 2$ by the next result.

THEOREM 2. *Let k be flat and for $n = 2$ let π_2 be in the class $\mathbf{A}$ of abelian groups defined below. Then there exists a square group Q such that the first k-invariant of $K((-)\otimes Q, n)$ is k.*

Let $\mathbf{A}$ be the smallest class of abelian groups which is closed under arbitrary direct sums and contains i) all cyclic groups, ii) all abelian groups A such that 2 is invertible in A and iii) all abelian groups A such that $\mathsf{Ext}(A, Sym^2 A) = 0$, where $Sym^2 A$ is the second symmetric power of A. It is clear that then $\mathbf{A}$ contains all finitely generated abelian groups as well as all free and all divisible abelian groups.

Theorem 1 for $n = 2$ is proved in Section 4.9 (see Theorem 17) and for $n \geq 3$ in Section 5.9 (see Corollary 36). Theorem 2 for $n = 2$ is proved in Section 5.8 (see Corollary 33) and for $n \geq 3$ in Section 5.9 (see Theorem 37).

Our approach is to use presquare groups. They are gadgets classifying quadratic functors from the category of finite pointed sets to the category of groups. If F is such a functor, we have

$$BF(S^1) \in \mathsf{types}(2, 3)$$

Thus one obtains a functor from the category of presquare groups to the category of spaces. To pass from presquare groups to square groups we then develop an appropriate obstruction theory.

2. On certain quadratic functors

In this section we consider few quadratic functors defined on the category Ab of abelian groups. Let $F : \mathsf{Ab} \to \mathsf{Ab}$ be a functor with $F(0) = 0$. Let us recall that the functor F is *additive* or *linear* if the natural projection

$$F(X \oplus Y) \to F(X) \oplus F(Y)$$

is an isomorphism. Furthermore, F is *quadratic* if the *second cross-effect*

$$F(X \mid Y) = \mathsf{Ker}(F(X \oplus Y) \to F(X) \oplus F(Y))$$

as a bifunctor is linear in X and Y. In this case one has a natural decomposition

$$F(X \oplus Y) \cong F(X) \oplus F(Y) \oplus F(X \mid Y).$$

2.1. Universal quadratic functor Let A and B be abelian groups. A map $f : A \to B$ is called *quadratic* if the *cross-effect*

$$(a \mid b)_f := f(a+b) - f(a) - f(b)$$

is linear in a and b. It follows then that $f(0) = 0$. It is well known that for any abelian group A there is a universal quadratic function $p : A \to P(A)$, meaning that for any quadratic map $f : A \to B$ there exists a unique homomorphism $h : P(A) \to B$ such that $f = h \circ p$. In this way one obtains the functor $A \mapsto P(A)$, which has the following alternative description. Let $I(A)$ be the augmentation ideal of the group algebra of A. Then one has the isomorphism

$$P(A) \cong I(A)/I(A)^3$$

induced by $p(a) \mapsto (a-1)(\mathsf{mod}\ I(A)^3)$ (see [14]). The following fact is well known [13]:

Lemma 3. *For any abelian group A one has the following short exact sequence*

$$0 \longrightarrow Sym^2(A) \xrightarrow{j} P(A) \xrightarrow{q} A \longrightarrow 0 \tag{1}$$

where Sym^2 is the second symmetric power, the homomorphisms j and q are given by

$$j(ab) = (a \mid b)_p = p(a+b) - p(a) - p(b)$$
$$q(p(a)) = a.$$

It follows that the functor P commutes with filtered colimits and one has the following natural isomorphism

$$P(A \oplus B) \cong P(A) \oplus P(B) \oplus (A \otimes B).$$

Furthermore, one has isomorphisms ([12])

$$P(\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z},$$

$$P(\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/4\mathbb{Z},$$

$$P(\mathbb{Z}/2^n\mathbb{Z}) \cong \mathbb{Z}/2^{n+1}\mathbb{Z} \oplus \mathbb{Z}/2^{n-1}\mathbb{Z}, \quad n > 1$$

and for any odd prime p, one has an isomorphism

$$P(\mathbb{Z}/p^n\mathbb{Z}) \cong \mathbb{Z}/p^n\mathbb{Z} \oplus \mathbb{Z}/p^n\mathbb{Z}$$

For an abelian group we let

$$\theta(A) \in \mathsf{Ext}(A, Sym^2(A)), \tag{2}$$

be the element corresponding to the exact sequence (1).

LEMMA 4. *The class $\theta(A)$ is represented by the canonical symmetric 2-cocycle f^s, given by*

$$f^s(a,b) = ab \in Sym^2(A), \quad a, b \in A.$$

Proof. The homomorphism $q : P(A) \to A$ has a set-section $p : A \to P(A)$ and the cocycle corresponding to this section is exactly f^s. ∎

The class θ is nontrivial in general. For example one has $\theta(\mathbb{Z}/2^n\mathbb{Z}) \neq 0$. However one has

LEMMA 5. *If 2 is invertible in A, then $\theta(A) = 0$.*

Proof. Let $g : A \to Sym^2(A)$ be the map given by $g(a) = a^2$. Then

$$(a \mid b)_g = 2ab$$

which shows that the coboundary of $\frac{g}{2}$ is f^s. ∎

2.2. A FUNCTOR Ψ For an abelian group A we let $\Psi(A)$ be the kernel of the natural projection $A \otimes A \to \Lambda^2(A)$ from the second tensor power to the second exterior power. Thus by the very definition one has the following exact sequence

$$0 \longrightarrow \Psi(A) \longrightarrow A \otimes A \longrightarrow \Lambda^2(A) \longrightarrow 0 \tag{3}$$

In this way one obtains the functor $\Psi : \mathsf{Ab} \to \mathsf{Ab}$. The functor Ψ commutes with filtered colimits and one has the following natural isomorphism

$$\Psi(A \oplus B) \cong \Psi(A) \oplus \Psi(B) \oplus (A \otimes B).$$

Furthermore, one has isomorphisms

$$\Psi(\mathbb{Z}) \cong \mathbb{Z}, \quad \Psi(\mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}.$$

2.3. WHITEHEAD Γ-FUNCTOR The functor Ψ is closely related with Whitehead Γ-functor, which is defined as follows. Let A and B be abelian groups. A quadratic map $f : A \to B$ is called *homogeneous* if $f(-a) = f(a)$. It follows then that

$$(a \mid a)_f = -(a \mid -a)_f = f(a) + f(-a) = 2f(a).$$

Based on this identity a simple induction argument shows that

$$f(na) = n^2 f(a).$$

It is well known [16] that for any abelian group A there is a universal homogeneous quadratic function $\gamma : A \to \Gamma(A)$, meaning that for any homogeneous quadratic map $f : A \to B$ there exists a unique homomorphism $h : \Gamma(A) \to B$, such that $f = h \circ \gamma$. The functor $A \mapsto \Gamma(A)$ is known as *the Whitehead's quadratic functor*. It is well known [16] that the functor Γ commutes with filtered colimits and one has the following natural isomorphism

$$\Gamma(A \oplus B) \cong \Gamma(A) \oplus \Gamma(B) \oplus (A \otimes B).$$

Furthermore, one has isomorphisms

$$\Gamma(\mathbb{Z}) \cong \mathbb{Z},$$

$$\Gamma(\mathbb{Z}/2^n\mathbb{Z}) \cong \mathbb{Z}/2^{n+1}\mathbb{Z}$$

and for any odd prime one has an isomorphism

$$\Gamma(\mathbb{Z}/p^n\mathbb{Z}) \cong \mathbb{Z}/p^n\mathbb{Z}.$$

It follows that if $a \in A$ is of order n, then $\gamma(a) \in \Gamma(A)$ is of order n if n is odd and it is of order $2n$, provided n is even.

We have a natural homomorphism $\tau : \Gamma(A) \to A \otimes A$, given by $\tau(\gamma(a)) = a \otimes a$. It is clear that the image of τ lies in $\Psi(A)$ and in this way one gets a natural homomorphism

$$\tau' : \Gamma(A) \to \Psi(A).$$

It is well known that τ' is an epimorphism, moreover it is an isomorphism provided $A = \mathbb{Z}$, or $A = \mathbb{Z}/n\mathbb{Z}$ with odd n. To identify the kernel of this map we need additional notations.

For each $n \geq 1$ and each abelian group A we put $t_n(A) = \{a \in A \mid 2^n a = 0\}$. Multiplication by 2 yields the natural transformation $t_{n+1} \to t_n$. We have also an inclusion $t_{n-1} \hookrightarrow t_n$. Here and elsewhere we assume that $t_0 = 0$. Thus one obtains a natural transformation $t_{n+1} \oplus t_{n-1} \to t_n$, whose cokernel is denoted by Φ_n. It follows that $\Phi_n : \mathsf{Ab} \to \mathsf{Ab}$ is a well-defined additive functor, which commutes with filtered colimits and

$$\Phi_n(\mathbb{Z}) = 0, \quad \Phi_n(\mathbb{Z}/p^k\mathbb{Z}) = 0$$

if p is an odd prime. It is also clear that

$$\Phi_n(\mathbb{Z}/2^k\mathbb{Z}) = 0, \quad \text{if } k \neq n$$

and

$$\Phi_n(\mathbb{Z}/2^n\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}.$$

LEMMA 6. *For each $n \geq 1$ there is a well-defined homomorphism $\iota_n : \Phi_n(A) \to \Gamma(A)$ given by $\iota_n(a) = 2^n\gamma(a)$.*

Proof. If $a, b \in t_n(A)$, then one has

$$2^n\gamma(a+b) = 2^n\gamma(a) + 2^n\gamma(b) + 2^n(a \mid b)_\gamma = 2^n\gamma(a) + 2^n\gamma(b)$$

Here we used the fact that $(a \mid b)_\gamma$ is linear in a, and therefore $2^n(a \mid b)_\gamma = (2^n a \mid b)_\gamma = 0$. On the other hand if $a = 2b$, then $2^n\gamma(a) = 2^n\gamma(2b) = 2^{n+2}\gamma(b) = 0$, because $2^{n+1}b = 0$. Similarly, if $a \in t_{n-1}$, then $2^{n-1}a = 0$ and therefore $2^n\gamma(a) = 0$. Thus ι_n is a well-defined homomorphism. ∎

The collection ι_n, $n \geq 1$, defines the natural transformation $\iota : \Phi \to \Gamma$, where $\Phi = \bigoplus_{n \geq 1} \Phi_n$.

PROPOSITION 7. *For any abelian group A the kernel of the natural map*

$$\tau' : \Gamma(A) \to \Psi(A)$$

is isomorphic to $\Phi(A)$, thus one has an exact sequence:

$$0 \longrightarrow \Phi(A) \xrightarrow{\iota} \Gamma(A) \xrightarrow{\tau} A \otimes A \longrightarrow \Lambda^2(A) \longrightarrow 0$$

Proof. Let us observe that $\Gamma \to \otimes^2$ induces a monomorphism on the second cross-effect and therefore $\Phi' := \mathsf{Ker}(\Gamma \to \otimes^2)$ is an additive functor. To show that ι yields an isomorphism $\Phi \to \Phi'$ it suffices to evaluate on cyclic groups, because both functors in question are additive and preserve filtered colimits. Since both functors vanish on $\mathbb{Z}$ and on $\mathbb{Z}/n\mathbb{Z}$ with odd n, we have to consider only the case, when $A = \mathbb{Z}/2^n\mathbb{Z}$. Since $\Gamma(\mathbb{Z}/2^n\mathbb{Z})$ is the cyclic group of order 2^{n+1} generated by $\gamma(1)$ it follows that $\Phi'(\mathbb{Z}/2^n\mathbb{Z})$ is the cyclic group of order two generated by $2^n\gamma(1)$. On the other hand $\Phi_k(\mathbb{Z}/2^n\mathbb{Z}) = 0$, provided $n \neq k$ and $\Phi_n(\mathbb{Z}/2^n\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ and the result follows. ∎

COROLLARY 8. *For any abelian group A the natural transformation $\Gamma(A) \to \mathbb{Z}/2\mathbb{Z} \otimes A$ induced by $\gamma(a) \mapsto a(\mathsf{mod}\ 2A)$ factors trough $\Psi(A)$.*

Proof. It suffices to note that the composite $\Phi_n(A) \to A/2A$ is induced by $a \mapsto 2^n a = 0$, $a \in t_n(A)$. ∎

The functors P and Γ are related via the natural transformation $\nu : P \to \Gamma$, which is given by $\nu(p(a)) = \gamma(a)$. Since any homogeneous quadratic function is quadratic it follows that this transformation is an epimorphism. To identify the kernel, let us observe that the map $f : A \to P(A)$ given by $f(a) = p(a) - p(-a)$ is linear. Indeed we have

$$(a \mid b)_f = (a \mid b)_p - (-a \mid -b)_p = 0$$

because $(- \mid -)_p$ is bilinear.

LEMMA 9. *One has the exact sequence*

$$0 \longrightarrow {}_2A \longrightarrow A \xrightarrow{f} P(A) \xrightarrow{\nu} \Gamma(A) \longrightarrow 0$$

where $f(a) = p(a) - p(-a)$ and ${}_2A = \{a \in A \mid 2a = 0\}$.

Proof. It is clear that the transformation $\nu : P \to \Gamma$ yields an isomorphism on the second cross-effects. Thus the kernel of ν is linear. It is clear that f yelds the transformation from the identity functor to the kernel of $\nu : P \to \Gamma$ and since both functors Id and $\mathsf{Ker}(\nu)$ are linear and preserve filtered colimits, it suffices to observe that the result is true for a cyclic group A. ∎

3. Algebraic models of one-connected two stage spaces

3.1. One-connected two stage spaces For any $n \geq 2$, we let $\mathsf{types}(n, n+1)$ be the homotopy category of such pointed CW-complexes X that $\pi_i X = 0$ for all $i \neq n, n+1$. If X is an object of $\mathsf{types}(n, n+1)$, then $\pi_i(\Sigma X) = 0$ if $i < n+1$, thus the $(n+2)$-th stage of the Postnikov tower of ΣX belongs to $\mathsf{types}(n+1, n+2)$. This yields the functor

$$P_{n+2}\Sigma : \mathsf{types}(n, n+1) \to \mathsf{types}(n+1, n+2)$$

which is known to be an equivalence of categories, provided $n \geq 3$.

This category is closely related to the category $\Pi(n, n+1)$ of k-invariants [1], whose objects are triples (π_n, π_{n+1}, k), where π_n and π_{n+1} are abelian groups and $k : \Gamma_n(\pi_n) \to \pi_{n+1}$ is a homomorphism. Here for an abelian group A and a natural number $n \geq 2$, we let $\Gamma_n(A)$ be $\Gamma(A)$ if $n = 2$ and $\mathbb{Z}/2\mathbb{Z} \otimes A$ if $n \geq 3$. A morphism f from (π_n, π_{n+1}, k) to (π'_n, π'_{n+1}, k') is a pair (f_n, f_{n+1}), where $f_n : \pi_n \to \pi'_n$ and $f_{n+1} : \pi_{n+1} \to \pi'_{n+1}$ are homomorphisms of abelian groups such that the diagram

$$\begin{array}{ccc} \Gamma_n(\pi_n) & \xrightarrow{k} & \pi_{n+1} \\ {\scriptstyle \Gamma_n(f_n)}\downarrow & & \downarrow{\scriptstyle f_{n+1}} \\ \Gamma_n(\pi'_n) & \xrightarrow[k']{} & \pi'_{n+1} \end{array}$$

commutes. Taking the nontrivial k-invariant yields the functor

$$\kappa : \mathsf{types}(n, n+1) \to \Pi(n, n+1), \quad n \geq 2$$

which fits in the following linear extension of categories [1], [3]

$$0 \to D_n \to \mathsf{types}(n, n+1) \to \Pi(n, n+1) \to 0,$$

where D_n is a bifunctor on $\Pi(n, n+1)$, given by

$$D_n((\pi_n, \pi_{n+1}, k), (\pi'_n, \pi'_{n+1}, k')) = \mathsf{Ext}(\pi_n, \pi'_{n+1}).$$

In particular $\kappa : \mathsf{types}(n, n+1) \to \Pi(n, n+1)$ yields a bijection on isomorphism classes of objects, moreover κ reflects isomorphisms and is surjective on morphisms.

3.2. Braided and symmetric categorical groups We also need the following well-known algebraic models for $\mathsf{types}(n, n+1)$, $n \geq 2$ [8],[10], [4].

Definition 10. A *braided categorical group*, shortly BCG, consists of the following data

$$C = (\partial : C_{ee} \to C_e, \quad \{-,-\} : C_e \times C_e \to C_{ee})$$

where C_e and C_{ee} are groups and ∂ is a homomorphism, while $\{-,-\}$ is a map such that the following equalities hold for $x, y, z \in C_e$ and $a, b \in C_{ee}$.

$$\partial\{x,y\} = x^{-1}y^{-1}xy$$

$$\{\partial a, \partial b\} = a^{-1}b^{-1}ab$$

$$\{\partial a, x\}\{x, \partial a\} = 1$$

$$\{x, yz\} = \{x,z\}\{x,y\}\{y^{-1}x^{-1}yx, z\}$$

$$\{xy, z\} = \{y^{-1}xy^{-1}, y^{-1}zy\}\{y,z\}.$$

A braided categorical group is called *symmetric categorical group*, shortly SCG, if

$$\{x,y\}\{y,x\} = 1. \tag{4}$$

It follows that $\mathsf{Ker}(\partial)$ is an abelian group and $\mathsf{Im}(\partial)$ is a normal subgroup of C_e and $\mathsf{Coker}(\partial)$ is an abelian group. One puts

$$\pi_0^C := \mathsf{Coker}(\partial), \quad \pi_1^C := \mathsf{Ker}(\partial).$$

The BCG's and SCG's form categories in an obvious way. A morphism of BCG's (resp. SCG's) is called *weak equivalence* if it induces an isomorphism on π_i, $i = 0, 1$. Let $\mathsf{Ho}(\mathsf{BCG})$ (resp. $\mathsf{Ho}(\mathsf{SCG})$) denote the localization of $\mathsf{Ho}(\mathsf{BCG})$ (resp. $\mathsf{Ho}(\mathsf{SCG})$) with respect to weak equivalences.

Let us note that BCG's are termed reduced 2-modules in [4], while SCG's are termed stable 2-modules in [4]. Thanks to [8] one knows that the category of braided categorical groups is equivalent to the category of such simplicial

groups G_*, that $N_iG_* = 0$, if $i \neq 1, 2$. Here N_*G_* denotes the Moore normalization of G_*. Similarly the category of symmetric categorical groups is equivalent to the category of such simplicial groups G_*, that $N_iG_* = 0$, if $i \neq n, n+1$ for a fixed $n > 1$. Therefore the classifying space functor induces the functors

$$b_2 : \mathsf{BCG} \to \mathsf{types}(2, 3)$$

and

$$b_n : \mathsf{SCG} \to \mathsf{types}(n, n+1),\ n \geq 3$$

such that $\pi_n b_n(C) = \pi_0^C$ and $\pi_{n+1} b_{n+1}(C) = \pi_1^C$, $n \geq 2$.

The inclusion functor $\mathsf{SCG} \subset \mathsf{BCG}$ has the left adjoint functor $\lambda : \mathsf{BCG} \to \mathsf{SCG}$, which is obtained by

$$\lambda(C) = (\partial : C'_{ee} \to C_e, \{-,-\} : C_e \times C_e \to C'_{ee}),$$

where C'_{ee} is the quotient of C_{ee} by the relation (4).

Then the functor λ makes the following diagram commute:

$$\begin{array}{ccc} \mathsf{BCG} & \longrightarrow & \mathsf{types}(2,3) \\ \downarrow{\scriptstyle \lambda} & & \downarrow{\scriptstyle P_4\Sigma} \\ \mathsf{SCG} & \longrightarrow & \mathsf{types}(3,4). \end{array}$$

According to [10],[4] one has the equivalence of categories

$$\mathsf{Ho}(\mathsf{BCG}) \cong \mathsf{types}(2, 3)$$

and

$$\mathsf{Ho}(\mathsf{SCG}) \cong \mathsf{types}(n, n+1), \quad n \geq 3.$$

4. Presquare groups

4.1. Quadratic functors on pointed finite sets Let Γ be the category of finite pointed sets and let Groups be the category of groups. We consider functors $F : \Gamma \to \mathsf{Groups}$ with the property $F([0]) = 0$. Here and elsewhere $[n]$ denotes the set $\{0, \cdots, n\}$, with basepoint 0. The functor F is *linear* if the map

$$(Fr_1, Fr_2) : F(X \vee Y) \to F(X) \times F(Y)$$

is an isomorphism, where $X \vee Y$ is the sum in the category Γ and $r_1 : X \vee Y \to X$, $r_2 : X \vee Y \to Y$ are the retractions. Furthermore, F is *quadratic* if the

second cross-effect $F(X \mid Y) = \mathsf{Ker}(F(r_1), F(r_2))$ as a bifunctor is linear in X and Y.

Let $H\mathbb{Z} : \Gamma \to \mathsf{Groups}$ be the functor which assigns to a pointed set S the free abelian group generated by S modulo the relation $* = 0$, where $*$ is the basepoint of S. For any abelian group A, we let $HA : \Gamma \to \mathsf{Groups}$ be the functor given by $HA(S) = A \otimes H\mathbb{Z}(S)$. It is clear that HA is a linear functor. It is easy to prove that any linear functor $\Gamma \to \mathsf{Groups}$ is isomorphic to one of the form HA. Thus the assignment $A \mapsto HA$ is an equivalence between the category of abelian groups and the category of linear functors $\Gamma \to \mathsf{Groups}$. The category of quadratic functors $\Gamma \to \mathsf{Groups}$ has the following description [15].

DEFINITION 11. A presquare group, shortly a PSG, consists of the following diagram

$$M = (\ M_e \times M_e \xrightarrow{\{-,-\}} M_{ee} \xrightarrow{\sigma} M_{ee} \xrightarrow{P} M_e\ ),$$

where M_{ee} is an abelian group and σ is a homomorphism with $\sigma^2 = \mathsf{Id}$. Moreover, M_e is a group written additively, P is a homomorphism and $\{-,-\}$ is a bilinear map, that is $\{x+y,z\} = \{x,z\} + \{y,z\}$ and $\{x,y+z\} = \{x,y\} + \{x,z\}$, for all $x, y, z \in M_e$. One requires that

(a) $P\sigma = P$,

(b) $\sigma\{x,y\} + \{y,x\} = 0$, $x, y \in M_e$,

(c) $P\{x,y\} = x + y - x - y$, $x, y \in M_e$,

(d) $\{x, Pa\} = 0$, $x \in M_e$, $a \in M_{ee}$.

It follows from (b) that for any PSG M one has $\{Pa, x\} = 0$. It follows from (c) and (d) that Pa lies in the centrum of M_e. Thus $\mathsf{Coker}(P)$ is well-defined and by (c) it is an abelian group. It follows that M_e is a group of nilpotency degree 2. It follows from the condition (a) that σ yields a well-defined involution on $\mathsf{Ker}(P)$.

If M and N are two PSG, then a morphism f from M to N consists of a pair of homomorphisms $f_e : M_e \to N_e, f_{ee} : M_{ee} \to N_{ee}$ such that f_{ee} commutes with involutions and the diagrams

$$\begin{array}{ccc} M_{ee} & \xrightarrow{P} & M_e \\ {\scriptstyle f_{ee}}\downarrow & & \downarrow{\scriptstyle f_e} \\ N_{ee} & \xrightarrow{P} & N_e \end{array} \qquad\qquad \begin{array}{ccc} M_e \times M_e & \xrightarrow{\{-,-\}} & M_{ee} \\ {\scriptstyle f_e \times f_e}\downarrow & & \downarrow{\scriptstyle f_{ee}} \\ N_e \times N_e & \xrightarrow{\{-,-\}} & N_{ee} \end{array}$$

commute. We let PSG be the category of presquare groups.

If M is a PSG and S is a pointed set with basepoint $*$, we let $S \odot M$ be the group generated by the symbols $s \odot x$ and $[s,t] \odot a$ with $s,t \in S$, $x \in M_e$, $a \in M_{ee}$ subject to the relations

$$[s,s] \odot a = s \odot P(a)$$

$$* \odot x = 0 = [*,s] \odot a$$

$$[s,t] \odot a = [t,s] \odot \sigma(a)$$

$$[s,t] \odot \{x,y\} = -t \odot x - s \odot y + t \odot x + s \odot y$$

where $s \odot x$ is linear in x and where $[s,t] \odot a$ is central and linear in a.

A result similar to [6] shows that the functor $S \mapsto S \odot M$ is a quadratic functor on Γ and in this way one gets the equivalence between the category PSG of presquare groups and the category of quadratic functors from Γ to Groups. Actually this is a very particular case of much more general results obtained in [15].

4.2. Homotopy and k-invariant of a presquare group Let M be a PSG. We set

$$\pi_1^M := \mathsf{Ker}(P : M_{ee} \to M_e) \quad \text{and} \quad \pi_0^M := \mathsf{Coker}(P : M_{ee} \to M_e).$$

The involution σ equips π_1^M with an involution, which is still denoted by σ.

For any $x \in M_e$ we let $\bar{x}$ be the class of x in π_0^M. It follows from the condition (d) of the definition of PSG, that $\{-,-\}$ factors through π_0^M and thanks to (b) it yields the homomorphism

$$\{-,-\} : \pi_0^M \otimes \pi_0^M \to M_{ee}^-,$$

where

$$M_{ee}^- := \{a \in M_{ee} \mid a + \sigma(a) = 0\}.$$

We also need the homomorphism $\omega : \Lambda^2(\pi_0^M) \to M_e$ which is induced by the commutator map:

$$\omega(\bar{x} \wedge \bar{y}) = x + y - x - y.$$

Consider the following diagram:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \Phi(\pi_0) & \xrightarrow{\iota} & \Gamma(\pi_0) & \xrightarrow{\tau} & \pi_0 \otimes \pi_0 & \longrightarrow & \Lambda^2(\pi_0) & \longrightarrow & 0 \\
 & & & & \vdots & & \downarrow{\scriptstyle \{-,-\}} & & \downarrow{\scriptstyle \omega} & & \\
 & & 0 & \longrightarrow & \pi_1 & \longrightarrow & M_{ee} & \xrightarrow{P} & M_e & \longrightarrow & \pi_0 \longrightarrow 0
\end{array}$$

where $\pi_i = \pi_i^M$, $i = 0, 1$. The diagram is commutative thanks to the property (c) of the definition of PSG. Since the columns are exact, we see that there is a well-defined morphism

$$k^M = k : \Gamma(\pi_0) \to \pi_1 \tag{5}$$

given by $k(\gamma(\bar{x})) = \{x, x\}$. A diagram-chase shows that $k \circ \iota = 0$. Furthermore, the condition (b) of the definition of PSG shows that the image of k lies in $\pi_1^- = \{b \in \pi_1 \mid b + \sigma(b) = 0\}$.

4.3. Stable homotopy and stable k-invariant of a presquare group We let PSG_s be the full subcategory consisting of objects M such that the involution on M_{ee} is trivial, that is $\sigma(a) = a$ for all $a \in M_{ee}$. In this case the bracket $\{-,-\} : \pi_0 \otimes \pi_0 \to M_{ee}$ factors through $\tilde{\Lambda}^2(\pi_0) \to M_{ee}$, where $\tilde{\Lambda}^2(A)$ is the quotient of $A \otimes A$ by the relation $a \otimes b + b \otimes a \sim 0$.

The inclusion $\mathsf{PSG}_s \subset \mathsf{PSG}$ has the left adjoint given by $M \mapsto \underline{M}$, where

$$\underline{M} = (\, M_e \times M_e \xrightarrow{\{-,-\}} M_{ee}/(\mathsf{Id} - \sigma) \xrightarrow{\mathsf{Id}} M_{ee}/(\mathsf{Id} - \sigma) \xrightarrow{P} M_e \,).$$

The fact that P is still well-defined follows from the property (a) of Definition 11. Moreover, the quotient map $M \to \underline{M}$ is a morphism in category PSG. We now put

$$\underline{\pi}_i^M := \pi_i^{\underline{M}}, \quad i = 0, 1.$$

Thus $\underline{\pi_0} = \pi_0$, while $\underline{\pi}_1 = \mathsf{Ker}(M_{ee}/(\mathsf{Id} - \sigma) \to M_e)$. In other words ${\underline{\pi_i}}^M$, $i = 0, 1$ is the i-th homology of the following chain complex

$$Q_*(M) := (\cdots \xrightarrow{\mathsf{Id}+\sigma} M_{ee} \xrightarrow{\mathsf{Id}-\sigma} M_{ee} \xrightarrow{\mathsf{Id}+\sigma} M_{ee} \xrightarrow{\mathsf{Id}-\sigma} M_{ee} \xrightarrow{P} M_e) \tag{6}$$

We define the homomorphism

$$\underline{k} : \mathbb{Z}/2\mathbb{Z} \otimes \pi_0^M \to {\underline{\pi}_1}^M$$

by

$$\underline{k}(\bar{x}) := \{x, x\}(\mathsf{mod}(\mathsf{Id} - \sigma)).$$

The homomorphism $\underline{k}$ fits in the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} \otimes \pi_0 & \xrightarrow{\nu} & \tilde{\Lambda}^2(\pi_0) & \longrightarrow & \Lambda^2(\pi_0) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \underline{k}} & & \downarrow & & \downarrow{\scriptstyle \omega} & & \\
0 & \longrightarrow & \underline{\pi}_1 & \longrightarrow & M_{ee}/(\mathsf{Id} - \sigma) & \xrightarrow{P} & M_e & \longrightarrow & \pi_0 \longrightarrow 0
\end{array}$$

where the homomorphism ν is induced by $\bar{x} \mapsto \bar{x} \otimes \bar{x}$, while the homomorphism $\tilde{\Lambda}^2(\pi_0) \to M_{ee}/(\mathsf{Id} - \sigma)$ is induced by $\{-,-\}$. The fact that the last map is well-defined can be checked as follows:

$$\{x,y\} + \{y,x\} = -\sigma\{y,x\} + \{y,x\} \in \mathsf{Im}(\mathsf{Id} - \sigma).$$

The commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \pi_1 & \longrightarrow & M_{ee} & \xrightarrow{P} & M_2 \\ & & \downarrow & & \downarrow & & \downarrow \mathsf{Id} \\ 0 & \longrightarrow & \underline{\pi}_1 & \longrightarrow & M_{ee}/(\mathsf{Id} - \sigma) & \xrightarrow{P} & M_e \end{array}$$

shows that there is a natural epimorphism $\epsilon^M : \pi_1^M \to \underline{\pi}_1^M$, which is an isomorphism provided $M \in \mathsf{PSG}_s$.

4.4. Product of presquare groups The category PSG possesses all limits and colimits. In the sequel we need the following explicit construction of the product in PSG. Let M and N be two PSG. Then $(M \times N)$ is the PSG given by

$$(M \times N)_e = M_e \times N_e,$$
$$(M \times N)_{ee} = M_{ee} \times N_{ee},$$
$$\sigma(a,c) = (\sigma(a), \sigma(c)),$$
$$P(a,c) = (Pa, Pc),$$
$$\{(x,u),(y,v)\} = (\{x,y\},\{u,v\}),$$

where $a \in M_{ee}, c \in N_{ee}$, $x, y \in M_e, u, v \in N_e$.

It is clear that the functors π_i, $i = 0, 1, \underline{\pi}_1$ (and the morphisms k, $\underline{k}$ as well) preserve the product.

4.5. Coproduct of presquare groups In the sequel we need also the following explicit construction of the coproduct in PSG. But first we recall few facts on the category Nil of nilpotent groups of class two. The inclusion functor

$$\mathsf{Nil} \to \mathsf{Groups}$$

has the left adjoint functor, which is given by the *nilization* functor:

$$G \mapsto G^{nil} = G/[G,[G,G]]$$

Let G_1 and G_2 be two objects in Nil, then the copruduct $G_1 \vee G_2$ in Nil is obtained by the nilization of the free product of the groups G_1 and G_2. It is well known that one has the following short exact sequence

$$0 \to G_1^{ab} \otimes G_2^{ab} \to G_1 \vee G_2 \to G_1 \times G_2 \to 0,$$

where $G^{ab} = G/[G,G]$ is the *abelization* of G. This shows that any element of $G_1 \vee G_2$ (all groups are written additively) can be written as a sum of elements $a+b+w$, where $a \in G_1$, $b \in G_2$ and w is a sum of commutators of the form $a_1 + b_1 - a_1 - b_1$, $a_1 \in G_1$ and $b_1 \in G_2$.

LEMMA 12. *Let*

$$0 \to A \to X \xrightarrow{\eta} G \to 0$$

and

$$0 \to B \to Y \xrightarrow{\xi} H \to 0$$

be central extensions in Nil *with abelian* G *and* H. *Define the group* Z *as the quotient* $(X \vee Y)/\sim$, *where the equivalence relation* $\sim$ *is generated by*

$$a + y \sim y + a$$

$$b + x \sim x + b$$

where $a \in A$, $b \in B$, $x \in X$ *and* $y \in Y$. *Then one has the following central extension of groups*

$$0 \to A \oplus B \oplus (G \otimes H) \xrightarrow{j} Z \to G \times H \to 0.$$

Here the homomorphism j *is given by* $j(a+b+g\otimes h) = a+b+(x+y-x-y)$, *where* $x \in X$ *and* $y \in Y$ *satisfy* $\eta(x) = g$ *and* $\xi(y) = h$.

Proof. It follows from the definition of the group Z that j is a well-defined homomorphism, whose image is a normal subgroup of Z. It is also clear that $\mathsf{Coker}(j) \cong G \times H$. It remains to show that j is a monomorphism. To this end let us recall that for any abelian group M which is considered as a trivial $(G \times H)$-module one has

$$H^2(G \times H, M) \cong H^2(G, M) \oplus H^2(Y, M) \oplus \mathsf{Hom}(G \otimes H, M).$$

We now take $M = A \oplus B \oplus (G \otimes H)$ and we let $cl(Z) \in H^2(G \times H, M)$ be the element whose components in the above decompositions are $i_{1*}(cl(X)), i_{2*}(cl(Y))$,

i_3. Here $cl(X) \in H^2(G, A)$ and $cl(Y) \in H^2(H, B)$ are elements defined by the given central extensions, while $i_1 : A \to M$, $i_e : B \to M$ and $i_3 : G \otimes H \to M$ are standard inclusions. The class $cl(Z)$ defines a central extension:

$$0 \to A \oplus B \oplus (G \otimes H) \to Z_1 \to G \times H \to 0$$

Since G and H are abelian groups, it follows that $Z_1 \in \mathsf{Nil}$. By our construction one has a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & X & \longrightarrow & G & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & Z_1 & \longrightarrow & G \times H & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & B & \longrightarrow & Y & \longrightarrow & H & \longrightarrow & 0
\end{array}$$

Thus we have a canonical morphism $X \vee Y \to Z_1$ and one easily shows that it yields the homomorphism $Z \to Z_1$ which makes the following diagram commute

$$\begin{array}{ccccccccc}
 & & A \oplus B \oplus (G \otimes H) & \longrightarrow & Z & \longrightarrow & G \times H & \longrightarrow & 0 \\
 & & \downarrow \mathsf{Id} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A \oplus B \oplus (G \otimes H) & \longrightarrow & Z_1 & \longrightarrow & G \times H & \longrightarrow & 0.
\end{array}$$

It follows that $j : A \oplus B \oplus (G \otimes H) \to Z$ is a monomorphism and the proof is finished. ∎

Now we construct coproducts in PSG. Let

$$M = (M_e \times M_e \overset{\{-,-\}_M}{\longrightarrow} M_{ee} \overset{\sigma_M}{\longrightarrow} M_{ee} \overset{P_M}{\longrightarrow} M_e)$$

$$N = (N_e \times N_e \overset{\{-,-\}_N}{\longrightarrow} N_{ee} \overset{\sigma_N}{\longrightarrow} N_{ee} \overset{N_M}{\longrightarrow} N_e)$$

be presquare groups. Let us recall that $M_e, N_e \in \mathsf{Nil}$. The coproduct

$$M \vee N = ((M \vee N)_e \times (M \vee N)_e \overset{\{-,-\}}{\longrightarrow} (M \vee N)_{ee} \overset{\sigma}{\longrightarrow} (M \vee N)_{ee} \overset{P}{\longrightarrow} (M \vee N)_e)$$

in the category PSG is given by

$$(M \vee N)_{ee} = M_{ee} \oplus N_{ee} \oplus \pi_0^M \otimes \pi_0^N \oplus \pi_0^N \otimes \pi_0^M$$

$$(M \vee N)_e = (M_e \vee N_e)/\sim$$

Here the equivalence relation is generated by

$$P_M(x) + c \sim c + P_M(x),$$

$$a + P_N(u) \sim P_N(u) + a,$$

for $x \in M_{ee}$, $c \in N_e$, $u \in N_{ee}$, $a \in M_e$. Let $\bar{x} \in \pi_0^M$, $\bar{u} \in \pi_0^N$ be the elements in cokernels represented by x and u respectively. The operators σ and P for $M \vee N$ are defined by

$$\sigma(x + u + \bar{a_1} \otimes \bar{c}_1 + \bar{c}_2 \otimes \bar{a}_2) = \sigma_M(x) + \sigma_N(u) + \bar{a_2} \otimes \bar{c_2} + \bar{c}_1 \otimes \bar{a}_1$$

$$P(x+u+\bar{a}_1\otimes\bar{c}_1+\bar{c}_2\otimes\bar{a}_2) = P_M(x)+P_N(u)+(a_1+c_1-a_1-c_1)+(c_2+a_2-c_2-a_2)$$

From this definition is it clear that $\pi_0^{M\vee N} \cong \pi_0^M \oplus \pi_0^N$. Now the map $\{-,-\} : \pi_0^{M\vee N} \otimes \pi_0^{M\vee N} \to (M \vee N)_{ee}$ is given by

$$\{\bar{a} + \bar{c}, \bar{a}_1 + \bar{c}_2\} = \{\bar{a}, \bar{a}_1\}_M + \{c_1, c_2\}_N + a \otimes c_1 + c \otimes a_1.$$

LEMMA 13. *For any $M, N \in \mathsf{PSG}$ one has the following isomorphisms*

$$\pi_0^{M\vee N} \cong \pi_0^M \oplus \pi_0^N, \quad \underline{\pi}_1^{M\vee N} \cong \underline{\pi}_1^M \oplus \underline{\pi}_1^N$$

$$\pi_1^{M\vee N} \cong \pi_1^M \oplus \pi_1^N \oplus (\pi_0^M \otimes \pi_0^N)$$

Proof. We already had the first isomorphism. To get other isomorphisms, one has to apply Lemma 12 to central extensions

$$0 \to \mathsf{Im}(P_M) \to M_e \to \pi_0^M \to 0$$

and

$$0 \to \mathsf{Im}(P_N) \to N_e \to \pi_0^N \to 0$$

to conclude that $\mathsf{Im}P_{M\vee N} \cong \mathsf{Im}(P_M) \oplus \mathsf{Im}(P_N) \oplus (\pi_0^M \otimes \pi_0^N)$ which obviously implies the result. ∎

4.6. A PUSHFORWARD CONSTRUCTION Let M be a PSG and let $f : \pi_1^M \to A$ be a homomorphism of abelian groups with involutions. We can form the pushout diagram in the category of abelian groups with involutions

$$\begin{array}{ccc} \pi_1^M & \xrightarrow{i} & M_{ee} \\ \downarrow{\scriptstyle f} & & \downarrow \\ A & \longrightarrow & f_*(M_{ee}). \end{array}$$

It follows from the properties of the pushout construction that we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_1^M & \xrightarrow{i} & M_{ee} & \xrightarrow{P} & M_e & \longrightarrow & \pi_0^M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow & & \downarrow{\scriptstyle \mathsf{Id}} & & \downarrow{\scriptstyle \mathsf{Id}} & & \\ 0 & \longrightarrow & A & \longrightarrow & f_*(M_{ee}) & \xrightarrow{f_*(P)} & M_e & \longrightarrow & \pi_0^M & \longrightarrow & 0 \end{array}$$

It is clear that $f_*(M)$ is also a PSG, where $f_*(M)_e = M_e$ and $f_*(M)_{ee} = f_*(M_{ee})$ and the map $M_e \times M_e \to f_*(M_{ee})$ is the composite of the map $M_e \times M_e \to M_{ee}$ and the homomorphism $M_{ee} \to f_*(M_{ee})$. Furthermore one has

$$\pi_0^{f_*(M)} = \pi_0^M, \quad \pi_1^{f_*(M)} = A$$

and $k^{f_*(M)} = f \circ k^M$.

4.7. PRESQUARE GROUPS AND THE UNIVERSAL COEFFICIENT THEOREM In this section we construct a collection of presquare groups using the universal coefficient theorem in group cohomology. Let us recall that for any abelian groups A and B there is a natural short exact sequence

$$0 \longrightarrow \mathsf{Ext}(A, B) \longrightarrow H^2(A, B) \xrightarrow{\mathsf{c}} \mathsf{Hom}(\Lambda^2(A), B) \longrightarrow 0$$

which has a splitting natural in B. Here we used the well-known isomorphism $H_2(A) \cong \Lambda^2(A)$. The homomorphism c is given by the commutator map: If

$$0 \to B \to G \to A \to 0$$

is a central extension, corresponding to an element $x \in H^2(A, B)$, then $\mathsf{c}(x) : \Lambda^2(A) \to B$ is given by $(a, b) \mapsto u + v - u - v$. Here u and v are liftings of a and b to the group G which is written additively.

Of special interest is the case when $B = \Lambda^2(A)$. We let T_A be the set of equivalence classes of central extensions

$$(N_A) = (\ 0 \longrightarrow \Lambda^2(A) \xrightarrow{\mu} N_A \xrightarrow{\rho} A \longrightarrow 0\ )$$

such that $\mathsf{c}(N_A) = \mathsf{Id}_{\Lambda^2(A)}$. The set T_A is nonempty and the group $\mathsf{Ext}(A, \Lambda^2(A))$ acts transitively and freely on T_A.

If $(N_A) \in \mathsf{T}_A$, then one can define the presquare group $\omega(N_A)$ as follows. By definition we put

$$\omega(N_A)_e = N_A,$$
$$\omega(N_A)_{ee} = A \otimes A,$$
$$P(a \otimes b) = \mu(a \wedge b),$$
$$\sigma(a \otimes b) = -b \otimes a,$$
$$\{x, y\} = \rho(x) \otimes \rho(y),$$

where $x, y \in N_A$ and $a, b \in A$. One easily checks that $\omega(N_A)$ is a PSG.

By our construction we have:

LEMMA 14. *For any abelian group A and any $(N_A) \in \mathsf{T}_A$ one has isomorphisms*

$$\pi_0^{\omega(N_A)} \cong A,$$
$$\pi_1^{\omega(N_A)} \cong \Psi(A)$$

under which $k^{\omega(N_A)}$ corresponds to the canonical homomorphism $\tau' : \Gamma(A) \to \Psi(A)$ induced by

$$\tau : \Gamma(A) \to A \otimes A, \quad x \mapsto x \otimes x.$$

Moreover, additionally one has

$$\underline{\pi}_1^{\omega(N_A)} \cong \mathbb{Z}/2\mathbb{Z} \otimes A$$

and $\underline{k}^{\omega(N_A)} = \mathsf{Id}_{\mathbb{Z}/2\mathbb{Z} \otimes A}$.

We can apply the functor $\mathsf{PSG} \to \mathsf{PSG}_s$, $M \mapsto \underline{M}$ to the presquare group $\omega(N_A)$. Here A is any abelian group and $N_A \in \mathsf{T}_A$. In this way one obtains an object $\underline{\omega}(N_A) \in \mathsf{PSG}$. By definition one has

$$\underline{\omega}(N_A)_e = N_e, \quad \underline{\omega}(N_A)_{ee} = \tilde{\Lambda}^2(A),$$

the structure homomorphisms are given by $\sigma = \mathsf{Id}_{\tilde{\Lambda}^2}$, $P(a\tilde{\wedge}b) = \mu(a \wedge b)$, and $\{x, y\} = \rho(x)\tilde{\wedge}\rho(y)$, where $x, y \in N_A$ and $a, b \in A$, compare with the definition of $\omega(N_A)$. By our construction we have:

LEMMA 15. *For any abelian group A and any $(N_A) \in \mathsf{T}_A$ one has isomorphisms*

$$\pi_0^{\omega(N_A)} \cong A,$$
$$\pi_1^{\omega(N_A)} \cong A/2A$$

which identify $k^{\omega(N_A)}$ with the canonical transformation $\Gamma(A) \to A/2A$ induced by $\gamma(x) \mapsto x(\mathsf{mod}\ 2A)$. Moreover, additionally one has

$$\underline{\pi}_1^{\omega(N_A)} \cong \mathbb{Z}/2\mathbb{Z} \otimes A$$

and $\underline{k}^{\omega(N_A)} = \mathsf{Id}_{\mathbb{Z}/2\mathbb{Z}\otimes A}$.

4.8. PRESQUARE GROUPS AND BRAIDED CATEGORICAL GROUPS Forgetting the involution one gets the functor

$$\Upsilon : \mathsf{PSG} \to \mathsf{BCG}$$

which is given by

$$(\ M_e \times M_e \xrightarrow{\{-,-\}} M_{ee} \xrightarrow{\sigma} M_{ee} \xrightarrow{P} M_e\ ) \mapsto ((P : M_{ee} \to M_e), \{-,-\}).$$

The same functor can be obtained in terms of functors on Γ as follows.

Let us recall that there is a standard way to prolong a functor $F : \Gamma \to \mathsf{Groups}$ to a functor from the category of pointed simplicial sets to the category of simplicial groups $\mathsf{s.Sets}_* \to \mathsf{s.Groups}$. First using direct limits one can prolong F to a functor from the category of pointed sets $\mathsf{Sets}_* \to \mathsf{Groups}$, then by degreewise action one obtains a functor from the category of pointed simplicial sets to the category of simplicial groups. By abuse of notation we will still denote this functor by F. In particular one can use this construction for the functor $F = (-) \odot M$ for a PSG M. In this paper we are particularly interested in the evaluation of $F = (-) \odot M$ on simplicial spheres and especially on S^1, which is the simplicial model of the circle with two nondegenerate simplices. Let us recall that S^1 is $[n]$ in dimension n. Moreover $s_i : [n] \to [n+1]$ is the unique monotone injection whose image does not contain $i+1$, while $d_i : [n] \to [n-1]$ is given by $d_i(j) = j$ if $j < i$, $d_i(i) = i$ if $i < n$, $d_n(n) = 0$ and $d_i(j) = j - 1$ if $j > i$.

LEMMA 16. *Let M be a PSG and $F = (-) \odot M : \Gamma \to \mathsf{Groups}$. Then the Moore complex associated to $F(S^1)$ is isomorphic to the following complex*

$$\cdots \to 0 \to M_{ee} \xrightarrow{P} M_e \to 0.$$

Proof. The fact that the Moore complex associated to $F(S^1)$ vanishes in dimensions > 2 is a particular case of Proposition 5.9 of [15] and the computations in dimensions 1 and 2 are trivial (compare also with (2.6) of [7]). ∎

Since the Moore complex of $F(S^1)$ is trivial in all dimensions except dimensions one and two, it follows from [8] that it corresponds to a BCG. Thanks to Lemma this particular BCG is nothing but $\Upsilon(M)$. In particular $BF(S^1)$ has only two nontrivial homotopy groups $\pi_2 B(S^1) \cong \pi_0^M$ and $\pi_3 B(S^1) \cong \pi_1^M$ and the unique nontrivial k-invariant is given by the map $k_M \in \mathsf{Hom}(\Gamma(\pi_0^M), \pi_1^M)$ constructed in equation (5). It follows that we have the following commutative diagram of categories and functors:

$$\begin{array}{ccccc} \mathsf{BCG} & \xleftarrow{\Upsilon} & \mathsf{PSG} & \xrightarrow{\kappa^*} & \Pi^*(2,3) \\ & \searrow & \downarrow{\scriptstyle ev(S^1)} & & \downarrow{\scriptstyle forgetful} \\ & & \mathsf{types}(2,3) & \xrightarrow[\kappa]{} & \Pi(2,3) \end{array}$$

Here $ev(S^1) : \mathsf{PSG} \to \mathsf{types}(2,3)$ is the functor which is given by

$$M \mapsto BF(S^1), \quad \text{where} \quad F = (-) \odot M,$$

while the category $\Pi^*(2,3)$ is defined as follows. An object of the category $\Pi^*(2,3)$ is a triple (π_2, π_3, k), where π_2 is an abelian group, π_3 is an abelian group with involution σ and $k : \Gamma(\pi_2) \to \pi_3^-$ is a homomorphism, where as usual we put

$$\pi_3^- := \{a \in \pi_3 \mid a + \sigma(a) = 0\}.$$

If (π_2, π_3, k) and (π_2', π_3', k') are objects of $\Pi^*(2,3)$, then a morphism f from (π_2, π_3, k) to (π_2', π_3', k') is a pair (f_2, f_3), where $f_2 : \pi_2 \to \pi_2'$ is a homomorphism of abelian groups, while $f_3 : \pi_3 \to \pi_3'$ is a homomorphism of abelian groups with involutions such that the diagram

$$\begin{array}{ccc} \Gamma(\pi_2) & \xrightarrow{k} & \pi_3^- \\ {\scriptstyle \Gamma(f_2)}\downarrow & & \downarrow{\scriptstyle f_3} \\ \Gamma(\pi_2') & \xrightarrow[k']{} & \pi_3'^- \end{array}$$

commutes. The functor

$$\kappa^* : \mathsf{PSQ} \to \Pi^*(2,3)$$

is given by

$$\kappa^*(M) = (\pi_0^M, \pi_1^M, k^M).$$

We have also the forgetful functor $\Pi^*(2,3) \to \Pi(2,3)$ which forgets the involution on π_3. This functor has the retraction given by the inclusion $\Pi(2,3) \hookrightarrow \Pi^*(2,3)$. Under this inclusion π_3 is considered as a group with involution, given by $\sigma(a) = -a$.

4.9. Realization of one-connected 3-types via presquare groups In this section we characterize objects of the categories $\mathsf{types}(2,3)$ and $\Pi^*(2,3)$ which are isomorphic to objects of the form $F(S^1)$ or $\kappa^*(M)$, where $M \in \mathsf{PSG}$ and $F = (-) \odot M$.

An object of $\Pi^*(2,3)$ (resp. $\Pi(2,3)$) is called *flat* if the composite

$$\Phi(\pi_2) \xrightarrow{\iota} \Gamma(\pi_2) \xrightarrow{k} \pi_3$$

is zero, where the functor Φ and the natural transformation ι were defined in Section 2.3, in other words k factors trough $\Psi(\pi_0)$. An object $X \in \mathsf{types}(2,3)$ is called *flat* provided $\kappa(X)$ is flat.

Theorem 17. i) *The values of κ^* (and therefore of κ as well) are flat.*

ii) *Let (π_2, π_3, k) be a flat object of the category $\Pi^*(2,3)$. Then there exist $M \in \mathsf{PSG}$ and an isomorphism $\kappa^*(M) \cong (\pi_2, \pi_3, k)$ in $\Pi^*(2,3)$.*

iii) *An object $X \in \mathsf{types}(2,3)$ is isomorphic to an object of the form $F(S^1)$, with quadratic $F : \Gamma \to \mathsf{Groups}$ iff X is flat.*

Proof. Part iii) is an immediate consequence of i) and ii) and properties of linear extensions of categories [3]. The first statement follows from the diagram chase based on the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Phi(\pi_0) & \xrightarrow{\iota} & \Gamma(\pi_0) & \xrightarrow{\tau} & \pi_0 \otimes \pi_0 \\ & & & & \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle \{-,-\}} \\ & & 0 & \longrightarrow & \pi_1 & \longrightarrow & M_{ee} \end{array}$$

For the second part we prove that a pushforward construction applied on $\omega(N_A)$ does the job. Here N_A is any element of T_A, where $A = \pi_2$. Indeed, we already observed that

$$\kappa^*(\omega(N_A)) = (A, \Psi(A), \tau')$$

where the involution on $\Psi(A)$ is given by $z \mapsto -z$, $z \in \Psi(A)$. Let us now take a flat object (π_2, π_3, k) of the category $\Pi^*(2,3)$. It follows that one has the commutative diagram

$$\begin{array}{ccc} \Gamma(\pi_2) & \xrightarrow{\tau'} & \Psi(\pi_2) \\ & \searrow^{k} & \downarrow{\scriptstyle k'} \\ & & \pi_3 \end{array}$$

Thus one can take the pushforward construction $M = k'_*(\omega(N_A))$, $A = \pi_2$. Then one has $\kappa^*(M) \cong (\pi_2, \pi_3, k)$ in $\Pi^*(2,3)$ and we are done. ∎

4.10. Presquare groups and symmetric categorical groups It is clear that $\Upsilon(M)$ is a symmetric categorical group provided $M \in \mathsf{PSG}_s$. On the other hand one can take the composite of functors $\Upsilon : \mathsf{PSG} \to \mathsf{BCG}$ and $\lambda : \mathsf{BCG} \to \mathsf{SCG}$ to get the functor

$$\lambda \circ \Upsilon : \mathsf{PSG} \to \mathsf{SCG}$$

It is clear that

$$\lambda(\Upsilon(M)) = ((P : M_{ee}/(\mathsf{Id} - \sigma) \to M_e), \{-,-\}).$$

Thus one has the following commutative diagram

$$\begin{array}{ccccc} \mathsf{PSG}_s & \xrightarrow{i} & \mathsf{PSG} & \xrightarrow{j} & \mathsf{PSG}_s \\ \downarrow{\scriptstyle \Upsilon} & & \downarrow{\scriptstyle \Upsilon} & \overset{\lambda\circ\Upsilon}{\searrow} & \downarrow{\scriptstyle \Upsilon} \\ \mathsf{SGC} & \xrightarrow{i_1} & \mathsf{BCG} & \xrightarrow[\lambda]{} & \mathsf{SCG} \end{array}$$

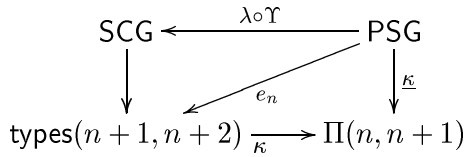
where i and i_1 are the inclusions, while $j(M) = \underline{M}$.

Let us fix a natural number $n \geq 2$ and let S^n be a simplicial model of the n-dimensional sphere, which has only two nondegenerate simplices. For any functor $F : \Gamma \to \mathsf{Groups}$ one obtains the simplicial group $F(S^n)$ by applying the functor F on S^n. If F is quadratic, then the Moore normalization of S^n is trivial in dimensions $> 2n$ and $< n$ and it is isomorphic to

$$\to \cdots 0 \to Q_n(M) \to \cdots \to Q_0(M) \to 0 \cdots \to 0$$

where $F = (-) \odot M$ and $Q_*(M)$ is defined in (6). As we see for $n \geq 2$ the space $BF(S^n)$ in general does not belong to $\mathsf{types}(n+1, n+2)$. However

one can take the $(n+2)$-th stage of the Postnikov tower of $BF(S^n)$, which is denoted by $e_n(M)$. It follows that one has the following commutative diagram of categories and functors:



where $\underline{\kappa} : \mathsf{PSG} \to \Pi(n+1, n+2)$ is given by $M \mapsto (\pi_0^M, \underline{\pi_1}^M, \underline{k}^M)$.

THEOREM 18. *Let $n \geq 2$. For any object X of the category $\mathsf{types}(n+1, n+2)$ there exists an object $M \in \mathsf{PSG}_s$ and an isomorphism $e_n(M) \cong X$ in $\mathsf{types}(n+1, n+2)$.*

Proof. Since the functor $\kappa : \mathsf{types}(n+1, n+2) \to \Pi(n+1, n+2)$ induces bijection on isomorphism classes of objects and realizes all morphisms in $\Pi(n+1, n+2)$ it suffices to prove that for any object $(\pi_{n+1}, \pi_{n+2}, k)$ of the category $\Pi(n+1, n+2)$ there exists an object $M \in \mathsf{PSG}$ and an isomorphism $\underline{\kappa}(M) \cong (\pi_{n+1}, \pi_{n+2}, k)$ in the category $\Pi(n+1, n+2)$. The proof of this statement is quite similar to the proof of Theorem 17. Let us recall that for any abelian group A and any element $N_A \in \mathsf{T}_A$ in Section 4.6 we constructed $\underline{\omega}(N_A) \in \mathsf{PSG}_s$ with the property

$$\underline{\kappa}(\underline{\omega}(N_A)) = (A, A/2A, \mathsf{Id}_{A/2A}).$$

Take now any object $(\pi_{n+1}, \pi_{n+2}, k) \in \Pi(n+1, n+2)$, where $k : \pi_n/2\pi_n \to \pi_{n+1}$ is a homomorphism. One can take the pushforward construction $k_*(\underline{\omega}(N_A))$, $A = \pi_n$ to get an object of expected kind. ∎

5. SQUARE GROUPS

5.1. QUADRATIC FUNCTORS ON THE CATEGORY OF FINITELY GENERATED FREE GROUPS We now consider functors $F : \mathsf{Gr_f} \to \mathsf{Groups}$, where $\mathsf{Gr_f}$ is the category of finitely generated free groups. For groups G_1 and G_2, we let $G_1 * G_2$ be the coproduct in Groups. The functor $F : \mathsf{Gr_f} \to \mathsf{Groups}$ is *linear* if the map

$$(Fr_1, Fr_2) : F(X * Y) \to F(X) \times F(Y)$$

is an isomorphism, where $r_1 : X * Y \to X$, $r_2 : X * Y \to Y$ are the retractions. Moreover F is *quadratic* if $F(X \mid Y) = \mathsf{Ker}(Fr_1, Fr_2)$ as a bifunctor is linear

in X and Y. The main result of [6] shows that the category of such quadratic functors $\mathsf{Gr_f} \to \mathsf{Groups}$ is equivalent to the category of square groups. Here *a square group* is a diagram

$$Q = (Q_e \xrightarrow{H} Q_{ee} \xrightarrow{P} Q_e)$$

where Q_{ee} is an abelian group and Q_e is a group. Both groups are written additively. Moreover P is a homomorphism and H is a map such that the cross effect

$$(x \mid y)_H := H(x+y) - H(x) - H(y)$$

is linear in $x, y \in Q_e$. In addition the following properties are satisfied

$$(Pa \mid x)_H = 0,$$

$$P(x \mid y)_H = x + y - x - y,$$

$$PHP(a) = P(a) + P(a),$$

where $x, y \in Q_e$ and $a, b \in Q_{ee}$. It follows from the first two identities that P maps to the center of Q_e. The second equation shows also that $\mathsf{Coker}(P)$ is abelian. Hence Q_e is a group of nilpotency degree 2. For square groups one has the following additional formulas (see [6]):

$$(x \mid Pa)_H = 0,$$

$$H(x+y-x-y) = -(y \mid x)_H + (x \mid y)_H.$$

Now we relate the square groups with presquare groups.

LEMMA 19. *Let Q be a square group. Then*

$$\wp(Q) = (Q_e, Q_{ee}, \sigma = HP - \mathsf{Id}, (-,-)_H, P)$$

is a presquare group.

Proof. The axioms (b) and (c) of the definition of PSG hold by the definition of square group. Let us observe that, HP is a homomorphism thanks to the identity $(Px \mid a)_H = 0$. Thus one has

$$\sigma^2 = HPHP - 2HP + \mathsf{Id} = H(2P) - 2HP + \mathsf{Id} = \mathsf{Id},$$

which shows that σ is an involution. We have also

$$P\sigma = P(HP - \mathsf{Id}) = PHP - P = P$$

and

$$\sigma(x \mid y)_H + (y,x)_H = HP(x \mid y)_H - (x,y)_H + (y,x)_H = 0.$$

Here we used the identity $P(x \mid y)_H = x + y - x - y$ and known expression for $H(x+y-x-y)$. ∎

We let SG be the category of square groups. The presquare group $\wp(Q)$ is called the underlying PSG of a square group Q. By abuse of notations we write $\pi_0^Q, \pi_1^Q, \underline{\pi}_1^Q$ and k^Q instead off $\pi_0^{\wp(Q)}, \pi_1^{\wp(Q)}, \underline{\pi}_1^{\wp(Q)}$ and $k^{\wp(Q)}$.

Let G be a group and let Q be a square group. We define the group $G \otimes Q$ by the generators $g \otimes x$ and $[g,h] \otimes a$ with $g, h \in G, x \in Q_e$ and $a \in Q_{ee}$ subject to the relations

$$(g+h) \otimes x = g \otimes x + h \otimes x + [g,h] \otimes H(x)$$

$$[g,g] \otimes a = g \otimes P(a)$$

where $g \otimes x$ is linear in x and where $[g,h] \otimes a$ is central and linear in each variable g, h and a. In this way one gets a bifunctor

$$\otimes : \mathsf{Gr_f} \times \mathsf{SG} \to \mathsf{Groups}$$

One can prove ([6]) that in addition the following identities hold:

$$[g,h] \otimes a = [h,g] \otimes \sigma(a), \quad \sigma = HP - \mathsf{Id}$$

$$-h \otimes x - g \otimes y + h \otimes x + g \otimes y = [g,h] \otimes (x \mid y)_H.$$

For any $Q \in \mathsf{SG}$ the functor $(-) \otimes Q : \mathsf{Gr_f} \to \mathsf{Groups}$ is quadratic and any quadratic functor is isomorphic to $(-) \otimes Q : \mathsf{Gr_f} \to \mathsf{Groups}$ with appropriate $Q \in \mathsf{SG}$ [6].

In terms of quadratic functors the relation between $(-) \otimes Q$ and $(-) \odot \wp(Q)$ can be seen as follows. For a pointed set S we let $\langle S \rangle$ be the free group generated by S modulo the relation $* = 0$, where $*$ is the base point of S. Then one has a natural isomorphism

$$S \odot \wp(Q) \ \cong \ \langle S \rangle \otimes Q.$$

In other words the following diagram commutes

$$\begin{array}{ccc} \Gamma & \xrightarrow{\langle - \rangle} & \mathsf{Gr_f} \\ & \searrow^{\odot \wp(Q)} & \downarrow{\scriptstyle \otimes Q} \\ & & \mathsf{Groups} \end{array}$$

5.2. Product and coproduct of square groups In the sequel we need the following explicit construction of the product and coproduct in SG. Let M and N be two SG. Then $(M \times N)$ is the SG given by

$$(M \times N)_e = M_e \times N_e,$$

$$(M \times N)_{ee} = M_{ee} \times N_{ee},$$

$$H(x, y) = (H_M(x), H_N(y))$$

$$P(a, c) = (P_M a, P_N c).$$

Thus the functor $\wp : \mathsf{SG} \to \mathsf{PSG}$ commutes with products. As our next construction shows it commutes also with coproducts. By abuse of notation we denote the underlying presquare groups of M and N still by M and N. We can consider the coproduct $M \vee N$ in PSG. Define

$$H : (M \vee N)_e \to (M \vee N)_{ee}$$

by

$$H(x + u + (a_1 + c_1 - a_1 - c_1)) = H_M(a) + H_N(c) + \bar{a} \otimes \bar{c}_1 - \bar{c}_1 \otimes \bar{a}_1$$

One checks that in this way one really gets the coproduct in SG (see 7.11 of [6]).

5.3. Lifting of PSG We are going to answer the following question. For a given $M \in \mathsf{PSG}$ under what conditions does there exists a square group Q such that $\wp(Q) \cong M$? If such Q exists it is called a lifting of M.

It is easy to see that not all PSG have liftings. Indeed, take $M_e = 0$ and $M_{ee} = \mathbb{Z}$. We let σ be the trivial involution on M_{ee} and $P = 0$, $\{-, -\} = 0$. Then one obtains a PSG. This particular PSG is not of the form $\wp(Q)$, because if $P = 0$ in a square group, then $\sigma = HP - \mathsf{Id} = -\mathsf{Id}$. This show that unlike the linear functors not any quadratic functor $\Gamma \to \mathsf{Groups}$ factors through $\mathsf{Gr_f}$.

As the following easy lemma shows if a PSG M has a lifting $Q \in \mathsf{SG}$ such a lifting in general is not unique. In fact the set of liftings is a torsor on an appropriate group.

Lemma 20. *a) Let Q be a square group and let $\alpha : \pi_0^Q \to Q_{ee}$ be a homomorphism. We set*

$$Q^\alpha_e = Q_e, \quad Q^\alpha_{ee} = Q_{ee}, \quad P^\alpha = P,$$

and

$$H^\alpha(x) = H(x) + \alpha(\bar{x})$$

where $x \in Q_e$ *and* $\bar{x}$ *denotes the class of* x *in* π_0^Q. *Then* Q^α *is a square group and* $\wp(Q) = \wp(Q^\alpha)$. *Conversely, if* Q *and* P *are two square groups with* $\wp(P) = \wp(Q)$ *then first of all* $P_e = Q_e$ *and* $P_{ee} = Q_{ee}$, *furthermore there exists a unique homomorphism* $\alpha : \pi_0^Q \to Q_{ee}$ *such that* $P = Q^\alpha$.

b) Let $Q, Q' \in \mathsf{SG}$ *and let* $f_e : Q_e \to Q'_e$ *and* $f_{ee} : Q_{ee} \to Q'_{ee}$ *be homomorphism of groups such that* $f = (f_e, f_{ee})$ *defines the morphism* $\wp(Q) \to \wp(Q')$ *in the category* PSG. *Then there exists a unique homomorphism* $\alpha(f) : \pi_0^Q \to Q'_{ee}$ *such that*

$$H' f_e(x) = f_{ee} H(x) + \alpha(f)(\bar{x}), \quad x \in Q_e.$$

In other words $\alpha(f) = 0$ *iff* f *is a morphism in* SG.

We now consider the problem under what conditions an object $M \in \mathsf{PSG}$ is isomorphic to one of the form $\wp(Q)$. Of course if such Q exists then $Q_e = M_e$ and $Q_{ee} = M_{ee}$. Moreover the map P in Q is the same as in M. Thus the problem is under what conditions does there exist H with appropriate properties.

5.4. THE CATEGORY PSG_0 We let PSG_0 be the full subcategory of the category PSG which consists of such M that

$$\pi_1^- = \pi_1.$$

In other words, one requires that if $Pa = 0$ for an element $a \in M_{ee}$, then $\sigma(a) = -a$.

LEMMA 21. *Let* $Q \in \mathsf{SG}$. *Then* $\wp(Q) \in \mathsf{PSG}_0$.

Proof. Let us recall that in $\wp(Q)$ the involution σ is given by $\sigma = HP - \mathsf{Id}$. Thus, if $Pa = 0$, then $\sigma(a) = -a$. ∎

LEMMA 22. *Let* $M \in \mathsf{PSG}_0$. *Then there exists the unique homomorphism*

$$h : \mathsf{Im}(P) \to M_{ee}$$

such that $hP(a) = a + \sigma(a)$.

Proof. Uniqueness is clear, because each element from $\mathsf{Im}(P)$ can be written as $P(a)$. To prove existence, we have to show that if $Pa = Pb$ then $a + \sigma(a) = b + \sigma(b)$. If this holds, then $a = b + c$ with $Pc = 0$. Thus

$$a + \sigma(a) = b + c + \sigma(b) + \sigma(c) = b + \sigma(b).$$

∎

LEMMA 23. *Let $M \in \mathsf{PSG}_0$. Then for the diagram*

$$A = (M_{ee} \xrightarrow{P} \mathsf{Im}(P) \xrightarrow{h} M_{ee})$$

one has $PhP = 2P$ and $hPh = 2$. In other words A is a quadratic $\mathbb{Z}$-module in the sense of [2].

Proof. For $a \in M_{ee}$ one has $PhP(a) = P(a) + P\sigma(a) = 2P(a)$. On the other hand we have $hPhP = h(2P) = 2hP$. Since $P : M_{ee} \to \mathsf{Im}(P)$ is an epimorphism it follows that $hPh = 2h$. ∎

5.5. A COHOMOLOGICAL OBSTRUCTION FOR LIFTING To each object $M \in \mathsf{PSG}$ one can associate two cohomological invariants. The first one is the class

$$[M_e] \in H^2(\pi_0, \mathsf{Im}(P)), \quad \pi_0 = \pi_0^M,$$

which is associated to the central extension of groups:

$$0 \to \mathsf{Im}(P) \to M_e \to \pi_0 \to 0.$$

The second one is the class

$$[M_{ee}] \in H^2(\pi_0, M_{ee})$$

which is represented by the 2-cocycle $f \in \mathsf{Z}^2(\pi_0, M_{ee})$, where

$$f(\bar{x}, \bar{y}) = \{x, y\}.$$

DEFINITION 24. Let $M \in \mathsf{PSG}_0$. Define the class

$$\vartheta(M) \in H^2(\pi_0, M_{ee})$$

by

$$\vartheta(M) := [M_{ee}] - h_*([M_e]).$$

Here $h_* : H^2(\pi_0, \mathsf{Im}(P)) \to H^2(\pi_0, M_{ee})$ is induced from the homomorphism h defined in Lemma 22.

THEOREM 25. *If $Q \in \mathsf{SG}$, then $\vartheta(\wp(Q)) = 0$. Conversely if $M \in \mathsf{PSG}_0$ is an object with $\vartheta(M) = 0$, then there exists a square group Q and an isomorphism $\wp(Q) \cong M$.*

Proof. Take $M \in \mathsf{PSG}_0$. Let us choose a set section $s : \pi_0 \to M_e$ of the quotient $M_e \to \pi_0$. One can assume that $s(0) = 0$. For any $x \in M_e$ one has $x - s(x) \in \mathsf{Im}(P)$. The class $[M_e]$ is represented by the 2-cocycle ξ, which is defined by

$$s(\bar{x}) + s(\bar{y}) = \xi(\bar{x}, \bar{y}) + s(\bar{x} + \bar{y}).$$

If $M = \wp(Q)$, then the map $h : \mathsf{Im}(P) \to M_{ee}$ is the restriction of H to $\mathsf{Im}(P)$. We set

$$g = H \circ s : \pi_0 \to M_{ee}.$$

One has

$$\begin{aligned} H(x) &= H(x - s\bar{x} + s\bar{x}) \\ &= h(x - s\bar{x}) + g(\bar{x}) + (x - s(x) \mid s(x))_H = h(x - s\bar{x}) + g(\bar{x}) \end{aligned}$$

because $x - s\bar{x} = P(a)$ for some $a \in M_{ee}$ and $\{P(a), s\bar{x}\} = 0 = (P(a) \mid s\bar{x})_H$. It follows that

$$H(x + y) = h(x + y - s(\bar{x} + \bar{y})) + g(\bar{x} + \bar{y}).$$

Since $y - s(\bar{y})$ lies in the center of M_e one can write

$$\begin{aligned} h(x + y - s(\bar{x} + \bar{y})) = h(x + y - s(\bar{y}) - s(\bar{x}) + \xi(\bar{x}, \bar{y})) = \\ h(x - s(\bar{x})) + h(y - s(\bar{y})) + h(\xi(\bar{x}, \bar{y})), \end{aligned}$$

because h is a homomorphism. Thus one obtains

$$\begin{aligned} (x \mid y)_H &= H(x + y) - H(x) - H(y) \\ &= h(x - s(\bar{x})) + h(y - s(\bar{y})) + h(\xi(\bar{x}, \bar{y}) + g(\bar{x} + \bar{y}) \\ &\quad - h(x - s(\bar{x})) - g(\bar{x}) - h(y - s(\bar{y})) - g(\bar{y}) \\ &= h(\xi(\bar{x}, \bar{y})) + (\bar{x} \mid \bar{y})_g. \end{aligned}$$

Since $(x \mid y)_H = (\bar{x} \mid \bar{y})_H$ represents the class $[M_{ee}]$, and the function $(\bar{x} \mid \bar{y})_g$ is the coboundary of g, we see that $\vartheta(M) = 0$. Conversely assume $M \in \mathsf{PSG}_0$ is such object that $\vartheta(M) = 0$. The first condition defines the homomorphism $h : \mathsf{Im}(P) \to M_{ee}$, while the second condition says that there exists a function $g : \pi_0 \to M_{ee}$ such that $\{x, y\} = (\bar{x} \mid \bar{y})_g + h(\xi(\bar{x}, \bar{y}))$. Now we can define $H : M_e \to M_{ee}$ by $H(x) = h(x - s\bar{x}) + g(\bar{x})$. One checks easily that M equipped with this H is indeed a square group. ∎

5.6. LIFTING PROBLEM FOR $\omega(N_A)$ In this section we consider the problem whether for a given abelian group A there exists an element $(N_A) \in \mathsf{T}_A$ such that $\omega(N_A)$ has a lifting as a square group. The answer to this question depends entirely on the element $\theta(A) \in \mathsf{Ext}(A, Sym^2(A))$, which was defined in (2) via the exact sequence (1), or equivalently via the canonical symmetric 2-cocycle f^s, given by $f^s(a,b) = ab \in Sym^2(A)$, $a, b \in A$. Let us recall that $\theta(A) = 0$ provided 2 is invertible in A (see Lemma 5).

As an application of Theorem 25 we obtain the following

THEOREM 26. *Let A be an abelian group. If there exists an element $(N_A) \in \mathsf{T}_A$ such that $\omega(N_A) \in \mathsf{PSG}$ has a lifting in SG then $\theta(A) = 0$. Conversely, if $\theta(A) = 0$, then there exists an element $(N_A) \in \mathsf{T}_A$ such that $\omega(N_A) \in \mathsf{PSG}$ has a lifting in SG. In particular such a lifting exists provided 2 is invertible in A or $\mathsf{Ext}(A, Sym^2 A) = 0$.*

Proof. First of all let us observe that for any abelian group A and any $(N_A) \in \mathsf{T}_A$ one has $\omega(N_A) \in \mathsf{PSG}_0$. For $\omega(N_A)$ we have $\mathsf{Im}(P) = \Lambda^2(A)$ and the homomorphism $h : \Lambda^2(A) \to A \otimes A$ is nothing but $h(a \wedge b) = a \otimes b - b \otimes a$. It follows then that the image of $\vartheta(\omega(N_A)) \in H^2(A, A \otimes A)$ under $\mathsf{c} : H^2(A, A \otimes A) \to \mathsf{Hom}(A, A \otimes A)$ is zero. Thanks to the universal coefficient theorem one has $\vartheta(\omega(N_A)) \in \mathsf{Ext}(A, A \otimes A)$. On the other hand one has also the short exact sequence

$$0 \to \Lambda^2(A) \to A \otimes A \to Sym^2(A) \to 0$$

where the first arrow is h. Thus one has exact sequences

$$\mathsf{Ext}(A, \Lambda^2(A)) \to \mathsf{Ext}(A, A \otimes A) \to \mathsf{Ext}(A, Sym^2(A)) \to 0$$

and

$$H^2(A, \Lambda^2(A)) \to H^2(A, A \otimes A) \to H^2(A, Sym^2(A))$$

Let us recall that $\vartheta(\omega(N_A)) = [M_{ee}] - h_*([M_e])$. One observes that the first term depends only on A and does not depend on $(N_A) \in \mathsf{T}_A$. It follows thus that the image of $\vartheta(\omega(N_A)) \in H^2(A, A \otimes A)$ in $H^2(A, S^2(A)$ is the same as the image of $[M_{ee}]$ in $H^2(A, A \otimes A)$. But $[M_{ee}]$ was represented by the cocycle $(a, b) \mapsto a \otimes b$ and therefore the image of $[M_{ee}]$ in $H^2(A, A \otimes A)$ lies in $\mathsf{Ext}(A, Sym^2(A))$ and it coincides with $\theta(A)$. If $(N_A) \in \mathsf{T}_A$ is such element that the presquare group $\omega(N_A)$ has lifting, then $\vartheta(\omega(N_A)) = 0$ and a fortiori $\theta(A) = 0$. Conversely, assume $\theta(A) = 0$, then the exact sequence for ext groups shows that there is an element $x \in \mathsf{Ext}(A, \Lambda^2(A))$ which maps to $\vartheta(\omega(N_A))$. But $\mathsf{Ext}(A, \Lambda^2(A))$ acts on T_A. Therefore using x we can correct

N to obtain another element $N' \in \mathsf{T}_A$ such that $\vartheta(\omega(N'_A)) = 0$ and we are done. ∎

COROLLARY 27. *If* $\mathsf{Ext}(A, A \otimes A) = 0$ *then the set* T_A *is a singleton and* $\omega(N_A) \in \mathsf{PSG}$ *has a lifting in* SG*, where* (N_A) *is the unique element of* T_A*.*

Proof. Since $\mathsf{Ext}(A, -) : \mathsf{Ab} \to \mathsf{Ab}$ is right exact, it follows that

$$\mathsf{Ext}(A, \Lambda^2(A)) = 0 = \mathsf{Ext}(A, Sym^2(A)).$$

The first equation shows that T_A is a singleton, while the second equations shows that such lifting exists. ∎

5.7. LIFTING PROBLEM FOR $\underline{\omega}(N_A)$ Let A be an abelian group. We let $\underline{\theta}(A) \in \mathsf{Ext}(A, Sym^2(A/2A))$ be the image of $\theta(A) \in \mathsf{Ext}(A, Sym^2(A))$ under the canonical map $Sym^2(A) \to Sym^2(A/2A)$. The following is a straightforward variation of the main result of the previous section:

LEMMA 28. *Let* A *be an abelian group. If there exists an element* $(N_A) \in \mathsf{T}_A$ *such that* $\underline{\omega}(N_A) \in \mathsf{PSG}$ *has a lifting in* SG *then* $\underline{\theta}(A) = 0$*. Conversely, if* $\underline{\theta}(A) = 0$*, then there exists an element* $(N_A) \in \mathsf{T}_A$ *such that* $\underline{\omega}(N_A) \in \mathsf{PSG}$ *has a lifting in* SG *.*

Proof. The only difference is to use the exact sequence

$$\Lambda^2(A) \to \tilde{\Lambda}^2(A) \to Sym^2(A/2A) \to 0$$

where the first map is induced by $a \wedge b \mapsto a\tilde{\wedge}b - b\tilde{\wedge}a = 2a\tilde{\wedge}b$. ∎

5.8. REALIZATION OF ONE-CONNECTED 3-TYPES VIA SQUARE GROUPS We have the functors

$$\mathsf{SG} \xrightarrow{\wp} \mathsf{PSG} \xrightarrow{\Upsilon} \mathsf{BCG} \xrightarrow{b_2} \mathsf{types}(2,3).$$

In this section we study the composite functor

$$e : \mathsf{SG} \to \mathsf{CW}(2,3).$$

From the homotopy theoretic point of view the functor e is the same as $Q \mapsto B((\Omega S^2) \otimes Q)$. Here ΩS^2 is the simplicial group, which is obtained by the degreewise action of the functor

$$\langle - \rangle : \Gamma \to \mathsf{Groups}$$

on S^1. Here S^1 is the standard simplicial model of the circle with two nondegenerate simplices. The fact that this particular simplicial functor is of the homotopy type of the loop space on the two-dimensional sphere ΩS^2 follows from the classical result of Milnor. The fact that the functor $Q \mapsto B((\Omega S^2) \otimes Q)$ is isomorphic to the composite $b_2 \circ \Upsilon \circ \wp$ follows from the following isomorphism of simplicial groups:

$$(\Omega\ S^2) \otimes Q \cong (S^1) \odot \wp(Q).$$

In this section we ask the following question: is every flat object of $\mathsf{types}(2,3)$ isomorphic to one of the form $B((\Omega\ S^2) \otimes Q)$, where $Q \in \mathsf{SG}$?

An object (π_2, π_3, k) of $\Pi(2,3)$ is called *realizable via SG* if there exists a square group Q and an isomorphism

$$\kappa(B((\Omega\ S^2) \otimes Q)) \cong (\pi_2, \pi_3, k).$$

An object $X \in \mathsf{types}(2,3)$ is called *realizable via SG* provided $\kappa(X)$ is realizable via SG. In other words X is isomorphic to $B((\Omega S^2) \otimes Q) \cong ev(S^1)(\wp(Q))$, where $ev(S^1) : \mathsf{PSG} \to \mathsf{CW}(2,3)$ is the same as in Section 2.7.

LEMMA 29. *If (π_2', π_3', k') and (π_2'', π_3'', k'') are realizable via SG, then $(\pi_2' \times \pi_2'', \pi_3' \times \pi_3'', k' \times k'')$ is also realizable via SG.*

Proof. Indeed, if Q' and Q'' realize (π_2', π_3', k') and (π_2'', π_3'', k'') respectively, then $Q' \times Q''$ realizes $(\pi_2' \times \pi_2'', \pi_3' \times \pi_3'', k' \times k'')$. ∎

An abelian group A is called *realizable via SG* provided $(A, \Psi(A), \tau')$ is realizable via SG.

LEMMA 30. *If π_2 is realizable via SG, then any flat object of the form (π_2, π_3, k) is also realizable via SG.*

Proof. Let Q realize $(\pi_2, \Psi(\pi_2), \tau')$. Since (π_2, π_3, k) is flat, the homomorphism k is the composite $k = k' \circ \tau'$, where $k' : \Psi(\pi_2) \to \pi_3$ is defined uniquely. Let us recall that in Section 4.6 we defined the pushforward construction for PSG's. It is clear that pushforward construction of a square group has a square group structure in an obvious way. It follows that $k'_*(Q) \in \mathsf{SG}$ realizes (π_2, π_3, k). ∎

LEMMA 31. *Let $(A_i)_{i \in I}$ be a family of abelian groups. If each A_i is realizable via SG, then $\oplus_{i \in I} A_i$ is also realizable via SG.*

Proof. Assume Q_i realizes A_i. We claim that the coproduct of Q_i in the category of square groups realizes $\oplus_{i\in I} A_i$. Since π_i respects filtered colimits, it suffices to assume that I is finite and therefore without loss of generality one can assume that I consists of two elements. In this case the result follows from the isomorphisms of Lemma 13. ∎

LEMMA 32. *a) If $\theta(A) = 0$, then A is realizable via SG. In particular any free abelian group, or any divisible abelian group is realizable via SG. Moreover, if 2 is invertible in an abelian group A, then A is realizable via SG.*

b) For any $n \geq 1$ the group $\mathbb{Z}/2^n\mathbb{Z}$ is realizable via SG.

Proof. If $\theta(A) = 0$ there exists (N_A) such that $\omega(N_A)$ has a square group structure (see Theorem 26) and this SG realizes A. b) Let us consider the following square group:

$$Q_e = \mathbb{Z}/2^{n+1}\mathbb{Z} = Q_{ee}$$

The homomorphism P is multiplication by 2^n. Define the quadratic map

$$H : \mathbb{Z}/2^{n+1} \to \mathbb{Z}/2^{n+1}$$

by $H(x) = x^2 - x$. One easily checks that in this way one obtains a SG which realizes $\mathbb{Z}/2^n$. ∎

Let $\mathbf{A}$ be the smallest class of abelian groups which is closed under arbitrary direct sums and contains i) $\mathbb{Z}/2\mathbb{Z}$, ii) all abelian groups A such that 2 is invertible in A and iii) all abelian groups A such that $\mathsf{Ext}(A, Sym^2 A) = 0$, where $Sym^2 A$ is the second symmetric power of A. It is clear that then $\mathbf{A}$ contains all cyclic groups, and hence all finitely generated abelian groups as well as all free and all divisible abelian groups.

COROLLARY 33. *Let $X \in \mathsf{types}(2,3)$ be a flat object. Then X is realizable via SG provided $\pi_2 X \in \mathbf{A}$.*

5.9. REALIZATION OF STABLE TWO-STAGE SPACES VIA SQUARE GROUPS Now we consider the corresponding stable problem. Let us fix an integer $n \geq 3$. We let $\underline{e}_n$ be the composite of the following functors:

$$\mathsf{SG} \xrightarrow{\wp} \mathsf{PSG} \xrightarrow{e_n} \mathsf{types}(n, n+1).$$

where e_n is the composite of the following functors:

$$\mathsf{PSG} \xrightarrow{\Upsilon} \mathsf{BCG} \xrightarrow{\lambda} \mathsf{SCG} \xrightarrow{b_n} \mathsf{types}(n, n+1).$$

From the homotopy theoretic point of view the functor $\underline{e}_n$ is the same as $Q \mapsto P_{n+2}B((\Omega S^n) \otimes Q)$. In this section we ask the following question: What sort of objects of $\mathsf{types}(n, n+1)$ are isomorphic to ones of the form $\underline{e}_n(Q)$, where $Q \in \mathsf{SG}$?

We start with few easy observations. We let SG_s be the full subcategory of the category SG consisting of objects Q such that for any $a \in Q_{ee}$ one has $HP(a) = 2a$.

LEMMA 34. *For any $Q \in \mathsf{SG}_s$ one has $\wp(Q) \in \mathsf{PSG}_s$.*

Proof. The involution σ on $(\wp(M))_{ee} = M_{ee}$ is defined by $\sigma = HP - \mathsf{Id}$. Thus $\sigma = \mathsf{Id}$ iff $M \in \mathsf{SG}_s$. ∎

LEMMA 35. *Let Q be a square group. Then there exists the unique square group structure on $\underline{Q}$ such that the quotient map $Q \to \underline{Q}$ is a morphism in SG, where*

$$(\underline{Q})_e = Q_e, \quad (\underline{Q})_{ee} = Q_{ee}/(HP - 2\mathsf{Id})$$

Moreover the functor $\mathsf{SG} \to \mathsf{SG}_s$ is the left adjoint functor to the inclusion functor $\mathsf{SG}_s \subset \mathsf{SG}$

Proof is immediate.

COROLLARY 36. *For any $Q \in \mathsf{SG}$ the group $\underline{\pi}_1^Q$ is a vector space over $\mathbb{Z}/2\mathbb{Z}$.*

Proof. By the definition we have $\underline{\pi}_1^Q = \pi_1^{\underline{Q}}$. Since $\wp(\underline{Q}) \in \mathsf{PSG}_s$ it suffices to show that if $M \in \mathsf{PSG}_0 \bigcap \mathsf{PSG}_s$ then 2 annihilates π_1^M. But by definition $M \in \mathsf{PSG}_0$ implies that the involution on π_1^M is multiplication by (-1), while $M \in \mathsf{PSG}_s$ implies that the involution on π_1^M is trivial, hence the result. ∎

We let $\mathsf{types}(n, n+1)_*$ be the full subcategory of $\mathsf{types}(n, n+1)$ consisting of spaces X such that $\pi_{n+1}X$ is a vector space over $\mathbb{Z}/2\mathbb{Z}$. Thus the values of the functor $\underline{e}_n$ lie in $\mathsf{types}(n, n+1)_*$.

THEOREM 37. *For any object $X \in \mathsf{types}(n, n+1)_*$, $n \geq 3$ there exists an object $Q \in \mathsf{SG}_s$ and an isomorphism $\underline{e}_n(Q) \cong X$ in $\mathsf{types}(n, n+1)$. Moreover, one can assume that Q_e is an abelian group.*

Proof. Still it suffices to realize objects like (π_n, π_{n+1}, k), where π_{n+1} is a vector space over $\mathbb{Z}/2\mathbb{Z}$. Using pushforward construction it suffices to consider the universal case $(A, A/2A, k)$, where $k : A \to A/2A$ is the canonical projection. We choose a basis $(b_i)_{i\in I}$ of $A/2A$. Let B a the free $\mathbb{Z}/4\mathbb{Z}$-module, with a basis $(\tilde{b}_i)_{i\in I}$. We have canonical epimorphisms $\epsilon : A \to A/2A$ $\epsilon(a) = \bar{a}$ and $\varepsilon : B \to A/2A$, $\varepsilon(\tilde{b}_i) = b_i$. It follows that one has the following exact sequence

$$0 \longrightarrow A/2A \xrightarrow{\alpha} B \xrightarrow{\varepsilon} A/2A \longrightarrow 0$$

where $\alpha(b_i) = 2\tilde{b}_i$. Let us consider the corresponding pullback diagram

$$\begin{array}{ccc} C & \xrightarrow{\rho} & A \\ {\scriptstyle \varrho}\downarrow & & \downarrow{\scriptstyle \epsilon} \\ B & \xrightarrow[\varepsilon]{} & A/2A \end{array}$$

It follows that one has the following exact sequence

$$0 \longrightarrow A/2A \xrightarrow{\iota} C \xrightarrow{\rho} A \longrightarrow 0.$$

We now put

$$Q_e = C, \quad Q_{ee} = B,$$
$$P = \iota \circ \varepsilon, \quad H = h \circ \rho,$$

where $h : B \to B$ is the quadratic map uniquely defined by the conditions: $h(\tilde{b}_i) = 0$ and $(\tilde{b}_i \mid \tilde{b}_j)_h = 0$, if $i \neq j$ and $(\tilde{b}_i \mid \tilde{b}_i)_h = 2\tilde{b}_i$. Here $i, j \in I$. A direct computation shows that in this way one really gets a PSG which realizes $(A, A/2A, k)$. ∎

5.10. The transformation Δ The homotopy groups π_i^Q, $i = 0, 1$ and the stable homotopy group $\underline{\pi}_1^Q$ of a square group Q depends only on the underlying presquare group $\wp(Q)$. In [6] a homomorphism $\Delta_Q : \pi_0^Q \to \pi_1^Q$ was constructed, which defines the natural transformation of functors defined on SG. Recall that

$$\Delta(\bar{x}) = HPH(x) + H(x + x) - 4H(x), \ x \in Q_e.$$

Since

$$\Delta P = HPHP + 2HP - 4HP = 2HP - 2HP = 0$$

we see that Δ is well-defined. Since $\sigma = HP - \mathsf{Id}$, one can rewrite

$$\Delta(\bar{x}) = \sigma(H(x)) - H(x) + (x \mid x)_H \tag{7}$$

Now it is clear that Δ is additive, because

$$(x \mid y)_\Delta = \sigma(x \mid y)_H - (x \mid y)_H + (x \mid y)_H + (y \mid x)_H = 0.$$

We have also

$$P\Delta = P\sigma H - PH + P(x \mid x)_H = 0.$$

Thus Δ really defines the natural transformation $\pi_0 \to \pi_1$.

It follows from the identity (7) that the following diagram is commutative:

$$\begin{array}{ccc} \pi_0 & \xrightarrow{\Delta} & \pi_1 \\ \downarrow & & \downarrow \epsilon \\ \pi_0/2\pi_0 & \xrightarrow[\underline{k}]{} & \underline{\pi}_1 \end{array}$$

Let $\alpha : \pi_0^Q \to Q_{ee}$ be a homomorphism; according to Lemma 20 we have also the square group Q^α which has the same underlying presquare group as Q and therefore the same homotopy groups as Q. One easily sees that

$$\Delta^\alpha = \Delta + \sigma\alpha - \alpha$$

which shows that Δ could not be constructed only in terms of presquare groups.

LEMMA 38. *Let A be a finitely generated abelian group and let B be any abelian group. Furthermore let $f : A \to B$ be any homomorphism. Then there exists a square group Q such that $\pi_0^Q = A$, $\pi_1^Q = B$ and $\Delta_Q = f$.*

Proof. Using pushforward construction it suffices to consider the universal case $B = A$ and $f = \mathsf{Id}_A$. An abelian group A is called Δ-*realizable* if there exists a square group Q such that $\pi_0^Q = A = \pi_1^Q$ and $\Delta_Q = \mathsf{Id}_A$. Since $\pi_i : \mathsf{SG} \to \mathsf{Ab}$, $i = 0, 1$ takes finite products to finite products and $\Delta_{M\times N} = (\Delta_M, \Delta_N)$ it suffices to show that any cyclic group is Δ-realizable. Assume 2 is invertible in A. Then we have the following square group

$$Q_e = A = Q_{ee}, \quad P = 0, \quad \text{and} \quad H(a) = -\frac{a}{2}$$

which realizes A. The square group $\mathbb{Z}_{nil}$ realizes $\mathbb{Z}$, where

$$(\mathbb{Z}_{nil})_e = \mathbb{Z} = (\mathbb{Z}_{nil})_{ee}, \quad P = 0 \quad \text{and} \quad H(a) = \frac{a^2 - a}{2}$$

Finally the square group constructed in the proof of the part b) of Lemma 32 realizes $\mathbb{Z}/2^n\mathbb{Z}$ for all $n \geq 1$. ∎

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