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Cohomology monoids of monoids with coefficients in semimodules I

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Dedicated to Hvedri Inassaridze on his 80th birthday

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Abstract Using the techniques of homological algebra of semimodules developed in our previous papers, we introduce new cohomology monoids of an arbitrary monoid M with coeffcients in semimodules over M, that is, with coeffcients in abelian monoids on which M acts. The construction is similar to the construction of the Eilenberg-Mac Lane cohomology groups of monoids. In particular, we use an M-semimodule analog of the classical normalized bar resolution. An explicit computation of these cohomology monoids in the case where M is a finite cyclic group is given.

Keywords Monoid · Semimodule · Chain complex · Cohomology monoid · Chain homotopy · Finite cyclic group

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1 Introduction

In [7,8], in order to give a cohomological description of the Schreier extensions of semimodules by monoids, we introduced cohomology monoids of an arbitrary monoid M with coefficients in semimodules over M. Later, in [5], developing the basic idea of Sweedler's two-cocycles of [13], Haile, Larson and Sweedler introduced the same cohomology monoids (in the case where M is a group) as a generalization of the usual

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Amitsur and Galois cohomology groups. Transferring the developed theory to topological spaces leads to inadequate cohomology theories in the sense that the resulting cohomology monoids of topological spaces do not satisfy the homotopy axiom (see Remark 3.6). Thus arose the problem of finding such a modification of the cohomology monoids, which would become a homotopy invariant of topological spaces. This has been done in [9], where a version of homological algebra for semimodules was given, which enables one to construct cohomology and homology monoids of topological spaces with coefficients in abelian monoids so that the homotopy axiom holds (see Example 3.5). The machinery of homological algebra introduced in [9] (and further developed in [10, 11]) gives rise to new cohomology monoids of an arbitrary monoid M with coefficients in semimodules over M. In the present paper, we define these cohomology monoids and show that they are more adequate for actual computation. In particular, we calculate them in the case where M is a finite cyclic group by using the technique of free resolutions. A connection of the new cohomology monoids with monoid extension theory will be discussed in a subsequent paper.

2 Preliminaries

A *semiring* $\Lambda = (\Lambda, +, 0, \cdot, 1)$ is an algebraic structure in which $(\Lambda, +, 0)$ is an abelian monoid, $(\Lambda, \cdot, 1)$ a monoid, and

$$\begin{aligned}
&\lambda \cdot (\lambda' + \lambda'') = \lambda \cdot \lambda' + \lambda \cdot \lambda'',\\
&(\lambda' + \lambda'') \cdot \lambda = \lambda' \cdot \lambda + \lambda'' \cdot \lambda,\\
&\lambda \cdot 0 = 0 \cdot \lambda = 0
\end{aligned}$$

for all $\lambda, \lambda', \lambda'' \in \Lambda$ (see e.g. [2]).

Let Λ be a semiring. An abelian monoid $A = (A, +, 0)$ together with a map $\Lambda \times A \longrightarrow A$, written as $(\lambda, a) \mapsto \lambda a$, is called a *left* Λ*-semimodule* if

$$\begin{aligned}
\lambda(a + a') &= \lambda a + \lambda a',\\
(\lambda + \lambda')a &= \lambda a + \lambda' a,\\
(\lambda \cdot \lambda')a &= \lambda(\lambda' a),\\
1a = a, &\quad 0a = 0
\end{aligned}$$

for all $\lambda, \lambda' \in \Lambda$ and $a, a' \in A$. It immediately follows that $\lambda 0 = 0$ for any $\lambda \in \Lambda$. A right Λ-semimodule A is defined similarly. In this paper, left Λ-semimodules are simply called Λ-semimodules.

A map $f : A \longrightarrow B$ between Λ-semimodules A and B is called a Λ-*homomorphism* if $f(a + a') = f(a) + f(a')$ and $f(\lambda a) = \lambda f(a)$ for all $a, a' \in A$ and $\lambda \in \Lambda$. It is obvious that any Λ-homomorphism carries 0 into 0.

A Λ-*subsemimodule* A of a Λ-semimodule B is a subsemigroup of $(B, +)$ such that $\lambda a \in A$ for all $a \in A$ and $\lambda \in \Lambda$. Clearly $0 \in A$. The *quotient* Λ*-semimodule* B/A is defined as the quotient Λ-semimodule of B by the smallest congruence on the

Λ-semimodule B some class of which contains A. Denote the congruence class of $b \in B$ by $[b]$. Then $[b_1] = [b_2]$ if and only if $a_1 + b_1 = a_2 + b_2$ for some $a_1, a_2 \in A$.

Let $\mathbb{N}$ be the semiring of nonnegative integers. An $\mathbb{N}$-semimodule A is simply an abelian monoid, and an $\mathbb{N}$-homomorphism $f : A \longrightarrow B$ is just a homomorphism of abelian monoids, and A is an $\mathbb{N}$-subsemimodule of an $\mathbb{N}$-semimodule B if and only if A is a submonoid of the monoid $(B, +, 0)$.

Next recall that the *group completion* of an abelian monoid M can be constructed in the following way. Define an equivalence relation $\sim$ on $M \times M$ as follows:

$$(u, v) \sim (x, y) \Leftrightarrow u + y + z = v + x + z \quad \text{for some} \quad z \in M.$$

Let $[u, v]$ denote the equivalence class of (u, v). The quotient set $(M \times M)/\sim$ with the addition $[x_1, y_1] + [x_2, y_2] = [x_1 + x_2, y_1 + y_2]$ is an abelian group $(0 = [x, x],\ -[x, y] = [y, x])$. This group, denoted by $K(M)$, is the group completion of M, and $k_M : M \longrightarrow K(M)$ defined by $k_M(x) = [x, 0]$ is the canonical homomorphism. If M is a semiring, then the multiplication $[x_1, y_1] \cdot [x_2, y_2] = [x_1x_2 + y_1y_2, x_1y_2 + y_1x_2]$ converts $K(M)$ into the *ring completion* of the semiring M, and k_M into the canonical semiring homomorphism. Now assume that A is a Λ-semimodule. Then $K(A, +, 0)$ with the multiplication $[\lambda_1, \lambda_2][a_1, a_2] = [\lambda_1 a_1 + \lambda_2 a_2, \lambda_1 a_2 + \lambda_2 a_1]$, $\lambda_1, \lambda_2 \in \Lambda$, $a_1, a_2 \in A$, becomes a $K(\Lambda)$-module. This $K(\Lambda)$-module, denoted by $K(A)$, is the $K(\Lambda)$-*module completion* of the Λ-semimodule A, and $k_A = k_{(A,+,0)}$ is the canonical Λ-homomorphism. Clearly, $K(A)$ is in fact an additive functor: for any homomorphism $f : A \longrightarrow B$ of Λ-semimodules, $K(f) : K(A) \longrightarrow K(B)$ defined by $K(f)([a_1, a_2]) = [f(a_1), f(a_2)]$ is a $K(\Lambda)$-homomorphism.

A Λ-semimodule A is said to be *cancellative* if whenever $a + a' = a + a''$, $a, a'a'' \in A$, one has $a' = a''$. Obviously, A is cancellative if and only if the canonical Λ-homomorphism $k_A : A \longrightarrow K(A)$ is injective. Consequently, for a cancellative Λ-semimodule A, one may assume that A is a Λ-subsemimodule of $K(A)$, and that each element b of $K(A)$ is a difference of two elements from A, i.e., $b = a_1 - a_2$, where $a_1, a_2 \in A$.

A Λ-semimodule A is called a Λ-*module* if $(A, +, 0)$ is an abelian group. One can easily see that A is a Λ-module if and only if A is a $K(\Lambda)$-module. Hence, if A is a Λ-module, then $K(A) = A$ and $k_A = 1_A$. For a Λ-semimodule A, by $U(A)$ we denote the *maximal* Λ-*submodule* of A, i.e.,

$$U(A) = \{a \in A \mid a + a' = 0 \ \text{ for some } \ a' \in A\}.$$

A subset T of a Λ-semimodule A is a set of Λ-*generators* for A if every element of A can be written as a finite sum $\sum \lambda_i t_i$, where $\lambda_i \in \Lambda$ and $t_i \in T$. A is a *free* Λ-*semimodule* on T, or T is a Λ-*basis* of A, if each element a of A has a unique representation of the form $a = \sum_{t \in T} \lambda_t t$, called the representation of a by the Λ-basis T, where $\lambda_t \in \Lambda$ and all but a finite number of the λ_t are zero.

Suppose M is an arbitrary, multiplicatively written, monoid. The free abelian monoid $\mathbb{N}[M]$ generated by the elements $x \in M$ consists of the finite formal sums

$\sum_{x \in M} n_x x$ with coefficients $n_x \in \mathbb{N}$. The product in M induces a product

$$\sum_{x \in M} n_x x \cdot \sum_{y \in M} n'_y y = \sum_{x,y \in M} (n_x n'_y) xy$$

of two such elements, and makes $\mathbb{N}[M]$ a semiring, the *monoid semiring of M with nonnegative integer coefficients*. Semimodules over $\mathbb{N}[M]$ are called M-semimodules.

3 Cohomology monoids

In order to introduce new cohomology monoids of monoids with coefficients in semimodules we need some definitions and facts from [9].

Definition 3.1 ([9]) We say that a sequence of Λ-semimodules and Λ-homomorphisms

$$X : \cdots \rightrightarrows X_{n+1} \overset{\partial^+_{n+1}}{\underset{\partial^-_{n+1}}{\rightrightarrows}} X_n \overset{\partial^+_{n}}{\underset{\partial^-_{n}}{\rightrightarrows}} X_{n-1} \rightrightarrows \cdots, \quad n \in \mathbb{Z},$$

written $X = \{X_n, \partial^+_n, \partial^-_n\}$ for short, is a *chain complex* if

$$\partial^+_n \partial^+_{n+1} + \partial^-_n \partial^-_{n+1} = \partial^+_n \partial^-_{n+1} + \partial^-_n \partial^+_{n+1}$$

for each integer n. For every chain complex X, we define the Λ-semimodule

$$Z_n(X) = \left\{x \in X_n | \partial^+_n(x) = \partial^-_n(x)\right\},$$

the *n-cycles*, and the *n-th homology Λ-semimodule*

$$H_n(X) = Z_n(X)/\rho_n(X),$$

where $\rho_n(X)$ is a congruence on $Z_n(X)$ defined as follows:

$$x \ \rho_n(X) \ y \quad \Leftrightarrow \quad x + \partial^+_{n+1}(u) + \partial^-_{n+1}(v) = y + \partial^+_{n+1}(v) + \partial^-_{n+1}(u)$$
$$\text{for some} \quad u, v \quad \text{in} \quad X_{n+1}.$$

The Λ-homomorphisms $\partial^+_n, \partial^-_n$ are called *differentials* of the chain complex X.

A chain complex X is nonnegative if $X_n = 0$ for $n < 0$.

One can think of an ordinary chain complex of Λ-semimodules

$$C : \cdots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \cdots, \quad \partial_n \partial_{n+1} = 0, \quad n \in \mathbb{Z},$$

as a chain complex in the sense of Definition 3.1; namely, we identify C with the chain complex

$$\cdots \rightrightarrows C_{n+1} \overset{\partial_{n+1}}{\underset{0}{\rightrightarrows}} C_n \overset{\partial_n}{\underset{0}{\rightrightarrows}} C_{n-1} \rightrightarrows \cdots .$$

Defining $H_k(C)$ to be $H_k(\{C_n, \partial_n, 0\})$, one has $H_k(C) = \mathrm{Ker}(\partial_k)/\partial_{k+1}(C_{k+1})$.

3.2 A sequence $G = \{G_n, d_n^+, d_n^-\}$ of Λ-modules and Λ-homomorphisms is a chain complex if and only if

$$\cdots \longrightarrow G_n \xrightarrow{d_n^+ - d_n^-} G_{n-1} \longrightarrow \cdots$$

is an ordinary chain complex of Λ-modules. Obviously, for any chain complex $G = \{G_n, d_n^+, d_n^-\}$ of Λ-modules, $H_*(G)$ coincides with the usual homology $H_*(\{G_n, d_n^+ - d_n^-\})$.

Definition 3.3 ([9]) Let $X = \{X_n, \partial_n^+, \partial_n^-\}$ and $X' = \{X'_n, \partial_n^{'+}, \partial_n^{'-}\}$ be chain complexes of Λ-semimodules. We say that a sequence $f = \{f_n\}$ of Λ-homomorphisms $f_n : X_n \longrightarrow X'_n$ is a $\pm$-*morphism* from X to X' if

$$f_{n-1}\partial_n^+ = \partial_n^{'+} f_n \quad \text{and} \quad f_{n-1}\partial_n^- = \partial_n^{'-} f_n \quad \text{for all} \quad n.$$

If $f = \{f_n\} : X \longrightarrow X'$ is a $\pm$-morphism of chain complexes, then $f_n(Z_n(X)) \subset Z_n(X')$, and the map

$$H_n(f) : H_n(X) \longrightarrow H_n(X'), \quad H_n(f)(\mathrm{cl}(x)) = \mathrm{cl}(f_n(x)),$$

is a homomorphism of Λ-semimodules. Thus H_n is a covariant additive functor from the category of chain complexes and their $\pm$-morphisms to the category of Λ-semimodules.

3.4 If $X = \{X_n, \partial_n^+, \partial_n^-\}$ is a chain complex of Λ-semimodules, then

$$K(X)\colon \cdots \longrightarrow K(X_{n+1}) \xrightarrow{K(\partial_{n+1}^+) - K(\partial_{n+1}^-)} K(X_n) \xrightarrow{K(\partial_n^+) - K(\partial_n^-)} K(X_{n-1}) \longrightarrow \cdots$$

is an ordinary chain complex of $K(\Lambda)$-modules (i.e., Λ-modules) (see 3.2). When each X_n is cancellative, then the converse is also true. Further, for any chain complex $X = \{X_n, \partial_n^+, \partial_n^-\}$ of Λ-semimodules, the canonical $\pm$-morphism $k_X = \{k_{X_n} : X_n \longrightarrow K(X_n)\}$ from X to the chain complex $\{K(X_n), K(\partial_n^+), K(\partial_n^-)\}$ induces the Λ-homomorphisms $H_n(k_X) : H_n(X) \longrightarrow H_n(K(X))$, $H_n(k_X)(\mathrm{cl}(x)) = \mathrm{cl}(k_{X_n}(x)) = \mathrm{cl}[x, 0]$. If X is a chain complex of cancellative Λ-semimodules, then $H_n(k_X)$ is injective and therefore $H_n(X)$ is a cancellative Λ-semimodule.

A cochain complex is a sequence of Λ-semimodules and Λ-homomorphisms

$$Y: \quad \cdots \rightrightarrows Y^{n-1} \underset{\delta_-^{n-1}}{\overset{\delta_+^{n-1}}{\rightrightarrows}} Y^n \underset{\delta_-^{n}}{\overset{\delta_+^{n}}{\rightrightarrows}} Y^{n+1} \rightrightarrows \cdots$$

with

$$\delta_+^n \delta_+^{n-1} + \delta_-^n \delta_-^{n-1} = \delta_+^n \delta_-^{n-1} + \delta_-^n \delta_+^{n-1}$$

for all n. One obviously defines the n*-cocycles* of Y, the n*-th cohomology* Λ*-semimodule* $H^n(Y)$, a $\pm$-morphism $g : Y \longrightarrow Y'$ of cochain complexes and a Λ-homomorphism $H^n(g) : H^n(Y) \longrightarrow H^n(Y')$.

For a chain complex $X = \{X_n, \partial_n^+, \partial_n^-\}$ of Λ-semimodules and a Λ-semimodule A, we define a cochain complex $\mathrm{Hom}_\Lambda(X, A) = \{\mathrm{Hom}_\Lambda(X, A)^n, \delta_+^n, \delta_-^n\}$ of abelian monoids by

$$\mathrm{Hom}_\Lambda(X, A)^n = \mathrm{Hom}_\Lambda(X_n, A), \quad \delta_+^n = \mathrm{Hom}_\Lambda(\partial_{n+1}^{(-)^{n+1}}, A),$$
$$\delta_-^n = \mathrm{Hom}_\Lambda(\partial_{n+1}^{(-)^{n}}, A),$$

where

$$(-)^j = \begin{cases} + & j \quad \text{even} \\ - & j \quad \text{odd.} \end{cases}$$

The n-th cohomology monoid of this cochain complex, denoted by $H^n(X, A)$, is called the n*-th cohomology monoid of* X *with coefficients in the* Λ*-semimodule* A. Further, let C be a right Λ-semimodule. Then the n*-th homology monoid of the chain complex* X *with coefficients in* C, denoted by $H_n(X, C)$, is defined as the n-th homology monoid of the chain complex $C \otimes_\Lambda X = \{C \otimes_\Lambda X_n,\ 1 \otimes \partial_n^+,\ 1 \otimes \partial_n^-\}$, the *tensor product* of C and X. It is clear that $H^n(X, A)$ and $H_n(X, C)$ are functorial in both variables.

Example 3.5 ([9]) Let S be a sequence of Λ-semimodules $S_0, S_1, S_2, \ldots$ together with Λ-homomorphisms

$$\partial_n^i : S_n \longrightarrow S_{n-1}, \qquad 0 \le i \le n,$$

satisfying $\partial_n^i \partial_{n+1}^j = \partial_n^{j-1} \partial_{n+1}^i$ for $0 \le i < j \le n+1$, i.e., S is a *presimplicial* Λ*-semimodule* (some authors would say that S is a semisimplicial Λ-semimodule). Then

$$\underline{S}: \quad \cdots \rightrightarrows S_n \underset{\partial_n^-}{\overset{\partial_n^+}{\rightrightarrows}} S_{n-1} \rightrightarrows \cdots \rightrightarrows S_2 \underset{\partial_2^-}{\overset{\partial_2^+}{\rightrightarrows}} S_1 \underset{\partial_1^-}{\overset{\partial_1^+}{\rightrightarrows}} S_0 \rightrightarrows 0$$

where

$$\partial_n^+ = \partial_n^0 + \partial_n^2 + \cdots, \qquad \partial_n^- = \partial_n^1 + \partial_n^3 + \cdots,$$

is a nonnegative chain complex of Λ-semimodules. We define the n-th homology Λ-semimodule of the presimplicial Λ-semimodule S by $H_n(S) = H_n(\underline{S})$, and also homology and cohomology monoids of S with coefficients as those of the chain complex $\underline{S}$. Since any presimplicial map $f : S \longrightarrow S'$ can be clearly regarded as a $\pm$-morphism from $\underline{S}$ to $\underline{S'}$, the functoriality of these constructions is obvious. Furthermore, presimplicially homotopic presimplicial maps induce the same maps on homology and cohomology (see [9] for details). These observstions enable one, in particular, to consruct singular homology and cohomology monoids of topological spaces with coefficients in abelian monoids so that the homotopy axiom holds. Indeed, if T is a topological space, we set $H_n(T) = H_n(\underline{\mathbf{FS}(T)})$, $H_n(T, A) = H_n(\underline{\mathbf{FS}(T)}, A)$ and $H^n(T, A) = H^n(\underline{\mathbf{FS}(T)}, A)$, where A is an abelian monoid, $\mathbf{S}$ the singular complex functor, and $\mathbf{F}$ the free abelian monoid functor. The homotopy axiom is satisfied since a topological homotopy between two continuous maps induces a simplicial homotopy between their images under the functor $\mathbf{S}$.

Remark 3.6 Another construction of homology of a chain complex $X = \{X_n, \partial_n^+, \partial_n^-\}$ comes from [5,8]. Namely, one defines

$$\mathscr{H}_n(X) = Z_n(X)/\widetilde{\rho}_n(X), \quad n \in \mathbb{Z},$$

where $Z_n(X)$ is as in Definition 3.1, i.e., $Z_n(X) = \left\{x \in X_n | \partial_n^+(x) = \partial_n^-(x)\right\}$, and a congruence $\widetilde{\rho}_n(X)$ on $Z_n(X)$ is given by

$$x \ \widetilde{\rho}_n(X) \ y \quad \Leftrightarrow \quad x = y + \partial_{n+1}^+(w) - \partial_{n+1}^-(w) \quad \text{for some} \quad w \quad \text{in} \quad U(X_{n+1}).$$

However, it can be easily seen that the n-th singular homology monoid $\mathscr{H}_n(T)$ of a topological space T defined by $\mathscr{H}_n(T) = \mathscr{H}_n(\underline{\mathbf{FS}(T)})$ is not a homotopy invariant.

Now let us introduce new cohomology monoids of monoids (and, in particular, of groups) with coefficients in semimodules.

Let M be a monoid and A be a (left) M-semimodule. Define

$$F^n(M, A) = \{f : M^n \longrightarrow A | f(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n) = 0, i = 1, 2, \ldots, n\},$$
$$n \geq 0.$$

Clearly, $F^n(M, A)$, together with the usual addition of functions, is an abelian monoid. Next, define monoid homomorphisms $d_-^n,\ d_+^n : F^n(M, A) \longrightarrow F^{n+1}(M, A)$ as follows:

$$(d_\pm^n f)(x_1, \ldots, x_{n+1}) = 0,\ n \geq 0,\ \text{if any } x_i = 1,\ i = 1, 2, \ldots, (n+1).$$

If each $x_i \neq 1, i = 1, 2, \dots, (n+1)$, then

$$(d_+^{2k} f)(x_1, \dots, x_{2k+1}) = \sum_{i=1}^{k} f(x_1, \dots, x_{2i-1}x_{2i}, \dots, x_{2k+1}) + f(x_1, \dots, x_{2k}),$$
$$k \geq 0,$$

$$(d_-^{2k} f)(x_1, \dots, x_{2k+1}) = x_1 f(x_2, \dots, x_{2k+1}) + \sum_{i=1}^{k} f(x_1, \dots, x_{2i}x_{2i+1}, \dots, x_{2k+1}),$$
$$k \geq 0,$$

$$(d_+^{2k-1} f)(x_1, \dots, x_{2k}) = x_1 f(x_2, \dots, x_{2k})$$
$$+ \sum_{i=1}^{k-1} f(x_1, \dots, x_{2i}x_{2i+1}, \dots, x_{2k}) + f(x_1, \dots, x_{2k-1}), \quad k \geq 1,$$

$$(d_-^{2k-1} f)(x_1, \dots, x_{2k}) = \sum_{i=1}^{k} f(x_1, \dots, x_{2i-1}x_{2i}, \dots, x_{2k}), \quad k \geq 1.$$

It is immediate from the definitions that the identity

$$d_+^n d_+^{n-1} + d_-^n d_-^{n-1} = d_+^n d_-^{n-1} + d_-^n d_+^{n-1}$$

holds for any $n \geq 1$. In other words, the sequence

$$F(M, A)\colon\ 0 \rightrightarrows F^0(M, A) \underset{d_-^0}{\overset{d_+^0}{\rightrightarrows}} F^1(M, A) \rightrightarrows \cdots \rightrightarrows F^n(M, A) \underset{d_-^n}{\overset{d_+^n}{\rightrightarrows}}$$
$$F^{n+1}(M, A) \rightrightarrows \cdots$$

is a nonnegative cochain complex of abelian monoids. The *n-th cohomology monoid* $H^n(M, A)$ *of* M *with coefficients in the* M*-semimodule* A is defined by

$$H^n(M, A) = H^n(F(M, A)), \quad n \geq 0.$$

It is obvious that $H^n(M, -)$ is a covariant additive functor from the category of M-semimodules to the category of abelian monoids.

In particular, one has

$$H^0(M, A) = \{a \in A \mid xa = a \text{ for any } x \in M\};$$

$$H^1(M, A) = \{f\colon M \longrightarrow A \mid f(1) = 0 \text{ and } xf(y) + f(x) = f(xy), x, y \in M\}/\rho^1,$$
$$f \rho^1 f' \Leftrightarrow \exists\, a_1, a_2 \in A\ :\ f(x) + xa_1 + a_2 = f'(x) + xa_2 + a_1,\ \forall\, x \in M;$$

$$H^2(M, A) = \{f\colon M \times M \longrightarrow A \mid f(x, 1) = 0 = f(1, y) \text{ and}$$
$$xf(y, z) + f(x, yz) = f(xy, z) + f(x, y),\ x, y, z \in M\}/\rho^2,$$

$$\begin{aligned} f\rho^2 f' \Leftrightarrow \exists\, g_1, g_2 \colon M \longrightarrow A \ :\ & g_1(1) = 0 \\ &= g_2(1) \text{ and } f(x, y) + x g_1(y) + g_1(x) + g_2(xy) \\ &= f'(x, y) + x g_2(y) + g_2(x) + g_1(xy),\ \forall\, x, y \in M. \end{aligned}$$

Remark 3.7 For an M-semimodule A, define

$$\mathscr{H}^n(M, A) = \left\{ f \in F^n(M, A) | d_+^n(f) = d_-^n(f) \right\} / \widetilde{\rho}^{\,n},$$

where a congruence $\widetilde{\rho}^{\,n}$ is given by

$$\begin{aligned} f\, \widetilde{\rho}^{\,n} f' \Leftrightarrow f &= f' + d_+^{n-1}(g) - d_-^{n-1}(g) \quad \text{for some} \quad g \quad \text{in} \quad U(F^{n-1}(M, A)) \\ &= F^{n-1}(M, U(A)) \end{aligned}$$

(see Remark 3.6). These cohomology monoids were introduced in [7,8] to describe the Schreier extensions of semimodules by monoids, and in [5] to generalize the usual Galois and Amitsur cohomology theories. They are adequate for many applications (see e.g. [1,3–5,7,8]), but difficult to compute in general. Even in the case where M is a group and A an abelian group with a zero o adjoined, i.e., $U(A) = A - \{o\}$, one faces significant problems when trying to compute them (see e.g. [12]). Also observe that if $U(A) = \{0\}$, then two cocycles f and f' are cohomologous if and only if $f = f'$. The cohomology monoid $H^n(M, A)$ is more computable alternative to $\mathscr{H}^n(M, A)$. In Sect. 4 it is shown that $H^n(M, A)$, unlike $\mathscr{H}^n(M, A)$, can be calculated via free resolutions. (Note that there is a natural surjective homomorphism of $\mathscr{H}^n(M, A)$ onto $H^n(M, A)$ given by $\widetilde{\mathrm{cl}}(f) \mapsto \mathrm{cl}(f)$.)

Next we present an equivalent construction of $H^n(M, A)$. Consider a sequence of M-semimodules and M-homomorphisms

$$(B, \varepsilon): \quad \cdots \rightrightarrows B_n \underset{\partial_n^-}{\overset{\partial_n^+}{\rightrightarrows}} B_{n-1} \rightrightarrows \cdots \rightrightarrows B_2 \underset{\partial_2^-}{\overset{\partial_2^+}{\rightrightarrows}} B_1 \underset{\partial_1^-}{\overset{\partial_1^+}{\rightrightarrows}} B_0 \xrightarrow{\ \varepsilon\ } \mathbb{N} \tag{$*$}$$

in which $\mathbb{N}$ is regarded as a trivial M-semimodule, $B_0 = \mathbb{N}(M)$ as a free M-semimodule on 1, B_n $(n > 0)$ is the free M-semimodule on the set of all symbols $[x_1|\cdots|x_n]$ with $x_i \in M - \{1\}$, and the M-homomorphisms $\varepsilon, \partial_1^+, \partial_1^-, \partial_n^+, \partial_n^-, n > 1$, are defined as follows:

$$\varepsilon(1) = 1, \quad \partial_1^+[x] = x, \quad \partial_1^-[x] = 1, \quad x \in M - \{1\},$$

and, making the convention that $[x_1|\cdots|x_n] = 0$ if any $x_j = 1$, we set

$$\begin{aligned} &\partial_{2k}^+[x_1|\cdots|x_{2k}] = x_1[x_2|\cdots|x_{2k}] + \sum_{i=1}^{k-1}[x_1|\cdots|x_{2i}x_{2i+1}|\cdots|x_{2k}] + [x_1|\cdots|x_{2k-1}], \\ &\quad k \geq 1, \end{aligned}$$

$$\partial_{2k}^{-}[x_1|\cdots|x_{2k}] = \sum_{i=1}^{k}[x_1|\cdots|x_{2i-1}x_{2i}|\cdots|x_{2k}], \quad k \geq 1,$$

$$\partial_{2k+1}^{+}[x_1|\cdots|x_{2k+1}] = x_1[x_2|\cdots|x_{2k+1}] + \sum_{i=1}^{k}[x_1|\cdots|x_{2i}x_{2i+1}|\cdots|x_{2k+1}], \quad k \geq 1,$$

$$\partial_{2k+1}^{-}[x_1|\cdots|x_{2k+1}] = \sum_{i=1}^{k}[x_1|\cdots|x_{2i-1}x_{2i}|\cdots|x_{2k+1}] + [x_1|\cdots|x_{2k}], \quad k \geq 1,$$
$$x_i \in M - \{1\}, \quad i = 1, 2, \ldots.$$

It is easy to see that $\partial_n^+ \partial_{n+1}^+ + \partial_n^- \partial_{n+1}^- = \partial_n^+ \partial_{n+1}^- + \partial_n^- \partial_{n+1}^+$, $n \geq 1$, and $\varepsilon\partial_1^+ = \varepsilon\partial_1^-$. In particular,

$$B: \quad \cdots \rightrightarrows B_n \underset{\partial_n^-}{\overset{\partial_n^+}{\rightrightarrows}} B_{n-1} \rightrightarrows \cdots \rightrightarrows B_2 \underset{\partial_2^-}{\overset{\partial_2^+}{\rightrightarrows}} B_1 \underset{\partial_1^-}{\overset{\partial_1^+}{\rightrightarrows}} B_0 \rightrightarrows 0$$

is a nonnegative chain complex. The n-th cohomology monoid $H^n(B, A)$ of B with coefficients in the M-semimodule A is naturally isomorphic to $H^n(M, A)$. Indeed, the canonical isomorphisms of abelian monoids

$$\xi^n\colon \operatorname{Hom}_{\mathbb{N}(M)}(B_n, A) \cong F^n(M, A), \quad n \geq 0,$$

assemble into a $\pm$-isomorphism of chain complexes

$$\begin{array}{ccccccccc}
0 \rightrightarrows \operatorname{Hom}_{\mathbb{N}(M)}(B_0, A) & \underset{\delta_-^0}{\overset{\delta_+^0}{\rightrightarrows}} & \operatorname{Hom}_{\mathbb{N}(M)}(B_1, A) & \rightrightarrows \cdots \rightrightarrows & \operatorname{Hom}_{\mathbb{N}(M)}(B_n, A) & \underset{\delta_-^n}{\overset{\delta_+^n}{\rightrightarrows}} & \operatorname{Hom}_{\mathbb{N}(M)}(B_{n+1}, A) \rightrightarrows \cdots \\
\downarrow{\scriptstyle \xi^0} \cong & & \downarrow{\scriptstyle \xi^1} \cong & & \downarrow{\scriptstyle \xi^n} \cong & & \downarrow{\scriptstyle \xi^{n+1}} \cong \\
0 \rightrightarrows F^0(M, A) & \underset{d_-^0}{\overset{d_+^0}{\rightrightarrows}} & F^1(M, A) & \rightrightarrows \cdots \rightrightarrows & F^n(M, A) & \underset{d_-^n}{\overset{d_+^n}{\rightrightarrows}} & F^{n+1}(M, A) \rightrightarrows \cdots,
\end{array}$$

$$\delta_+^n = \operatorname{Hom}_{\mathbb{N}(M)}(\partial_{n+1}^{(-)^{n+1}}, A), \quad \delta_-^n = \operatorname{Hom}_{\mathbb{N}(M)}(\partial_{n+1}^{(-)^n}, A), \quad n \geq 0.$$

Hence

$$H^n(B, A) \cong H^n(M, A), \quad n \geq 0.$$

Let A be an M-module, i.e., a module over $K(\mathbb{N}(M)) = \mathbb{Z}(M)$, the integral monoid ring of the monoid M. Then the cohomology monoids of $F(M, A)$ are the cohomology groups of the nonnegative (ordinary) cochain complex $\{F^n(M, A), \ d_+^n - d_-^n\}$ of abelian groups (see 3.2). Hence, $H^n(M, A)$ coincides with the usual n-th cohomology group of M with coefficients in A. This is also clear from the latter construction of the cohomology monoids. Indeed, the $\mathbb{Z}(M)$-module completion of (∗) leads to the normalized bar resolution of the trivial M-module $\mathbb{Z}$

$$\cdots \longrightarrow K(B_n) \xrightarrow{K(\partial_n^+)-K(\partial_n^-)} K(B_{n-1}) \longrightarrow \cdots \longrightarrow K(B_1) \xrightarrow{K(\partial_1^+)-K(\partial_1^-)}$$
$$K(B_0) \xrightarrow{K(\varepsilon)} \mathbb{Z} \longrightarrow 0,$$

and $\mathrm{Hom}_{\mathbb{Z}(M)}(K(B_n), A) = \mathrm{Hom}_{\mathbb{N}(M)}(B_n, A)$ for any M-module A.

4 Cohomology monoids of finite cyclic groups

In this section Λ always denotes an additively cancellative semiring, and all (left) Λ-semimodules are cancellative.

Definition 3.1 ([10]) Let $X = \{X_n, \partial_n^+, \partial_n^-\}$ and $X' = \{X'_n, \partial_n^{'+}, \partial_n^{'-}\}$ be chain complexes of Λ-semimodules. We say that a sequence $f = \{f_n\}$ of Λ-homomorphisms $f_n : X_n \longrightarrow X'_n$ is a *morphism* from X to X' if

$$\partial_n^{'+} f_n + f_{n-1}\partial_n^- = \partial_n^{'-} f_n + f_{n-1}\partial_n^+ \quad \text{for all} \quad n.$$

A sequence $f = \{f_n : X_n \longrightarrow X'_n\}$ of Λ-homomorphisms is a morphism from X to X' if and only if $K(f) = \{K(f_n) : K(X_n) \longrightarrow K(X'_n)\}$ is the usual chain map from $K(X)$ to $K(X')$ (see 3.4).

If $f = \{f_n\} : X \longrightarrow X'$ and $g = \{g_n\} : X' \longrightarrow X''$ are morphisms, then $gf = \{g_n f_n\} : X \longrightarrow X''$ is also a morphism. On the other hand, for every morphism $f = \{f_n\} : X \longrightarrow X'$, one has $f_n(Z_n(X)) \subset Z_n(X')$, and $H_n(f) : H_n(X) \longrightarrow H_n(X')$ defined by $H_n(f)(\mathrm{cl}(x)) = \mathrm{cl}(f_n(x))$ is a Λ-homomorphism. Therefore H_n is a covariant additive functor from the category of chain complexes of cancellative Λ-semimodules and their morphisms to the category of cancellative Λ-semimodules (see 3.4).

Note that any $\pm$-morphism between chain complexes of Λ-semimodules is a morphism in the sense of 4.1.

Definition 3.2 ([10]) Let $f = \{f_n\}$ and $g = \{g_n\}$ be morphisms from X to X'. We say that f is *homotopic* to g if there exist Λ-homomorphisms $s_n^+, s_n^- : X_n \longrightarrow X'_{n+1}$ such that

$$\partial_{n+1}^{'+} s_n^- + \partial_{n+1}^{'-} s_n^+ + s_{n-1}^- \partial_n^+ + s_{n-1}^+ \partial_n^- + f_n = \partial_{n+1}^{'+} s_n^+ + \partial_{n+1}^{'-} s_n^- + s_{n-1}^+ \partial_n^+ \\ + s_{n-1}^- \partial_n^- + g_n$$

for all n. The family $\{s_n^+, s_n^-\}$ is called a *chain homotopy* from f to g and we write $(s^+, s^-) : f \simeq g$.

A family $\{s_n^+, s_n^- : X_n \longrightarrow X'_{n+1}\}$ of Λ-homomorphisms is a chain homotopy from f to g if and only if $\{K(s_n^+) - K(s_n^-) : K(X_n) \longrightarrow K(X'_{n+1})\}$ is the usual chain homotopy from $K(f) : K(X) \longrightarrow K(X')$ to $K(g) : K(X) \longrightarrow K(X')$ (see 3.4).

Proposition 3.3 *Let $f, g : X \longrightarrow X'$ be morphisms between chain complexes of Λ-semimodules. If f is homotopic to g, then*

$$H_n(f) = H_n(g) : H_n(X) \longrightarrow H_n(X') \quad \textit{for all} \quad n.$$

We say that a morphism $f : X \longrightarrow X'$ is a *chain homotopy equivalence* if there exists a morphism $g : X' \longrightarrow X$ and chain homotopies $(s^+, s^-) : gf \simeq 1_X$ and $(t^+, t^-) : fg \simeq 1_{X'}$.

Corollary 3.4 *If $f : X \longrightarrow X'$ is a chain homotopy equivalence, then $H_n(f) : H_n(X) \longrightarrow H_n(X')$ is an isomorphism of Λ-semimodules for each n.*

One obviously defines a morphism of cochain complexes, a cochain homotopy, and a cochain homotopy equivalence. The formulation of the statements dual to 4.3 and 4.4 is also obvious.

If a morphism $f : X \longrightarrow X'$ is homotopic to a morphism $g : X \longrightarrow X'$, then $K(f) : K(X) \longrightarrow K(X')$ is homotopic to $K(g) : K(X) \longrightarrow K(X')$ in the usual sense. The converse is not always true. However, the following proposition is valid.

Proposition 3.5 ([10]) *Suppose that $X = \{X_n, \partial_n^+, \partial_n^-\}$ is a chain complex of free Λ-semimodules, $X' = \{X'_n, \partial_n'^+, \partial_n'^-\}$ a chain complex of Λ-semimodules and $f, g : X \longrightarrow X'$ are morphisms, and let $K(f)$ be homotopic to $K(g)$. Then f is homotopic to g.*

Proof Suppose that $\{s_n : K(X_n) \longrightarrow K(X'_{n+1})\}$ is a chain homotopy from $K(f)$ to $K(g)$, i.e., $K(f_n) - K(g_n) = (K(\partial_{n+1}'^+) - K(\partial_{n+1}'^-))s_n + s_{n-1}(K(\partial_n^+) - K(\partial_n^-))$. Since X_n is a free Λ-semimodule, one has a representation $s_n = K(s_n^+) - K(s_n^-)$, where s_n^+ and s_n^- are Λ-homomorphisms from X_n to X'_{n+1}. Indeed, let T be a Λ-basis of X_n. For any $t \in T$, $s_n(t) = x'_t - y'_t,\ x'_t,\ y'_t \in X'_{n+1}$. Defining s_n^+ and s_n^- by $s_n^+(t) = x'_t$ and $s_n^-(t) = y'_t$, respectively, we obtain the desired representation. Clearly, the family $\{s_n^+, s_n^-\}$ is a chain homotopy from f to g. $\square$

Definition 3.6 Let C be a Λ-semimodule. A sequence

$$(X, \varepsilon) : \quad \cdots \rightrightarrows X_n \underset{\partial_n^-}{\overset{\partial_n^+}{\rightrightarrows}} X_{n-1} \rightrightarrows \cdots \rightrightarrows X_2 \underset{\partial_2^-}{\overset{\partial_2^+}{\rightrightarrows}} X_1 \underset{\partial_1^-}{\overset{\partial_1^+}{\rightrightarrows}} X_0 \xrightarrow{\varepsilon} C$$

of Λ-semimodules and Λ-homomorphisms, shortly denoted by $X \xrightarrow{\varepsilon} C$, is called an *augmented chain complex over* C, or simply a *complex over* C, if

$$X : \quad \cdots \rightrightarrows X_n \underset{\partial_n^-}{\overset{\partial_n^+}{\rightrightarrows}} X_{n-1} \rightrightarrows \cdots \rightrightarrows X_2 \underset{\partial_2^-}{\overset{\partial_2^+}{\rightrightarrows}} X_1 \underset{\partial_1^-}{\overset{\partial_1^+}{\rightrightarrows}} X_0 \rightrightarrows 0$$

is a nonnegative chain complex and $\varepsilon\partial_1^+ = \varepsilon\partial_1^-$. A morphism from $X \xrightarrow{\varepsilon} C$ to $X' \xrightarrow{\varepsilon'} C'$ is a morphism $f = \{f_n\}$ from X to X' together with a Λ-homomorphism $\gamma : C \longrightarrow C'$ satisfying $\gamma\varepsilon = \varepsilon' f_0$. We also say that f is a morphism over γ.

Definition 3.7 A chain complex

$$X : \cdots \rightrightarrows X_{n+1} \overset{\partial^+_{n+1}}{\underset{\partial^-_{n+1}}{\rightrightarrows}} X_n \overset{\partial^+_{n}}{\underset{\partial^-_{n}}{\rightrightarrows}} X_{n-1} \rightrightarrows \cdots$$

is called K-*exact* if the ordinary chain complex of $K(\Lambda)$-modules

$$K(X) \colon \cdots \longrightarrow K(X_{n+1}) \xrightarrow{K(\partial^+_{n+1})-K(\partial^-_{n+1})} K(X_n) \xrightarrow{K(\partial^+_{n})-K(\partial^-_{n})} K(X_{n-1}) \longrightarrow \cdots$$

is an exact sequence. Similarly, an augmented chain complex

$$\cdots \rightrightarrows X_n \overset{\partial^+_{n}}{\underset{\partial^-_{n}}{\rightrightarrows}} X_{n-1} \rightrightarrows \cdots \rightrightarrows X_1 \overset{\partial^+_{1}}{\underset{\partial^-_{1}}{\rightrightarrows}} X_0 \xrightarrow{\varepsilon} C$$

is said to be K-*exact* if the ordinary chain complex of $K(\Lambda)$-modules

$$\cdots \longrightarrow K(X_n) \xrightarrow{K(\partial^+_{n})-K(\partial^-_{n})} K(X_{n-1}) \longrightarrow \cdots \longrightarrow K(X_1) \xrightarrow{K(\partial^+_{1})-K(\partial^-_{1})}$$
$$K(X_0) \xrightarrow{K(\varepsilon)} K(C) \longrightarrow 0$$

is exact.

Proposition 3.8 *Suppose we are given a diagram*

$$\begin{tikzcd} F \arrow[r, "\varepsilon"] \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] & C \arrow[d, "\gamma"] \\ F' \arrow[r, "\varepsilon'"] & C' \end{tikzcd}$$

in which $F \xrightarrow{\varepsilon} C$ *is a complex of free* Λ*-semimodules over a* Λ*-semimodule* C *and* $F' \xrightarrow{\varepsilon'} C'$ *is a* K*-exact augmented chain complex of (not necessarily free)* Λ*-semimodules over a* Λ*-semimodule* C'*, and* $f, g : F \longrightarrow F'$ *are morphisms over a* Λ*-homomorphism* $\gamma : C \longrightarrow C'$*. Then* f *is homotopic to* g*.*

Proof From the given diagram we get the diagram

$$\begin{tikzcd} K(F) \arrow[r, "K(\varepsilon)"] \arrow[d, shift right, "K(f)"'] \arrow[d, shift left, "K(g)"] & K(C) \arrow[d, "K(\gamma)"] \\ K(F') \arrow[r, "K(\varepsilon')"] & K(C'), \end{tikzcd}$$

where $K(F) \xrightarrow{K(\varepsilon)} K(C)$ is an ordinary chain complex of free $K(\Lambda)$-modules over the $K(\Lambda)$-module $K(C)$ and $K(F') \xrightarrow{K(\varepsilon')} K(C')$ is a resolution of the $K(\Lambda)$-module $K(C')$, and $K(f)$, $K(g)$ are chain maps lifting the $K(\Lambda)$-homomorphism $K(\gamma)$. By the Comparison Theorem (see, for example, [6, Theorem III.6.1]), $K(f)$ is homotopic to $K(g)$. Therefore, by Proposition 4.5, f is homotopic to g. ◻

Note that if $F \xrightarrow{\varepsilon} C$, $F' \xrightarrow{\varepsilon'} C'$ and $\gamma : C \longrightarrow C'$ are as in 4.8, then a lifting of γ, i.e., a morphism $f : F \longrightarrow F'$ over γ, may not exist at all.

As an immediate consequence of Proposition 4.8, we have

Corollary 4.9 *Let $F \xrightarrow{\varepsilon} C$ and $F' \xrightarrow{\varepsilon'} C$ be K-exact complexes of free Λ-semimodules over a Λ-semimodule C, and assume that $f : F \longrightarrow F'$ and $g : F' \longrightarrow F$ are morphisms over $1_C : C \longrightarrow C$. Then $gf \simeq 1_F$ and $fg \simeq 1_{F'}$.*

Finally, we come to the desired calculations of cohomology monoids of finite cyclic groups.

Let $C_m(t)$ be the multiplicative cyclic group of order m with generator t. Using four particular elements 0, 1, t and $N = 1 + t + \cdots + t^{m-1}$ in $\mathbb{N}(C_m(t))$, we obtain a K-exact augmented chain complex of free $\mathbb{N}(C_m(t))$-semimodules over the trivial $\mathbb{N}(C_m(t))$-semimodule $\mathbb{N}$

$$(W, \varepsilon) : \quad \cdots \rightrightarrows \mathbb{N}(C_m(t)) \underset{1}{\overset{t_*}{\rightrightarrows}} \mathbb{N}(C_m(t)) \underset{0}{\overset{N_*}{\rightrightarrows}} \mathbb{N}(C_m(t)) \underset{1}{\overset{t_*}{\rightrightarrows}}$$

$$\mathbb{N}(C_m(t)) \xrightarrow{\varepsilon} \mathbb{N},$$

$$\varepsilon(1) = 1, \quad t_*(u) = tu, \quad N_*(u) = (1 + t + \cdots + t^{m-1})u.$$

On the other hand, for $C_m(t)$ we have the augmented chain complex $(*)$ of free $C_m(t)$-semimodules over $\mathbb{N}$ which is also K-exact and which we denote by $B(C_m(t)) \xrightarrow{\varepsilon} \mathbb{N}$. There are morphisms

$$f = (\ldots, f_n, \ldots, f_3, f_2, f_1, 1) : W \longrightarrow B(C_m(t))$$

and

$$g = (\ldots, g_n, \ldots, g_3, g_2, g_1, 1) : B(C_m(t)) \longrightarrow W$$

over $1_{\mathbb{N}} : \mathbb{N} \longrightarrow \mathbb{N}$ defined as follows:

$$f_1(1) = [t],$$

$$f_{2k}(1) = \sum_{i_1, i_2, \ldots, i_k = 1}^{m-1} [t^{i_1}|t|t^{i_2}|t| \cdots |t^{i_k}|t] \in B_{2k}(C_m(t)), \qquad k \geq 1,$$

$$f_{2k+1}(1) = \sum_{i_1, i_2, \ldots, i_k = 1}^{m-1} [t|t^{i_1}|t|t^{i_2}|t|\cdots|t^{i_k}|t] \in B_{2k+1}(C_m(t)), \quad k \geq 1,$$

and

$$g_1[t^i] = 1 + t + \cdots + t^{i-1}, \quad 0 < i < m,$$

$$g_{2k}[t^{i_1}|\cdots|t^{i_{2k}}] = \begin{cases} 1, & \text{if } (i_1 + i_2 \geq m) \,\&\cdots\&\, (i_{2k-1} + i_{2k} \geq m), \\ 0, & \text{otherwise}, \end{cases}$$

$$0 < i_1, \ldots, i_{2k} < m, \qquad k \geq 1,$$

$$g_{2k+1}[t^{i_1}|\cdots|t^{i_{2k+1}}]$$

$$= \begin{cases} 1 + t + \cdots t^{i_1 - 1}, & \text{if } (i_2 + i_3 \geq m) \,\&\cdots\&(i_{2k} + i_{2k+1} \geq m), \\ 0, & \text{otherwise}, \end{cases}$$

$$0 < i_1, i_2, \ldots, i_{2k+1} < m, \quad k \geq 1.$$

By Corollary 4.9, $gf \simeq 1_W$ and $fg \simeq 1_{B(Cm(t))}$. Let A be a cancellative $C_m(t)$-semimodule. The additive (contravariant) functor $\mathrm{Hom}_{\mathbb{N}(Cm(t))}(-, A)$ carries chain homotopies to cochain homotopies. Besides, one has a natural isomorphism (of abelian monoids) $\mathrm{Hom}_{\mathbb{N}(Cm(t))}(\mathbb{N}(Cm(t)), A) \cong A$ given by $h \mapsto h(1)$. Consequently, by the statement dual to Corollary 3.4, the cohomology monoid $H^n(C_m(t), A)$ $(n \geq 0)$ is isomorphic to the n-th cohomology monoid of the cochain complex

$$0 \rightrightarrows A \underset{t^*}{\overset{1}{\rightrightarrows}} A \underset{0}{\overset{N^*}{\rightrightarrows}} A \underset{t^*}{\overset{1}{\rightrightarrows}} A \underset{0}{\overset{N^*}{\rightrightarrows}} A \rightrightarrows \cdots,$$

$$t^*(a) = ta, \quad N^*(a) = (1 + t + \cdots + t^{m-1})a.$$

Hence

$$H^0(C_m(t), A) \cong \{a \in A \mid ta = a\},$$

$$H^{2k}(C_m(t), A) \cong \{a \in A \mid ta = a\}/(1 + t + \cdots + t^{m-1})A, \quad k > 0,$$

$$H^{2k+1}(C_m(t), A) \cong \{a \in A \mid (1 + t + \cdots + t^{m-1})a = 0\}/\rho^{2k+1},$$

$$a \; \rho^{2k+1} \; a' \iff \exists a_1, a_2 \in A : a + a_1 + ta_2 = a' + a_2 + ta_1, \quad k \geq 0,$$

for any cancellative $C_m(t)$-semimodule A.

Further, since A is a cancellative $C_m(t)$-semimodule, one has the natural injective homomorphism

$$\{a \in A \mid ta = a\}/(1 + t + \cdots + t^{m-1})A \longrightarrow$$

$$\{b \in K(A) \mid tb = b\}/(1 + t + \cdots + t^{m-1})K(A),$$

$$\mathrm{cl}(a) \mapsto \mathrm{cl}(a),$$

which is in fact an isomorphism. Indeed, suppose $b \in K(A)$, i.e., $b = a_1 - a_2$, $a_1, a_2 \in A$, and assume that $tb = b$, i.e., $ta_1 + a_2 = ta_2 + a_1$. Consider $a = a_1 + (t + t^2 + \cdots + t^{m-1})a_2 \in A$. Clearly, $ta = a$ and $a - b = (1 + t + \cdots + t^{m-1})a_2$. Hence $\mathrm{cl}(a) \mapsto \mathrm{cl}(a) = \mathrm{cl}(b)$. Thus, the correspondence given by $\mathrm{cl}(a) \mapsto \mathrm{cl}(a)$ is an isomorphism. Consequently, $H^{2k}(C_m(t), A) \cong H^{2k}(C_m(t), K(A))$ for $k > 0$.

Next, it is obvious that

$$\{a \in A \mid (1 + t + \cdots + t^{m-1})a = 0\} = \{a \in U(A) \mid (1 + t + \cdots + t^{m-1})a = 0\}$$

and, for a $\in U(A)$,

$$a\ \rho^{2k+1}\ 0 \Longleftrightarrow a \in U(A) \cap (t - 1)K(A), \quad k = 0, 1, \ldots.$$

Thus, we come to the following result (cf. e.g. [6, Theorem IV.7.1]).

Theorem 4.10 *Let $C_m(t)$ be the multiplicative cyclic group of order m on generator t. If A is a cancellative $C_m(t)$-semimodule, then*

$$H^0(C_m(t), A) \cong \{a \in A \mid ta = a\},$$

$$H^{2k}(C_m(t), A) \cong H^{2k}(C_m(t), K(A)), \quad k > 0,$$

$$H^{2k+1}(C_m(t), A) \cong \{a \in U(A) | (1 + t + \cdots + t^{m-1})a = 0\}/U(A) \cap (t - 1)K(A), \quad k \geq 0.$$

In particular, we see that $H^n(C_m(t), A)$ is an (abelian) group for $n > 0$.

Remark 4.11 Let M be a monoid and A be a right M-semimodule. The *n-th homology monoid $H_n(M, A)$ of M with coefficients in* A is that of the chain complex B, i.e.,

$$H_n(M, A) = H_n(A \otimes_M B), \quad n = 0, 1, \ldots.$$

If A is a cancellative $C_m(t)$-semimodule, then, by applying $A \otimes_{C_m(t)} -$ to W, we obtain

$$H_0(C_m(t), A) \cong A/\rho_0, \quad a\ \rho_0\ a' \Longleftrightarrow \exists a_1, a_2 \in A : a + ta_1 + a_2 = a' + ta_2 + a_1,$$

$$H_{2k}(C_m(t), A) \cong \{a \in U(A) \mid (1 + t + \cdots + t^{m-1})a = 0\}/U(A) \cap (t - 1)K(A), \quad k > 0,$$

$$H_{2k+1}(C_m(t), A) \cong H_{2k+1}(C_m(t), K(A)), \quad k \geq 0.$$

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